ECE 7360    Optimal and Robust Control

Focused Independent Study Project

Presentation

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Reduced-interval model and edge theory

My focused independent study project is about interval system modeling and using edge theory to evaluate the robust controller.

• If the number of intervals are large in the model, reduced-interval method is used to reduce the numbers.
• Although $\mu$ synthesis can be used to evaluate the robust performance of the controller, edge theory gives another way to evaluate the system performance.
Reduced-interval model and edge theory Cont’d

Reduced-interval modeling:

Assumption:

A plant is modeled into different models by different set of sensor data. Then we present how to model an interval plant model from the multiple models for the same plant.

To simplify the controller design, we use a method similar to balanced order reduction in our text book to reduce the interval order of the model.
Reduced-interval model and edge theory Cont’d

• The approach uses singular value decomposition technique to reduce the number of uncertainty parameters.

• Each interval represents one bounded uncertainty parameter. The interval lengths can be used to determine the order of reduced-interval transfer function. Interval length indicates the uncertainty distribution. If the interval length is large, we need to increase the order of the interval model.
Reduced-interval model and edge theory Cont’d

Example: the plant is:

\[ g_i(s) = \frac{p_{i1}s^6 + p_{i10}s^5 + p_{i9}s^4 + p_{i8}s^3 + p_{i7}s^2}{s^6 + p_{i6}s^5 + p_{i5}s^4 + p_{i4}s^3 + p_{i3}s^2 + p_{i2}s + p_{i1}} \quad , i = 1, 2, \cdots, 25 \]

with the \( i \) th parameter vector

\[ p_i = [p_{i1}, p_{i2}, p_{i3}, \cdots, p_{i11}]^T, \]

\[ i = 1, \cdots, 25 \]
Reduced-interval model and edge theory  Cont’d

• First, we compute the average parameter vector over all the identified models as:

\[ p_0 = \frac{1}{25} \sum_{i=1}^{25} p_i \]

• Then, the deviation of the \( i \)th parameter vector about the average is defined as:

\[ \Delta p_i = p_i - p_0, i = 1, \cdots, 25. \]
Reduced-interval model and edge theory Cont’d

Then, an uncertainty matrix is generated from the deviation as:

$$\Delta p = [\Delta p_1, \cdots \Delta p_{25}].$$

To reflect the degree of variation in each coefficient, we define the weighted uncertainty matrix as:

$$\Delta p^W = W^{-1}\Delta p$$
Reduced-interval model and edge theory Cont’d

An interval modeling technique will be used to generate a linear interval model, which represents the original 25 systems, as:

\[
p = \left\{ p \mid p = \sum_{j=1}^{11} \alpha_j \hat{p}_j, \alpha_j \in [\alpha_j^-, \alpha_j^+] \right\}
\]

where \( \hat{p} \) is the nominal model, and \( \alpha_j \) are the identified bounded uncertainty parameters corresponding to the basis vector \( \hat{p}_i \).
Reduced-interval model and edge theory  Cont’d

The corresponding interval transfer function can be expressed as:

\[
G(s, \alpha) = \left\{ \begin{array}{ll}
\hat{n}(s) + \sum_{j=1}^{11} \alpha_j n_j(s) \\
g(s) : \frac{\hat{n}(s) + \sum_{j=1}^{11} \alpha_j n_j(s)}{\hat{d}(s) + \sum_{j=1}^{11} \alpha_j d_j(s)} , \alpha_j \in [\alpha^-_j, \alpha^+_j] 
\end{array} \right\}
\]

with

\[
\alpha=[\alpha_1, \cdots, \alpha_{11}]^T.
\]
Reduced-interval model and edge theory Cont’d

The steps using interval modeling is:

First, use SVD to obtain the basis matrix $U^W$ for $\Delta P^W$

$$\Delta P^w = U^W S V^T, \quad S = \text{diag}[s_1 \cdots s_2]$$

and compute the basic $U$ for $\Delta P$:

$$U = W U^W, \quad U = [\hat{p}_1, \cdots \hat{p}_{11}].$$
The singular values \( s_j \) are in descending order, this leads to a descending order of perturbation distribution in \( \hat{p}_j \).

then compute the base polynomials corresponding to \( \hat{p}_j \).

\[
n_j(s) = \sum \hat{p}_j (k + 6) s^{k+1},
\]

\[
d_j(s) = \sum_{k=1}^{11} \hat{p}_j (k) s^{k-1}, j = 1, \ldots, 11
\]
Reduced-interval model and edge theory Cont’d

Compute the coordinate vector of $\Delta \hat{p}_j$, corresponding to the basis vectors $\hat{p}_j$. 

$$\beta_i = U^{-1} \Delta p_i, i = 1, \ldots, 25$$

Obtain the parameter bounds corresponding to the average $p_0$ as:

$$\beta^+(j) = \max\{\beta_1(j), \ldots, \beta_{25}(j)\},$$

$$\beta^-(j) = \min\{\beta_1(j), \ldots, \beta_{25}(j)\},$$

$j = 1, 2, \ldots, 11$. 
Reduced-interval model and edge theory  Cont’d

Fourth, compute the nominal parameter vector as

\[ \hat{p} = p_0 + \sum_{j=1}^{11} \frac{\beta^+(j) + \beta^-(j)}{2} \]

\( \hat{n}(s) \) and \( \hat{d}(s) \) correspond to the nominal vector \( \hat{p} \). The parameter bounds with respect to the nominal vector are computed as:

\[ \alpha^+_j = \frac{\beta^+(j) - \beta^-(j)}{2} \]

\[ \alpha^-_j = \frac{\beta^-(j) - \beta^+(j)}{2} \]
Reduced-interval model and edge theory  Cont’d

The interval length of each identified uncertainty parameter $\alpha_j$ is defined as

$$\delta\alpha_j = \alpha_j^+ - \alpha_j^-$$

$\delta\alpha_j$ represents the distribution of the uncertainty in the direction of $\hat{p}_j$. The interval length can be used as a metric for choosing the order of the system interval model.
Reduced-interval model and edge theory Cont’d

• Following the examples above, the model uncertainty is dominated by the identified first three uncertainty parameters, $\alpha_j, i=1,2,3$, The reduced three interval model is used for the study, and it can be written as:

\[
G_r(s, \alpha) = \left\{ \begin{array}{l}
\hat{n}(s) + \sum_{j=1}^{3} \alpha_j n_j(s) \\
g(s) : \frac{\alpha_i}{\sum_{j=1}^{3} \alpha_j} \alpha_j \in [\alpha_j^-, \alpha_j^+] \\
\hat{d}(s) + \sum_{j=1}^{3} \alpha_j d_j(s)
\end{array} \right\},
\]

• By using $H_\infty$ design method and $D-K$ iteration, we can find a robust $H_\infty$ for our interval nominal model.
Reduced-interval model and edge theory Cont’d

Diagram of feedback system with reduced interval parametric uncertainty.
Reduced-interval model and edge theory  Cont’d

We can use $\mu$ analysis to evaluate the closed control system. The other way to evaluate our controller is using edge theory.

Consider now the set $I(s)$ of real polynomials of degree $n$ of the form:

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \delta_4 s^4 + \cdots \delta_n s^n$$

where the coefficient lie within given ranges

$$\delta_0 \in [x_0, y_0], \delta_1 \in [x_1, y_1], \cdots \delta_n \in [x_n, y_n]$$

Such a set of polynomials is called a real interval family and denoted as $I(s)$
Reduced-interval model and edge theory  Cont’d

Kharitonov’s Theorem:
Every polynomial in the family \( I(s) \) is Hurwitz if and only if the following four extreme polynomials are Hurwitz:

\[
K^1(s) = x_0 + x_1s + y_2s^2 + y_3s^3 + x_4s^4 + x_5s^5 + y_6s^6 + \cdots
\]

\[
K^2(s) = x_0 + y_1s + y_2s^2 + x_3s^3 + x_4s^4 + y_5s^5 + y_6s^6 + \cdots
\]

\[
K^3(s) = y_0 + x_1s + x_2s^2 + y_3s^3 + y_4s^4 + x_5s^5 + x_6s^6 + \cdots
\]

\[
K^4(s) = y_0 + y_1s + x_2s^2 + x_3s^3 + y_4s^4 + y_5s^5 + x_6s^6 + \cdots
\]

We can use Routh criterion to check if the above four polynomial Hurwitz or not.
Reduced-interval model and edge theory  Cont’d

To find a controller and make the state-feed back system stable, we have theorem below

Given a set of $n$ nominal parameters

$\{a_0^0, a_1^0, \ldots, a_{n-1}^0\}$, together with a set of prescribed uncertainty ranges

$\Delta a_0, \Delta a_1 \ldots \Delta a_{n-1}$ and consider the family $I_k(s)$ of monic polynomials,

$$
\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \delta_4 s^4 + \cdots s^n
$$

$$
\delta_0 \in [x_0, y_0], \delta_1 \in [x_1, y_1], \cdots \delta_n \in [x_n, y_n], \quad \Delta a_i = y_i - x_i
$$

Then, it is possible to find a vector $k$ such that the entire family $I_k(s)$ is stable.
Reduced-interval model and edge theory Cont’d

For real parameter vector \( p = [p_1, p_2, \cdots p_l]^T \) where each component \( p_i \) can vary independently of the other components, we assume the \( p \) lies in an uncertainty set which is box-like:

\[
\Pi = \left\{ P : p_i^- \leq p_i \leq p_i^+, i = 1,2,\cdots l \right\}
\]

The set \( \Pi \) is in fact an example of a special kind of polytope. In general, a polytope in \( n \)-dimensional space is the convex hull of a set of points called generators in this space. The vertices \( V \) of \( \Pi \) are obtained by setting each \( p_i \) to \( p_i^+ \) or \( p_i^- \)

\[
V := \left\{ P : p_i = p_i^-, p_i^+, i = 1,2,\cdots \right\}
\]
Reduced-interval model and edge theory Cont’d

- The exposed edges $E$ of the box $\Pi$ are defined as follows

\[
E_i = \left\{ p : p_i^- \leq p_i \leq p_i^+, p_j = p_j^-, p_j^+ , j \neq i \right\}
\]

\[
E = \bigcup_{i=1}^{l} E_i
\]

- Let $\Omega \subset \mathbb{R}^{n+1}$ be an $m$-dimensional polytope, that is, the convex hull of a finite number of points.

- Consider

\[
\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \delta_4 s^4 + \cdots \delta_n s^n
\]

- Assumption The sign of $\delta_n$ is constant over $\Omega$, either always positive or always negative.
Reduced-interval model and edge theory Cont’d

• Consider any $W \subset \Omega$, then $R(W)$ is said to be the root space of $W$, if

$$R(W) = \{ s : \delta(s) = 0 \}$$

• Edge Theorem:
• Let $\Omega \subset \mathbb{R}^n$ be a polytope of polynomials which satisfies assumption above, then the boundary of $R(\omega)$ is contained in the root space of the exposed edges of $\Omega$. 
Reduced-interval model and edge theory Cont’d

It is clear that those exposed edges are part of all pairwise convex combinations of \( Q \), and it enough to check those.

In the representation

\[
p := \{ p(s) : p(s) = a_1 Q_1(s) + a_2 Q_2(s) + a_3 Q_3(s) + \cdots + a_m Q_m(s), a \in A \}
\]

where the exposed edges of the polytope \( p \) are obtained from the exposed edges of the hypercube \( A \) to which \( a \) belongs. This can be done by fixing all \( a_i \) except one, say \( a_k \) at a vertex \( a_i \), and letting \( a_k \) vary in the interval \([a_k, \bar{a}_k]\) and repeating this for \( k=1 \) to \( m \). This procedure captures all the exposed edges.

The problem of plotting the boundaries of root clusters is nothing but a \( m2^{m-1} \) root locus problem.

Edge can be used to check the stability of the system when we designed the controller for the reduce-order interval system.
Reduced-interval model and edge theory  Cont’d

Example:
Consider the interval plant: \( G(s) = \frac{s + a}{s^2 + bs + c} \)

where \( a \in [1,2], b \in [9,11], c \in [15,18]. \)

The controller is:
\[
C(s) = \frac{3s + 2}{s + 5}
\]

From the closed loop characteristic polynomial, the boundary of the root space can be obtained by plotting the root loci along the 12 exposed edges.
- It can be seen that the system is stable since the root space is in the left half plane. From the root set we can evaluate the performance of the controller in terms of the worst case damping ratio, the minimum stability degree and maximum frequency of oscillation.
Reduced-interval model and edge theory Cont’d

Reference:
