Lecture Notes: Week 1b

Topic: Linear Multivariable System Theory - Review

ECE/MAE 7360
Optimal and Robust Control
(Fall 2003 Offering)

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Linear Systems

- dynamical systems
- controllability and stabilizability
- observability and detectability
- observer theory
- system interconnections
- realizations
- poles and zeros

Dynamical Systems

- Linear equations:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(t_0) = x_0 \\ y &= Cx + Du \end{aligned}$$

- transfer matrix:

$$Y(s) = G(s)U(s)$$

$$G(s) = C(sI - A)^{-1}B + D.$$

- notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D$$

- solution:

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

- impulse matrix

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = Ce^{At}B1_{+}(t) + D\delta(t)$$

- input/output relationship:

$$y(t) = (g * u)(t) := \int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau.$$

Controllability

- Controllability: (A, B) is *controllable* if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1, there exists a (piecewise continuous) input $u(\cdot)$ such that satisfies $x(t_1) = x_1$.
- The matrix
$$W_c(t) := \int_0^t e^{A\tau} B B^* e^{A^*\tau} d\tau$$
is positive definite for any $t > 0$.
- The controllability matrix
$$\mathcal{C} = \left[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array} \right]$$
has full row rank, i.e., $\langle A \,|\mathrm{Im} B\rangle := \Sigma_{i=1}^n \mathrm{Im}(A^{i-1}B) = \mathbb{R}^n$.
- The eigenvalues of $A + BF$ can be freely assigned by a suitable F.

PBH test:

- The matrix $[A - \lambda I,\ B]$ has full row rank for all λ in $\mathbb{C}$.
- Let λ and x be any eigenvalue and *any* corresponding left eigenvector of A, i.e., $x^* A = x^* \lambda$, then $x^* B \neq 0$.

Stability and Stabilizability

A is *stable* if $\text{Re}\lambda(A) < 0$.

- (A, B) is stabilizable.
- $A + BF$ is stable for some F.

PBH test:

- The matrix $[A - \lambda I, \ B]$ has full row rank for all $\text{Re}\lambda \geq 0$.
- For all λ and x such that $x^*A = x^*\lambda$ and $\text{Re}\lambda \geq 0$, $x^*B \neq 0$.

Observability

- (C, A) is *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval of $[0, t_1]$.
- The matrix
$$W_o(t) := \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau$$
is positive definite for any $t > 0$.
- The observability matrix
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$
has full column rank, i.e., $\cap_{i=1}^n \operatorname{Ker}(CA^{i-1}) = 0$.
- The eigenvalues of $A + LC$ can be freely assigned by a suitable L.
- (A^*, C^*) is controllable.

PBH test:

- The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all λ in $\mathbb{C}$.
- Let λ and y be *any* eigenvalue and any corresponding right eigenvector of A, i.e., $Ay = \lambda y$, then $Cy \neq 0$.

Detectability

The following are equivalent:

- (C, A) is detectable.
- $A + LC$ is stable for a suitable L.
- (A^*, C^*) is stabilizable.

PBH test:

- The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all $\mathrm{Re}\lambda \geq 0$.
- For all λ and x such that $Ax = \lambda x$ and $\mathrm{Re}\lambda \geq 0$, $Cx \neq 0$.

an example:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 & \alpha \\ 0 & 0 & 0 & \lambda_2 & 1 \\ \hline 1 & 0 & 0 & \beta & 0 \end{array}\right]$$

Observers and Observer-Based Controllers

An observer is a dynamical system with input of (u, y) and output of, say $\hat{x}$, which asymptotically estimates the state x, i.e., $\hat{x}(t) - x(t) \to 0$ as $t \to \infty$ for all initial states and for every input.

An observer exists iff (C, A) is detectable. Further, if (C, A) is detectable, then a full order Luenberger observer is given by

$$\dot{q} = Aq + Bu + L(Cq + Du - y) \tag{0.1}$$

$$\hat{x} = q \tag{0.2}$$

where L is any matrix such that $A + LC$ is stable.

Observer-based controller:

$$\begin{aligned} \dot{\hat{x}} &= (A + LC)\hat{x} + Bu + LDu - Ly \\ u &= F\hat{x}. \end{aligned}$$

$$u = K(s)y$$

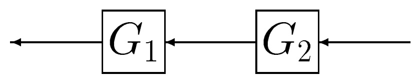
and

$$K(s) = \left[\begin{array}{c|c} A + BF + LC + LDF & -L \\ \hline F & 0 \end{array} \right].$$

Operations on Systems

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right].$$

- cascade:

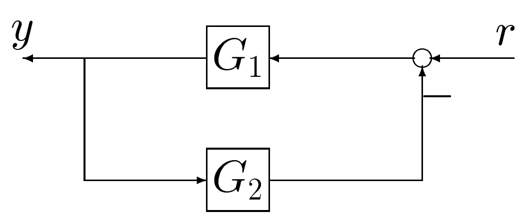


$$\begin{aligned} G_1 G_2 &= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right]. \end{aligned}$$

- addition:

$$G_1 + G_2 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right].$$

- feedback:



$$T = \left[\begin{array}{cc|c} A_1 - B_1 D_2 R_{12}^{-1} C_1 & -B_1 R_{21}^{-1} C_2 & B_1 R_{21}^{-1} \\ B_2 R_{12}^{-1} C_1 & A_2 - B_2 D_1 R_{21}^{-1} C_2 & B_2 D_1 R_{21}^{-1} \\ \hline R_{12}^{-1} C_1 & -R_{12}^{-1} D_1 C_2 & D_1 R_{21}^{-1} \end{array}\right]$$

where $R_{12} = I + D_1 D_2$ and $R_{21} = I + D_2 D_1$.

- *transpose* or *dual system*

$$G \longmapsto G^T(s) = B^*(sI - A^*)^{-1}C^* + D^*$$

or equivalently

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \longmapsto \left[\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right].$$

- *conjugate* system

$$G \longmapsto G^\sim(s) := G^T(-s) = B^*(-sI - A^*)^{-1}C^* + D^*$$

or equivalently

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \longmapsto \left[\begin{array}{c|c} -A^* & -C^* \\ \hline B^* & D^* \end{array}\right].$$

In particular, we have $G^*(j\omega) := [G(j\omega)]^* = G^\sim(j\omega)$.

- Let $D^\dagger$ denote a right (left) inverse of D if D has full row (column) rank. Then

$$G^\dagger = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array}\right]$$

is a right (left) inverse of G.

State Space Realizations

Given $G(s)$, find (A, B, C, D) such that

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

which is a *state space realization* of $G(s)$.

- A state space realization (A, B, C, D) of $G(s)$ is minimal if and only if (A, B) is controllable and (C, A) is observable.
- Let (A_1, B_1, C_1, D) and (A_2, B_2, C_2, D) be two minimal realizations of $G(s)$. Then there exists a unique nonsingular T such that

  $$A_2 = TA_1T^{-1}, \quad B_2 = TB_1, \quad C_2 = C_1T^{-1}.$$

  Furthermore, T can be specified as

  $$T = (\mathcal{O}_2^*\mathcal{O}_2)^{-1}\mathcal{O}_2^*\mathcal{O}_1$$

  or

  $$T^{-1} = \mathcal{C}_1\mathcal{C}_2^*(\mathcal{C}_2\mathcal{C}_2^*)^{-1}.$$

  where $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{O}_1$, and $\mathcal{O}_2$ are e the corresponding controllability and observability matrices, respectively.

SIMO and MISO

<u>SIMO Case:</u> Let

$$G(s) = \begin{pmatrix} g_1(s) \\ g_2(s) \\ \vdots \\ g_m(s) \end{pmatrix} = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} + d,$$

where $\beta_i \in \mathbb{R}^m$ and $d \in \mathbb{R}^m$. Then

$$G(s) = \left[\begin{array}{c|c} A & b \\ \hline C & d \end{array}\right], \quad b \in \mathbb{R}^n,\ C \in \mathbb{R}^{m\times n},\ d \in \mathbb{R}^m$$

where

$$A := \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad b := \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}$$

<u>MISO Case:</u> Let

$$G(s) = \begin{pmatrix} g_1(s) & g_2(s) & \ldots & g_p(s) \end{pmatrix}$$

$$= \frac{\eta_1 s^{n-1} + \eta_2 s^{n-2} + \cdots + \eta_{n-1} s + \eta_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} + d$$

with $\eta_i^*, d^* \in \mathbb{R}^p$. Then

$$G(s) = \left[\begin{array}{ccccc|c} -a_1 & 1 & 0 & \cdots & 0 & \eta_1 \\ -a_2 & 0 & 1 & \cdots & 0 & \eta_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 & \eta_{n-1} \\ -a_n & 0 & 0 & \cdots & 0 & \eta_n \\ \hline 1 & 0 & 0 & \cdots & 0 & d \end{array}\right]$$

Realizing Each Elements

To illustrate, consider a 2×2 (block) matrix

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) \\ G_3(s) & G_4(s) \end{bmatrix}$$

and assume that $G_i(s)$ has a state space realization of

$$G_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad i = 1, \ldots, 4.$$

Note that $G_i(s)$ may itself be a MIMO transfer matrix.

Then a realization for $G(s)$ can be given by

$$G(s) = \left[\begin{array}{cccc|cc} A_1 & 0 & 0 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 \\ 0 & 0 & A_3 & 0 & B_3 & 0 \\ 0 & 0 & 0 & A_4 & 0 & B_4 \\ \hline C_1 & C_2 & 0 & 0 & D_1 & D_2 \\ 0 & 0 & C_3 & C_4 & D_3 & D_4 \end{array}\right].$$

Problem: minimality.

Gilbert's Realization

Let $G(s)$ be a $p \times m$ transfer matrix

$$G(s) = \frac{N(s)}{d(s)}$$

with $d(s)$ a scalar polynomial. For simplicity, we shall assume that $d(s)$ has only real and distinct roots $\lambda_i \neq \lambda_j$ if $i \neq j$ and

$$d(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_r).$$

Then $G(s)$ has the following partial fractional expansion:

$$G(s) = D + \sum_{i=1}^{r} \frac{W_i}{s - \lambda_i}.$$

Suppose

$$\operatorname{rank} W_i = k_i$$

and let $B_i \in \mathbb{R}^{k_i \times m}$ and $C_i \in \mathbb{R}^{p \times k_i}$ be two constant matrices such that

$$W_i = C_i B_i.$$

Then a realization for $G(s)$ is given by

$$G(s) = \left[\begin{array}{ccc|c} \lambda_1 I_{k_1} & & & B_1 \\ & \ddots & & \vdots \\ & & \lambda_r I_{k_r} & B_r \\ \hline C_1 & \cdots & C_r & D \end{array}\right].$$

This realization is controllable and observable (minimal) by PBH tests.

Repeated Poles

Note that

$$G(s) = \left[\begin{array}{cc|c} \lambda & 1 & b_1 \\ 0 & \lambda & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right]$$

$$= \frac{c_1[b_2 + (s-\lambda)b_1]}{(s-\lambda)^2} + \frac{c_2 b_2}{s-\lambda}$$

$$= \frac{\mathbf{c_1 b_2}}{(s-\lambda)^2} + \frac{\mathbf{c_1} b_1 + c_2 \mathbf{b_2}}{s-\lambda}$$

A realization procedure:

- Let $G(s)$ be a $p \times q$ matrix and have the following partial fractional expansion:

$$G(s) = \frac{R_1}{(s-\lambda)^2} + \frac{R_2}{s-\lambda}$$

- Suppose $\text{rank}(R_1) = 1$ and write

$$R_1 = c_1 b_1, \quad c_1 \in \mathbb{R}^p, \ b_1 \in \mathbb{R}^q$$

- Find c_2 and b_1 if possible such that

$$\mathbf{c_1} b_1 + c_2 \mathbf{b_2} = R_2$$

  Otherwise find also matrices C_3 and B_3 such that

$$\mathbf{c_1} b_1 + c_2 \mathbf{b_2} + C_3 B_3 = R_2$$

  and $[c_1 \ C_3]$ full column rank and $\begin{bmatrix} b_2 \\ B_3 \end{bmatrix}$ full row rank.

- if $\text{rank}(R_1) > 1$ then write

$$R_1 = c_1 b_1 + \tilde{c}_1 \tilde{b}_1 + \ldots$$

  and repeated the above process.

Consider a 3×3 transfer matrix:

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{2s+1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{1}{(s+1)^2} & \frac{s^2+5s+3}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{1}{(s+1)^2(s+2)} & \frac{2s+1}{(s+1)^2(s+2)} & \frac{s}{(s+1)^2(s+2)} \end{bmatrix}.$$

$$G(s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(s+1)^2} \overbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{c_1} \overbrace{\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}}^{b_2}$$

$$+\frac{1}{s+1}\left(\overbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{c_1} \overbrace{\begin{bmatrix} 0 & 3 & 1 \end{bmatrix}}^{b_1} + \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{c_2} \overbrace{\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}}^{b_2} \right)$$

$$+\frac{1}{s+1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} + \frac{1}{s+2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & -2 \end{bmatrix}$$

So a 4-th order minimal state space realization is given by

$$G(s) = \left[\begin{array}{cccc|ccc} -1 & 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & -3 & -2 \\ \hline 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

Let

$$G_3(s) = \left[\begin{array}{ccc|c} \lambda & 1 & 0 & b_1 \\ 0 & \lambda & 1 & b_2 \\ 0 & 0 & \lambda & b_3 \\ \hline c_1 & c_2 & c_3 & 0 \end{array}\right]$$

$$= \frac{c_1[b_3 + (s-\lambda)b_2 + (s-\lambda)^2 b_1]}{(s-\lambda)^3}$$

$$+\frac{c_2[b_3 + (s-\lambda)b_2]}{(s-\lambda)^2} + \frac{c_3 b_3}{s-\lambda}$$

$$= \frac{\mathbf{c_1 b_3}}{(s-\lambda)^3} + \frac{\mathbf{c_1} b_2 + c_2 \mathbf{b_3}}{(s-\lambda)^2} + \frac{\mathbf{c_1} b_1 + c_2 b_2 + c_3 \mathbf{b_3}}{s-\lambda}$$

Example: Let

$$G(s) = \begin{bmatrix} \frac{1}{(s+2)^3(s+5)} & \frac{1}{s+5} \\ \frac{1}{s+2} & 0 \end{bmatrix}$$

$$= \frac{1}{(s+2)^3} \overbrace{\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}}^{c_1} \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{b_3} + \frac{1}{(s+2)^2} \overbrace{\begin{bmatrix} -\frac{1}{9} \\ 0 \end{bmatrix}}^{c_2} \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{b_3}$$

$$+\frac{1}{s+2} \overbrace{\begin{bmatrix} \frac{1}{27} \\ 1 \end{bmatrix}}^{c_3} \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{b_3} + \frac{1}{s+5} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{27} & 1 \end{bmatrix}$$

Take $b_1 = 0$ and $b_2 = 0$, we get

$$G(s) = \left[\begin{array}{cccc|cc} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 & -\frac{1}{27} & 1 \\ \hline \frac{1}{3} & -\frac{1}{9} & \frac{1}{27} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

Example: Let

$$G(s) = \frac{\overbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}^{c_1}\overbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}^{b_3}}{(s+p)^3} + \frac{\overbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}^{\tilde{c}_1}\overbrace{\begin{bmatrix}0 & 1 & 0\end{bmatrix}}^{\tilde{b}_3}}{(s+p)^3}$$

$$+\frac{\overbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}^{c_2}\overbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}^{b_3}}{(s+p)^2} + \frac{\overbrace{\begin{bmatrix}2\\5\\2\end{bmatrix}}^{2c_1+3\tilde{c}_1}\overbrace{\begin{bmatrix}0 & 0 & 1\end{bmatrix}}^{\frac{1}{2}b_2 \ or \ \frac{1}{3}\tilde{c}_1}}{(s+p)^2}$$

$$+\frac{\overbrace{\begin{bmatrix}2\\2\\3\end{bmatrix}}^{-c_3 \ or \ -\tilde{c}_3}\overbrace{\begin{bmatrix}-1 & -1 & 0\end{bmatrix}}^{-b_3-\tilde{b}_3}}{s+p}$$

Hence

$$G(s) = \left[\begin{array}{cccccc|ccc} -p & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -p & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -p & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & -p & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

System Poles and Zeros

An example:

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{2}{s+2} & \dfrac{1}{s+1} \end{bmatrix}$$

which is stable and each element of $G(s)$ has no finite zeros. Let

$$K = \begin{bmatrix} \dfrac{s+2}{s-\sqrt{2}} & -\dfrac{s+1}{s-\sqrt{2}} \\ 0 & 1 \end{bmatrix}$$

which is unstable. However,

$$KG = \begin{bmatrix} -\dfrac{s+\sqrt{2}}{(s+1)(s+2)} & 0 \\ \dfrac{2}{s+2} & \dfrac{1}{s+1} \end{bmatrix}$$

is stable. This implies that $G(s)$ must have an unstable zero at $\sqrt{2}$ that cancels the unstable pole of K.

Smith Form

- a square polynomial matrix $Q(s)$ is *unimodular* if and only if $\det Q(s)$ is a constant.
- Let $Q(s)$ be a $(p \times m)$ polynomial matrix. Then the *normal rank* of $Q(s)$, denoted *normalrank* $(Q(s))$, is the maximally possible rank of $Q(s)$ for at least one $s \in \mathbb{C}$.

  an example:

  $$Q(s) = \begin{bmatrix} s & 1 \\ s^2 & 1 \\ s & 1 \end{bmatrix}.$$

  $Q(s)$ has normal rank 2 since rank $Q(2) = 2$. However, $Q(0)$ has rank 1.
- *Smith form:* Let $P(s)$ be any polynomial matrix, then there exist unimodular matrices $U(s), V(s) \in \mathbb{R}[s]$ such that

  $$U(s)P(s)V(s) = S(s) := \begin{bmatrix} \gamma_1(s) & 0 & \cdots & 0 & 0 \\ 0 & \gamma_2(s) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_r(s) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

  and $\gamma_i(s)$ divides $\gamma_{i+1}(s)$.

  $S(s)$ is called the *Smith form* of $P(s)$. r is the normal rank of $P(s)$.

an example:

$$P(s) = \begin{bmatrix} s+1 & (s+1)(2s+1) & s(s+1) \\ s+2 & (s+2)(s^2+5s+3) & s(s+2) \\ 1 & 2s+1 & s \end{bmatrix}.$$

$P(s)$ has normal rank 2 since $\det(P(s)) \equiv 0$ and

$$\det \begin{bmatrix} s+1 & (s+1)(2s+1) \\ s+2 & (s+2)(s^2+5s+3) \end{bmatrix} = (s+1)^2(s+2)^2 \not\equiv 0.$$

Let

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -(s+2) \\ 1 & 0 & -(s+1) \end{bmatrix}.$$

$$V(s) = \begin{bmatrix} 1 & -(2s+1) & -s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$S(s) = U(s)P(s)V(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Smith-McMillan Form

- Let $G(s)$ be any proper real rational transfer matrix, then there exist unimodular matrices $U(s), V(s) \in \mathbb{R}[s]$ such that

$$U(s)G(s)V(s) = M(s) := \begin{bmatrix} \frac{\alpha_1(s)}{\beta_1(s)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\alpha_2(s)}{\beta_2(s)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_r(s)}{\beta_r(s)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

  and $\alpha_i(s)$ divides $\alpha_{i+1}(s)$ and $\beta_{i+1}(s)$ divides $\beta_i(s)$.

- Write $G(s)$ as $G(s) = N(s)/d(s)$ such that $d(s)$ is a scalar polynomial and $N(s)$ is a $p \times m$ polynomial matrix.

  Let the Smith form of $N(s)$ be $S(s) = U(s)N(s)V(s)$.

  Then $M(s) = S(s)/d(s)$.

- *McMillan degree* of $G(s) = \sum_i \deg(\beta_i(s))$ where $\deg(\beta_i(s))$ denotes the degree of the polynomial $\beta_i(s)$.

- McMillan degree of $G(s) =$ the dimension of a minimal realization of $G(s)$.

- *poles* of $G =$ roots of $\beta_i(s)$

- *transmission zeros* of $G(s) =$ the roots of $\alpha_i(s)$

  $z_0 \in \mathbb{C}$ is a *blocking zero* of $G(s)$ if $G(z_0) = 0$.

An example:

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{2s+1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{1}{(s+1)^2} & \frac{s^2+5s+3}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{1}{(s+1)^2(s+2)} & \frac{2s+1}{(s+1)^2(s+2)} & \frac{s}{(s+1)^2(s+2)} \end{bmatrix}.$$

Then $G(s)$ can be written as

$$G(s) = \frac{1}{(s+1)^2(s+2)} \begin{bmatrix} s+1 & (s+1)(2s+1) & s(s+1) \\ s+2 & (s+2)(s^2+5s+3) & s(s+2) \\ 1 & 2s+1 & s \end{bmatrix}.$$

$G(s)$ has the McMillan form

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)} & 0 & 0 \\ 0 & \frac{s+2}{s+1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

McMillan degree of $G(s) = 4$.
poles of the transfer matrix: $\{-1, -1, -1, -2\}$.
transmission zero: $\{-2\}$.

The transfer matrix has pole and zero at the same location $\{-2\}$*; this is the unique feature of multivariable systems.*

Alternative Characterizations

- Let $G(s)$ have <u>full column normal rank</u>. Then $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ if and only if there exists a vector $0 \neq u_0$ such that $G(z_0)u_0 = 0$.

  not true if $G(s)$ does not have full column normal rank.

  an example

  $$G(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

  G has no transmission zero but $G(s)u_0 = 0$ for all s.

  z_0 can be a pole of $G(s)$ although $G(z_0)$ is not defined. (however $G(z_0)u_0$ may be well defined.) For example,

  $$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s+2}{s-1} \end{bmatrix}, \quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

  Then $G(1)u_0 = 0$. Therefore, 1 is a transmission zero.

- Let $G(s)$ have <u>full row normal rank</u>. Then $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ if and only if there exists a vector $\eta_0 \neq 0$ such that $\eta_0^* G(z_0) = 0$.

- Suppose $z_0 \in \mathbb{C}$ is not a pole of $G(s)$. Then z_0 is a transmission zero if and only if $\text{rank}(G(z_0)) < \text{normalrank}(G(s))$.

- Let $G(s)$ be a square $m \times m$ matrix and $\det G(s) \not\equiv 0$. Suppose $z_o \in \mathbb{C}$ is not a pole of $G(s)$. Then $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ if and only if $\det G(z_0) = 0$.

  $$\det \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix} = \frac{2 - s^2}{(s+1)^2(s+2)^2}.$$

Invariant Zeros

The poles and zeros of a transfer matrix can also be characterized in terms of its state space realizations:

$$G(s) \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

consider the following system matrix

$$Q(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}.$$

$z_0 \in \mathbb{C}$ is an *invariant zero* of the realization if it satisfies

$$\text{rank} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} < normalrank \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}.$$

- Suppose $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full column normal rank. Then $z_0 \in \mathbb{C}$ is an invariant zero iff there exist $0 \neq x \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$ such that

  $$\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0.$$

  Moreover, if $u = 0$, then z_0 is also a non-observable mode.

- Suppose $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full row normal rank. Then $z_0 \in \mathbb{C}$ is an invariant zero iff there exist $0 \neq y \in \mathbb{C}^n$ and $v \in \mathbb{C}^p$ such that

  $$\begin{bmatrix} y^* & v^* \end{bmatrix} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = 0.$$

  Moreover, if $v = 0$, then z_0 is also a non-controllable mode.

- $G(s)$ has full column (row) normal rank if and only if $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full column (row) normal rank.

This follows by noting that

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(A - sI)^{-1} & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ 0 & G(s) \end{bmatrix}$$

and

$$\text{normalrank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \text{normalrank}(G(s)).$$

- Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization. Then z_0 is a transmission zero of $G(s)$ iff it is an invariant zero of the minimal realization.

- Let $G(s)$ be a $p \times m$ transfer matrix and let (A, B, C, D) be a minimal realization. Let the input be $u(t) = u_0 e^{\lambda t}$, where $\lambda \in \mathbb{C}$ is not a pole of $G(s)$ and $u_0 \in \mathbb{C}^m$ is an arbitrary constant vector, then the output with the initial state $x(0) = (\lambda I - A)^{-1} B u_0$ is $y(t) = G(\lambda) u_0 e^{\lambda t}$, $\forall t \geq 0$.

- Let $G(s)$ be a $p \times m$ transfer matrix and let (A, B, C, D) be a minimal realization. Suppose that $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ and is not a pole of $G(s)$. Then for any nonzero vector $u_0 \in \mathbb{C}^m$ such that $G(z_0) u_0 = 0$, the output of the system due to the initial state $x(0) = (z_0 I - A)^{-1} B u_0$ and the input $u = u_0 e^{z_0 t}$ is identically zero: $y(t) = G(z_0) u_0 e^{z_0 t} = 0$.