Lecture Notes for ECE/MAE 7360

Robust and Optimal Control

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What will we study?

- Optimal Control Theory:
  - The calculus of variations (a little bit)
  - Solution of general optimization problems
  - Optimal closed-loop control (LQR problem)
  - Pontryagin's minimum principle

What will we study? (cont.)

- RIOTS:
  - Introduction to numerical optimal control
  - Introduction to RIOTS
    - Background
    - Usage and demo

Warm-up: An Static Optimization Problem:

$$\text{min} : L(x, u),\ \ x \in R^n,\ \ u \in R^m$$

$$\text{subject to: } f(x, u) = 0,\ \ f \in R^n$$

Define *Hamiltonian* function

$$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u),\ \ \lambda \in R^n$$

Necessary conditions:

$$\frac{\partial H}{\partial \lambda} = f = 0$$

$$\frac{\partial H}{\partial x} = L_x + f_x^T \lambda = 0$$

$$\frac{\partial H}{\partial u} = L_u + f_u^T \lambda = 0$$

What is an optimal control problem?

System model:

$$\dot{x}(t) = f(x, u, t),\ \ x(t) \in R^n,\ \ u(t) \in R^m$$

Performance index (cost function):

$$J(u, T) = \phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)\mathrm{d}t$$

Final-state constraint:

$$\psi(x(T), T) = 0,\ \ \psi \in R^p$$

Some cost function examples

- Minimum-fuel problem

$$J = \int_{t_0}^{T} u^2(t)\mathrm{d}t$$

- Minimum-time problem

$$J = \int_{t_0}^{T} 1\mathrm{d}t$$

- Minimum-energy problem

$$J = x^T(T)Rx(T) + \int_{t_0}^{T} \{x^T(t)Qx(t) + u^T(t)Ru(t)\}\mathrm{d}t$$

A little bit of calculus of variations

- Why calculus of variation?

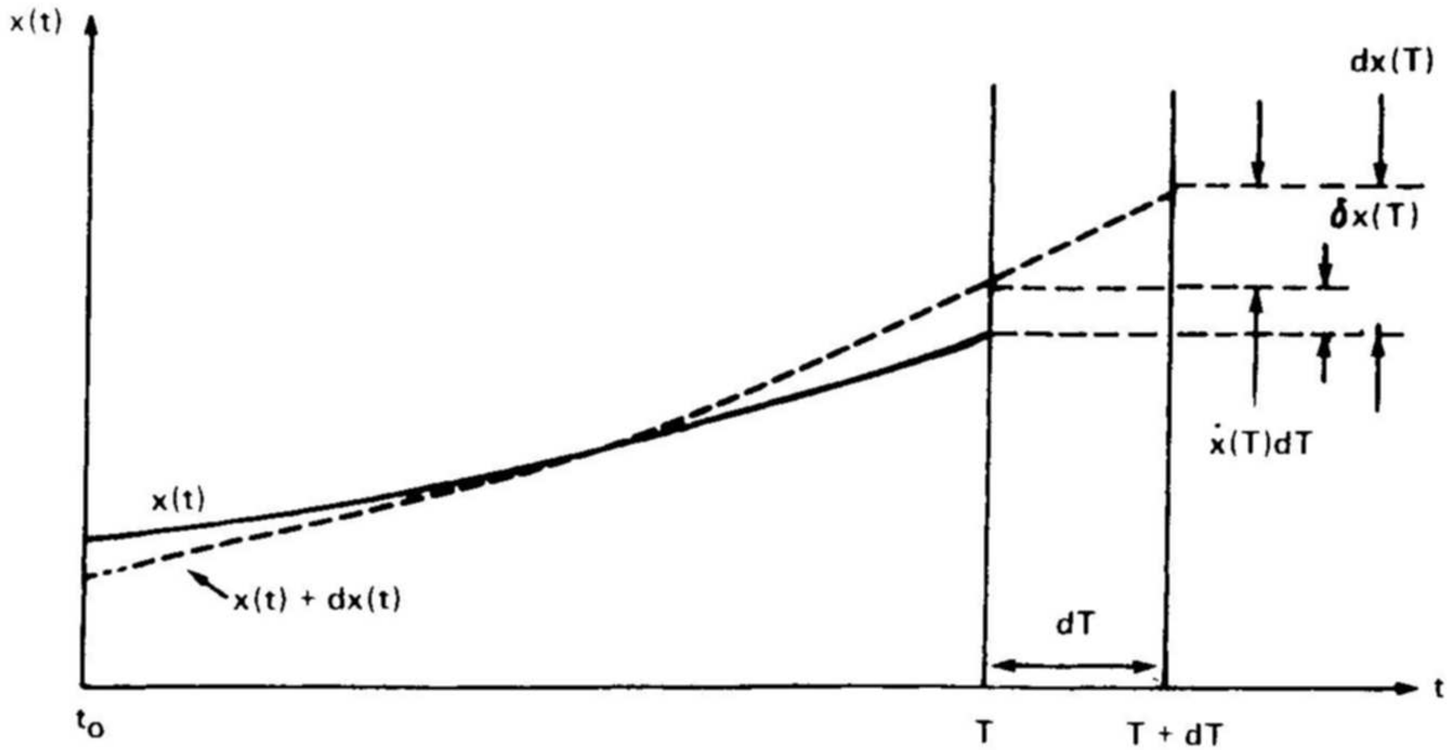
  We are dealing with a function of functions $L(u(t), T)$, called *functional*, rather than a function of scalar variables $L(x, u)$. We need a new mathematical tool.

- What is $\delta x(t)$?

- Relationship between $dx(T)$, $\delta x(T)$, and dT

Relationship between $dx(T)$, $\delta x(T)$, and dT:

$$dx(T) = \delta x(T) + \dot{x}(T)dT$$



Leibniz's rule:

$$J(x(t)) = \int_{x_0}^{T} h(x(t), t)\mathrm{d}t$$

$$dJ = h(x(T), T)dT + \int_{t_0}^{T} [h_x^T(x(t), t)\delta x]\mathrm{d}t$$

where $h_x \triangleq \frac{\partial h}{\partial x}$.

Solution of the general optimization problem:

$$J(u,T) = \phi(x(T),T) + \int_{t_0}^{T} L(x(t),u(t),t)\mathrm{d}t \qquad (1)$$

$$\dot{x}(t) = f(x,u,t),\ \ x(t) \in R^n,\ \ u(t) \in R^m \qquad (2)$$

$$\psi(x(T),T) = 0 \qquad (3)$$

Using Lagrange multipliers $\lambda(t)$ and ν to join the constraints (2) and (3) to the performance index (1):

$$J' = \phi(x(T),T)+\nu^T\psi(x(T),T)+\int_{t_0}^{T}[L(x,u,t)+\lambda^T(t)(f(x,u,t)-\dot{x})]\mathrm{d}t$$

Note:

ν: constant, $\lambda(t)$: function

Define the *Hamiltonian function*:

$$H(x,u,t) = L(x,u,t) + \lambda^T f(x,u,t),$$

then

$$J' = \phi(x(T),T) + \nu^T \psi(x(T),T) + \int_{t_0}^{T} [H(x,u,t) - \lambda^T \dot{x}]\mathrm{d}t.$$

Using Leibniz's rule, the increment in J' as a function of increments in x,λ, ν, u, and t is

$$\begin{aligned} dJ' \;=\; & (\phi_x + \psi_x^T \nu)^T dx|_T + (\phi_t + \psi_t^T \nu) dt|_T + \psi^T|_T d\nu \\ & +(H - \lambda^T \dot{x}) dt|_T \\ & + \textstyle\int_{t_0}^{T} [H_x^T \delta x + H_u^T \delta u - \lambda^T \delta \dot{x} + (H_\lambda - \dot{x})^T \delta\lambda]\mathrm{d}t \end{aligned} \qquad (4)$$

To eliminate the variation in $\dot{x}$, integrate by parts:

$$-\int_{t_0}^{T} \lambda^T \delta\dot{x}\mathrm{d}t = -\lambda^T \delta x|_T + \int_{t_0}^{T} \dot{\lambda}^T \delta x \mathrm{d}t$$

Remember

$$dx(T) = \delta x(T) + \dot{x}(T)dT,$$

so

$$\begin{aligned} dJ' &= (\phi_x + \psi_x^T \nu - \lambda)^T dx|_T + (\phi_t + \psi_t^T \nu + H)dt|_T + \psi^T|_T d\nu \\ &\quad + \int_{t_0}^{T} [(H_x + \dot{\lambda})^T \delta x + H_u^T \delta_u + (H_\lambda - \dot{x})^T \delta\lambda] \mathrm{d}t. \end{aligned}$$

According to the Lagrange theory, the constrained minimum of J is attained at the unconstrained minimum of J'. This is achieved when $dJ' = 0$ for all independent increments in its arguments. Setting to zero the coefficients of the independent increments $d\nu$, δx, δu, and $\delta\lambda$ yields following necessary conditions for a minimum.

System model:

$$\dot{x} = f(x, u, t), \;\; t \geq t_0, \;\; t_0 \text{ fixed}$$

Cost function:

$$J(u, T) = \phi(x(T), T) + \int_{t_0}^{T} L(x, u, t)\mathrm{d}t$$

Final-state constraint:

$$\psi(x(T), T) = 0$$

State equation:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f$$

Costate (adjoint) equation:

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^T}{\partial x}\lambda + \frac{\partial L}{\partial x}$$

Stationarity condition:

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \frac{\partial f^T}{\partial u}\lambda$$

Boundary (transversality) condition:

$$(\phi_x + \psi_x^T \nu - \lambda)^T|_T dx(T) + (\phi_t + \psi_t^T \nu + H)|_T dT = 0$$

Note that in the boundary condition, since $dx(T)$ and dT are not independent, we cannot simply set the coefficients of $dx(T)$ and dT equal to zero. If $dx(T) = 0$ (fixed final state) or $dT = 0$ (fixed final time), the boundary condition is simplified. What if neither is equal to zero?

An optimal control problem example: temperature control in a room

It is desired to heat a room using the least possible energy. If $\theta(t)$ is the temperature in the room, θ_a the ambient air temperature outside (a constant), and $u(t)$ the rate of heat supply to the room, then the dynamics are

$$\dot{\theta} = -a(\theta - \theta_a) + bu$$

for some constants a and b, which depend on the room insulation and so on. By defining the state as

$$x(t) \triangleq \theta(t) - \theta_a,$$

we can write the state equation

$$\dot{x} = -ax + bu.$$

In order to control the temperature on the fixed time interval $[0, T]$ with the least supplied energy, define the cost function as

$$J(u) = \frac{1}{2}s(x(T)) + \frac{1}{2}\int_0^T u^2(t)dt,$$

for some weighting s.

The Hamiltonian is

$$H = \frac{u^2}{2} + \lambda(-ax + bu)$$

The optimal control $u(t)$ is determined by solving:

$$\dot{x} = H_\lambda = -ax + bu, \tag{5}$$

$$\dot{\lambda} = -H_x = a\lambda, \tag{6}$$

$$0 = H_u = u + b\lambda \tag{7}$$

From the stationarity condition (7), the optimal control is given by

$$u(t) = -b\lambda(t), \tag{8}$$

so to determine $u^*(t)$ we need to only find the optimal costate $\lambda^*(t)$.

Substitute (8) into (5) yields the state-costate equations

$$\begin{aligned} \dot{x} &= -ax - b^2\lambda && (9) \\ \dot{\lambda} &= a\lambda && (10) \end{aligned}$$

Sounds trivial? Think about the boundary condition!

From the boundary condition:

$$(\phi_x + \psi_x^T \nu - \lambda)^T|_T dx(T) + (\phi_t + \psi_t^T \nu + H)|_T dT = 0,$$

dT is zero, $dx(T)$ is free, and there is no final-state constraint. So

$$\lambda(T) = \frac{\partial \phi}{\partial x}|_T = s(x(T) - 10).$$

So the boundary condition of x and λ are specified at t_0 and T, respectively. This is called two-point boundary-value (TPBV) problem.

Let's assume $\lambda(T)$ is known. From (10), we have

$$\lambda(t) = e^{-a(T-t)}\lambda(T).$$

So

$$\dot{x} = -ax - b^2\lambda(T)e^{-a(T-t)}.$$

Solving the above ODE, we have

$$x(t) = x(0)e^{-at} - \frac{b^2}{a}\lambda(T)e^{-aT}sinh(at).$$

Now we have the second equation about $x(T)$ and $\lambda(T)$

$$x(T) = x(0)e^{-aT} - \frac{b^2}{2a}\lambda(T)(1 - e^{-2aT})$$

Assuming $x(0) = 0^\circ$, $\lambda(T)$ can now be solved:

$$\lambda(T) = \frac{-20as}{2a + b^2s(1 - e^{-2aT})}$$

Now the costate equation becomes

$$\lambda^*(t) = \frac{-10ase^{at}}{ae^{aT} + sb^2 sinh(aT)}$$

Finally we obtain the optimal control

$$u^*(t) = \frac{10abse^{at}}{ae^{aT} + sb^2 sinh(aT)}$$

Conclusion:

TPBV makes it hard to solve even for simple OCP problems. In most cases, we have to rely on numerical methods and dedicated OCP software package, such as RIOTS.

Closed-loop optimal control: LQR problem

Problems with the optimal controller obtained so far:

- solutions are hard to compute.
- open-loop

For Linear Quadratic Regulation (LQR) problems, a closed-loop controller exists.

System model:

$$\dot{x} = A(t)x + B(t)u$$

Objective function:

$$J(u) = \frac{1}{2}x^T(T)S(T)x(T) + \frac{1}{2}\int_{t_0}^{T}(x^T Q(t)x + u^T R(t)u)\mathrm{d}t$$

where $S(T)$ and $Q(t)$ are symmetric and positive semidefinite weighting matrices, $R(t)$ is symmetric and positive definite, for all $t \in [t_0, T]$. We are assuming T is fixed and the final state $x(T)$ is free.

State and costate equations:

$$\begin{aligned} \dot{x} &= Ax - BR^{-1}B^T\lambda, \\ -\dot{\lambda} &= Qx + A^T\lambda \end{aligned}$$

Control input:

$$u(t) = -R^{-1}B^T\lambda$$

Terminal condition:

$$\lambda(T) = S(T)x(T)$$

Considering the terminal condition, let's assume that $x(t)$ and $\lambda(t)$ satisfy a linear relation for all $t \in [t_0, T]$ for some unknown matrix $S(t)$:

$$\lambda(t) = S(t)x(t)$$

To find $S(t)$, differentiate the costate to get

$$\dot{\lambda} = \dot{S}x + S\dot{x} = \dot{S}x + S(Ax - BR^{-1}B^T Sx).$$

Taking into account the costate equation, we have

$$-\dot{S}x = (A^T S + SA - SBR^{-1}B^T S + Q)x.$$

Since the above equation holds for all $x(t)$, we have the *Riccati equation*:

$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q, \ t \leq T.$$

Now the optimal controller is given by

$$u(t) = -R^{-1}B^T S(t)x(t),$$

and $K(t) = R^{-1}B^T S(t)x(t)$ is called *Kalman gain*.

Note that solution of $S(t)$ does not require $x(t)$, so $K(t)$ can be computed off-line and stored.

Pontryagin's Minimum Principle: a bang-bang control case study

So far, the solution to an optimal control problem depends on the stationarity condition $\frac{\partial H}{\partial u} = 0$. What if the control $u(t)$ is constrained to lie in an admissible region, which is usually defined by a requirement that its magnitude be less than a given value?

Pontryagin's Minimum Principle: **the Hamiltonian must be minimized over all admissible u for optimal values of the state and costate**

$$H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t), \text{ for all admissible } u$$

Example: bang-bang control of systems obeying Newton's laws:

System model:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u, \end{aligned}$$

Objective function (time-optimal:

$$J(u) = T = \int_0^T 1 \mathrm{d}t$$

Input constraints:

$$|u(t)| \leq 1$$

End-point constraint:

$$\psi(x(T), T) = \begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = 0.$$

The Hamiltonian is:

$$H = 1 + \lambda_1 x_2 + \lambda_2 u,$$

where $\lambda = [\lambda_1, \lambda_2]^T$ is the costate.

Costate equation:

$$\begin{aligned} \dot{\lambda}_1 &= 0 \quad \Rightarrow \lambda_1 = \text{constant} \\ \dot{\lambda}_2 &= -\lambda_1 \quad \Rightarrow \lambda_2(t) \text{ is a linear function of } t \text{ (remember it!)} \end{aligned}$$

Boundary condition:

$$\lambda_2(T)u(T) = -1$$

Pontryagin's minimum principle requires that

$$\lambda_2^*(t)u^*(t) \leq \lambda_2^*(t)u(t)$$

How to make sure $\lambda_2^*(t)u^*(t)$ is less or equal than $\lambda_2^*(t)u(t)$ for any admissible $u(t)$?

Answer:

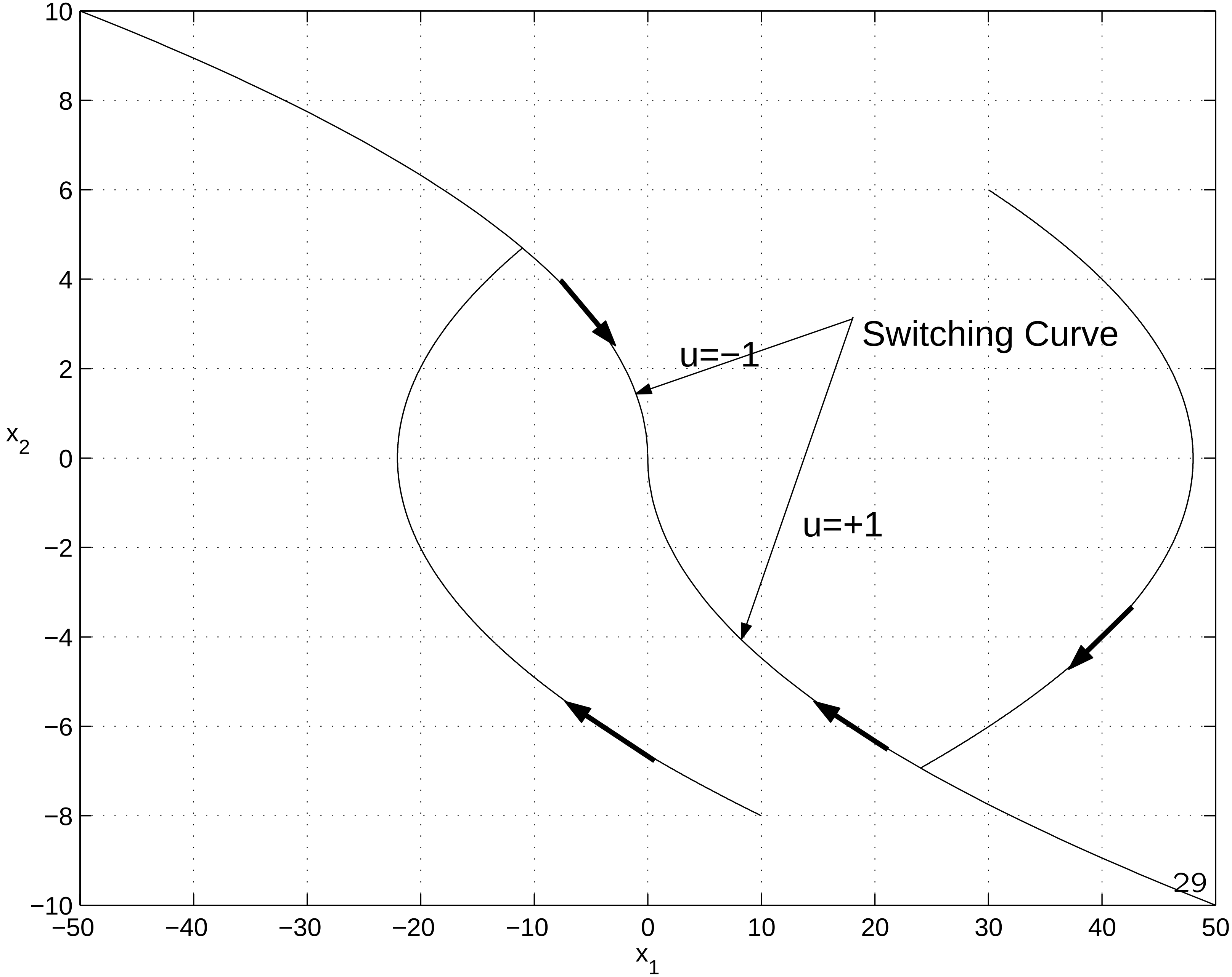
$$u^*(t) = -sgn(\lambda_2^*(t)) = \begin{cases} 1, & \lambda_2^*(t) < 0 \\ -1, & \lambda_2^*(t) > 0 \end{cases}$$

What if $\lambda_2^*(t) = 0$? Then u^* is undetermined.

Since $\lambda_2^*(t)$ is linear, it changes sign at most once. So does $u^*(t)$!

Since rigorous derivation of $u^*(t)$ is still a little bit complicated, an intuitive method using phase plane will be shown below.

Going backward ($x_1 = 0$ and $x_2 = 0$) in time from T, with $u(t) = +1$ or $u(t) = -1$, we obtain a trajectories, or *switching curve*, because switching of control (if any) must occur on this curve. Why?


Switching Curve
u=−1
u=+1
x2
x1
10
8
6
4
2
0
−2
−4
−6
−8
−10
−50
−40
−30
−20
−10
0
10
20
30
40
50

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