Lecture Notes: Week 3a

Topic: Norms,H-2,H_infinity Spaces, Internal Stability

ECE/MAE 7360
Optimal and Robust Control
(Fall 2003 Offering)

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Chapter 4: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Spaces

- Hilbert space
- $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Functions
- State Space Computation of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Hilbert Spaces

Inner product on $\mathbb{C}^n$:

$$\langle x, y\rangle := x^* y = \sum_{i=1}^{n} \bar{x}_i y_i \quad \forall x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{C}^n.$$

$$\|x\| := \sqrt{\langle x, x\rangle},$$

$$\cos\angle(x, y) = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}, \quad \angle(x, y) \in [0, \pi].$$

orthogonal if $\angle(x, y) = \frac{\pi}{2}$.

Definition 0.1 Let V be a vector space over $\mathbb{C}$. An *inner product* on V is a complex valued function,

$$\langle \cdot, \cdot\rangle :\ V \times V \longmapsto \mathbb{C}$$

such that for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) $\langle x, x\rangle > 0$ if $x \neq 0$.

A vector space V with an inner product is called an *inner product space*.

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inner product induced norm $\|x\| := \sqrt{\langle x, x\rangle}$
distance between vectors x and y: $d(x, y) = \|x - y\|$.
Two vectors x and y *orthogonal* if $\langle x, y\rangle = 0$, denoted $x \perp y$.

- $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ (Cauchy-Schwarz inequality). Equality holds iff $x = \alpha y$ for some constant α or $y = 0$.
- $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Parallelogram law) .
- $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$.

Hilbert space: a complete inner product space.
Examples:

- $\mathbb{C}^n$ with the usual inner product.
- $\mathbb{C}^{n\times m}$ with the inner product

$$\langle A, B\rangle := \operatorname{Trace} A^* B = \sum_{i=1}^{n}\sum_{j=1}^{m} \bar{a}_{ij} b_{ij} \quad \forall A, B \in \mathbb{C}^{n\times m}$$

- $\mathcal{L}_2[a, b]$: all square integrable and Lebesgue measurable functions defined on an interval $[a, b]$ with the inner product

$$\langle f, g\rangle := \int_a^b f(t)^* g(t) dt$$

  Matrix form: $\langle f, g\rangle := \int_a^b \operatorname{Trace}\left[f(t)^* g(t)\right] dt$.
- $\mathcal{L}_2 = \mathcal{L}_2(-\infty, \infty)$: $\langle f, g\rangle := \int_{-\infty}^{\infty} \operatorname{Trace}\left[f(t)^* g(t)\right] dt$.
- $\mathcal{L}_{2+} = \mathcal{L}_2[0, \infty)$: subspace of $\mathcal{L}_2(-\infty, \infty)$.
- $\mathcal{L}_{2-} = \mathcal{L}_2(-\infty, 0]$: subspace of $\mathcal{L}_2(-\infty, \infty)$.

Analytic Functions

Let $S \subset \mathbb{C}$ be an open set, and let $f(s)$ be a complex valued function defined on S:

$$f(s) : S \longmapsto \mathbb{C}.$$

Then $f(s)$ is *analytic at a point* z_0 in S if it is differentiable at z_0 and also at each point in some neighborhood of z_0.

It is a fact that if $f(s)$ is analytic at z_0 then f has continuous derivatives of all orders at z_0. Hence, a function analytic at z_0 has a power series representation at z_0.

A function $f(s)$ is said to be *analytic in* S if it has a derivative or is analytic at each point of S.

Maximum Modulus Theorem: If $f(s)$ is defined and continuous on a closed-bounded set S and analytic on the interior of S, then

$$\max_{s \in S} |f(s)| = \max_{s \in \partial S} |f(s)|$$

where ∂S denotes the boundary of S.

$\mathcal{L}_2$ and $\mathcal{H}_2$ Spaces

$\mathcal{L}_2(j\mathbb{R})$ Space: all complex matrix functions F such that the integral below is bounded:

$$\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega < \infty$$

with the inner product

$$\langle F, G \rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)G(j\omega)\right] d\omega$$

and the inner product induced norm is given by

$$\|F\|_2 := \sqrt{\langle F, F \rangle}.$$

$\mathcal{RL}_2(j\mathbb{R})$ or simply $\mathcal{RL}_2$: all real rational strictly proper transfer matrices with no poles on the imaginary axis.

$\mathcal{H}_2$ Space: a (closed) subspace of $\mathcal{L}_2(j\mathbb{R})$ with functions $F(s)$ analytic in $Re(s) > 0$.

$$\begin{aligned} \|F\|_2^2 &:= \sup_{\sigma>0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(\sigma + j\omega)F(\sigma + j\omega)\right] d\omega\right\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega. \end{aligned}$$

$\mathcal{RH}_2$ (real rational subspace of $\mathcal{H}_2$): all strictly proper and real rational stable transfer matrices.

$\mathcal{H}_2^{\perp}$ Space: the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$, i.e., the (closed) subspace of functions in $\mathcal{L}_2$ that are analytic in $Re(s) < 0$. .

$\mathcal{RH}_2^{\perp}$ (the real rational subspace of $\mathcal{H}_2^{\perp}$): all strictly proper rational antistable transfer matrices.

Parseval's relations:

$$\mathcal{L}_2(-\infty, \infty) \cong \mathcal{L}_2(j\mathbb{R}) \qquad \mathcal{L}_2[0, \infty) \cong \mathcal{H}_2 \qquad \mathcal{L}_2(-\infty, 0] \cong \mathcal{H}_2^{\perp}.$$

$$\|G\|_2 = \|g\|_2 \quad \text{where } G(s) = \mathcal{L}[g(t)] \in \mathcal{L}_2(j\mathbb{R})$$

$\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ Spaces

$\mathcal{L}_\infty(j\mathbb{R})$ Space

$\mathcal{L}_\infty(j\mathbb{R})$ or simply $\mathcal{L}_\infty$ is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on $j\mathbb{R}$, with norm

$$\|F\|_\infty := \operatorname{ess}\sup_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

$\mathcal{RL}_\infty(j\mathbb{R})$ or simply $\mathcal{RL}_\infty$: all proper and real rational transfer matrices with no poles on the imaginary axis.

$\mathcal{H}_\infty$ Space

$\mathcal{H}_\infty$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. The $\mathcal{H}_\infty$ norm is defined as

$$\|F\|_\infty := \sup_{\operatorname{Re}(s)>0} \overline{\sigma}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

The second equality can be regarded as a generalization of the maximum modulus theorem for matrix functions. See Boyd and Desoer [1985] for a proof.

$\mathcal{RH}_\infty$: all proper and real rational stable transfer matrices.

$\mathcal{H}_\infty^-$ Space

$\mathcal{H}_\infty^-$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open left-half plane. The $\mathcal{H}_\infty^-$ norm is defined as

$$\|F\|_\infty := \sup_{\operatorname{Re}(s)<0} \overline{\sigma}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

$\mathcal{RH}_\infty^-$: all proper real rational antistable transfer matrices.

$\mathcal{H}_\infty$ Norm as Induced $\mathcal{H}_2$ Norm

Let $G(s) \in \mathcal{L}_\infty$ be a $p \times q$ transfer matrix. Then a multiplication operator is defined as

$$M_G : \mathcal{L}_2 \longmapsto \mathcal{L}_2$$

$$M_G f := Gf.$$

Then $\|M_G\| = \sup_{f \in \mathcal{L}_2} \dfrac{\|Gf\|_2}{\|f\|_2} = \|G\|_\infty$.

$$\begin{aligned} \|Gf\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega) G^*(j\omega) G(j\omega) f(j\omega)\; d\omega \\ &\leq \|G\|_\infty^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(j\omega)\|^2 \; d\omega \\ &= \|G\|_\infty^2 \|f\|_2^2. \end{aligned}$$

To show that $\|G\|_\infty$ is the least upper bound, first choose a frequency ω_0 where $\overline{\sigma}\left[G(j\omega)\right]$ is maximum, i.e.,

$$\overline{\sigma}\left[G(j\omega_0)\right] = \|G\|_\infty$$

and denote the singular value decomposition of $G(j\omega_0)$ by

$$G(j\omega_0) = \overline{\sigma} u_1(j\omega_0) v_1^*(j\omega_0) + \sum_{i=2}^{r} \sigma_i u_i(j\omega_0) v_i^*(j\omega_0)$$

where r is the rank of $G(j\omega_0)$ and u_i, v_i have unit length.

If $\omega_0 < \infty$, write $v_1(j\omega_0)$ as

$$v_1(j\omega_0) = \begin{bmatrix} \alpha_1 e^{j\theta_1} \\ \alpha_2 e^{j\theta_2} \\ \vdots \\ \alpha_q e^{j\theta_q} \end{bmatrix}$$

where $\alpha_i \in \mathbb{R}$ is such that $\theta_i \in (-\pi, 0]$. Now let $0 \le \beta_i \le \infty$ be such that

$$\theta_i = \angle \left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0} \right)$$

(with $\beta_i = \infty$ if $\theta_i = 0$) and let f be given by

$$f(s) = \begin{bmatrix} \alpha_1 \frac{\beta_1 - s}{\beta_1 + s} \\ \alpha_2 \frac{\beta_2 - s}{\beta_2 + s} \\ \vdots \\ \alpha_q \frac{\beta_q - s}{\beta_q + s} \end{bmatrix} \hat{f}(s)$$

(with 1 replacing $\frac{\beta_i - s}{\beta_i + s}$ if $\theta_i = 0$) where a scalar function $\hat{f}$ is chosen so that

$$|\hat{f}(j\omega)| = \begin{cases} c & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

where ϵ is a small positive number and c is chosen so that $\hat{f}$ has unit 2-norm, i.e., $c = \sqrt{\pi/2\epsilon}$. This in turn implies that f has unit 2-norm. Then

$$\begin{aligned} \|Gf\|_2^2 &\approx \frac{1}{2\pi} \left[\overline{\sigma} \left[G(-j\omega_0) \right]^2 \pi + \overline{\sigma} \left[G(j\omega_0) \right]^2 \pi \right] \\ &= \overline{\sigma} \left[G(j\omega_0) \right]^2 = \|G\|_\infty^2 . \end{aligned}$$

Similarly, if $\omega_0 = \infty$, the conclusion follows by letting $\omega_0 \to \infty$ in the above.

Computing $\mathcal{L}_2$ and $\mathcal{H}_2$ Norms

Let $G(s) \in \mathcal{L}_2$ and $g(t) = \mathcal{L}^{-1}[G(s)]$. Then

$$\begin{aligned}\|G\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G^*(j\omega)G(j\omega)\}\, d\omega = \frac{1}{2\pi j}\oint \text{Trace}\{G^\sim(s)G(s)\}\, ds. \\ &= \sum \begin{array}{c}\text{the residues of Trace}\{G^\sim(s)G(s)\} \\ \text{at its poles in the left half plane.}\end{array} \\ &= \int_{-\infty}^{\infty} \text{Trace}\{g^*(t)g(t)\}\, dt = \|g\|_2^2\end{aligned}$$

Consider $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_2$. Then we have

$$\|G\|_2^2 = \text{trace}(B^* L_o B) = \text{trace}(C L_c C^*)$$

where L_o and L_c are observability and controllability Gramians:

$$AL_c + L_c A^* + BB^* = 0 \qquad A^* L_o + L_o A + C^* C = 0.$$

Note that $g(t) = \mathcal{L}^{-1}(G) = \begin{cases} Ce^{At}B, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$L_o = \int_0^\infty e^{A^*t}C^*Ce^{At}\, dt, \quad L_c = \int_0^\infty e^{At}BB^*e^{A^*t}\, dt,$$

$$\begin{aligned}\|G\|_2^2 &= \int_0^\infty \text{Trace}\{g^*(t)g(t)\}\, dt = \int_0^\infty \text{Trace}\{B^*e^{A^*t}C^*Ce^{At}B\}\, dt \\ &= \text{Trace}\{B^*\int_0^\infty e^{A^*t}C^*Ce^{At}dtB\} = \text{trace}(B^*L_oB) \\ &= \int_0^\infty \text{Trace}\{g(t)g^*(t)\}\, dt = \int_0^\infty \text{Trace}\{Ce^{At}BB^*e^{A^*t}C^*\}\, dt.\end{aligned}$$

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hypothetical input-output experiments: Apply the impulsive input $\delta(t)e_i$ ($\delta(t)$ is the unit impulse and e_i is the i^{th} standard basis vector) and denote the output by $z_i(t)(= g(t)e_i)$. Then $z_i \in \mathcal{L}_{2+}$ (assuming $D = 0$) and

$$\|G\|_2^2 = \sum_{i=1}^m \|z_i\|_2^2.$$

Can be used for nonlinear time varying systems.

Example

Consider a transfer matrix

$$G = \begin{bmatrix} \dfrac{3(s+3)}{(s-1)(s+2)} & \dfrac{2}{s-1} \\ \dfrac{s+1}{(s+2)(s+3)} & \dfrac{1}{s-4} \end{bmatrix} = G_s + G_u$$

with

$$G_s = \left[\begin{array}{cc|cc} -2 & 0 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad G_u = \left[\begin{array}{cc|cc} 1 & 0 & 4 & 2 \\ 0 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Then the command $\mathbf{h2norm}(\mathbf{G_s})$ gives $\|G_s\|_2 = 0.6055$ and $\mathbf{h2norm}(\mathbf{cjt}(\mathbf{G_u}))$ gives $\|G_u\|_2 = 3.182$. Hence $\|G\|_2 = \sqrt{\|G_s\|_2^2 + \|G_u\|_2^2} = 3.2393$.

- $\gg$ $\mathbf{P} = \mathbf{gram}(\mathbf{A}, \mathbf{B})$; $\mathbf{Q} = \mathbf{gram}(\mathbf{A'}, \mathbf{C'})$; **or** $\mathbf{P} = \mathbf{lyap}(\mathbf{A}, \mathbf{B} * \mathbf{B'})$;
- $\gg$ $[\mathbf{G_s}, \mathbf{G_u}] = \mathbf{sdecomp}(\mathbf{G})$; % decompose into stable and antistable parts.

Computing $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ Norms

Let $G(s) \in \mathcal{L}_\infty$

$$\|G\|_\infty := \text{ess} \sup_\omega \overline{\sigma}\{G(j\omega)\}.$$

- the farthest distance the Nyquist plot of G from the origin
- the peak on the Bode magnitude plot
- estimation: set up a fine grid of frequency points, $\{\omega_1, \cdots, \omega_N\}$.

$$\|G\|_\infty \approx \max_{1\le k\le N} \overline{\sigma}\{G(j\omega_k)\}.$$

Let $\gamma > 0$ and $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RL}_\infty$.

$$\|G\|_\infty < \gamma \Longleftrightarrow \overline{\sigma}(D) < \gamma \ \ \& \ \ H \ \ \text{has no j}\omega \text{ eigenvalues}$$

where $H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix}$
and $R = \gamma^2 I - D^*D$.

Let $\Phi(s) = \gamma^2 I - G^\sim(s)G(s)$.

$\|G\|_\infty < \gamma$

$\Longleftrightarrow \Phi(j\omega) > 0, \ \forall \omega \in \mathbb{R}$.

$\Longleftrightarrow \det \Phi(j\omega) \neq 0$ since $\Phi(\infty) = R > 0$ and $\Phi(j\omega)$ is continuous

$\Longleftrightarrow \Phi(s)$ has no imaginary axis zero.

$\Longleftrightarrow \Phi^{-1}(s)$ has no imaginary axis pole.

$$\Phi^{-1}(s) = \left[\begin{array}{c|c} H & \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} & R^{-1} \end{array}\right].$$

$\Longleftrightarrow$ H has no $j\omega$ axis eigenvalues if the above realization has neither uncontrollable modes nor unobservable modes on the imaginary axis.

Assume that $j\omega_0$ is an eigenvalue of H but not a pole of $\Phi^{-1}(s)$. Then $j\omega_0$ must be either an unobservable mode of $(\begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix}, H)$ or an uncontrollable mode of $(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix})$. Suppose $j\omega_0$ is an unobservable mode of $(\begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix}, H)$. Then there exists an $x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ such that

$$Hx_0 = j\omega_0 x_0, \ \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} x_0 = 0.$$

$$\Updownarrow$$

$$\begin{aligned} (j\omega_0 I - A)x_1 &= 0 \\ (j\omega_0 I + A^*)x_2 &= -C^*Cx_1 \\ D^*Cx_1 + B^*x_2 &= 0. \end{aligned}$$

Since A has no imaginary axis eigenvalues, we have $x_1 = 0$ and $x_2 = 0$. Contradiction!!!

Similarly, a contradiction will also be arrived if $j\omega_0$ is assumed to be an uncontrollable mode of $(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix})$.

Bisection Algorithm

(a) select an upper bound γ_u and a lower bound γ_l such that $\gamma_l \leq \|G\|_\infty \leq \gamma_u$;

(b) if $(\gamma_u - \gamma_l)/\gamma_l \leq$specified level, stop; $\|G\| \approx (\gamma_u + \gamma_l)/2$. Otherwise go to next step;

(c) set $\gamma = (\gamma_l + \gamma_u)/2$;

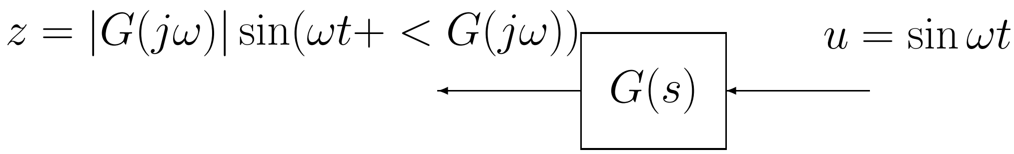
(d) test if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of H for the given γ;

(e) if H has an eigenvalue on $j\mathbb{R}$ set $\gamma_l = \gamma$; otherwise set $\gamma_u = \gamma$; go back to step (b).

WLOG assume $\gamma = 1$ since $\|G\|_\infty < \gamma$ iff $\|\gamma^{-1}G\|_\infty < 1$

Estimating the $\mathcal{H}_\infty$ norm

Estimating the $\mathcal{H}_\infty$ norm experimentally: the maximum magnitude of the steady-state response to all possible unit amplitude sinusoidal input signals.



Let the sinusoidal inputs be

$$u(t) = \begin{bmatrix} u_1 \sin(\omega_0 t + \phi_1) \\ u_2 \sin(\omega_0 t + \phi_2) \\ \vdots \\ u_q \sin(\omega_0 t + \phi_q) \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}.$$

Then the steady-state response of the system can be written as

$$y(t) = \begin{bmatrix} y_1 \sin(\omega_0 t + \theta_1) \\ y_2 \sin(\omega_0 t + \theta_2) \\ \vdots \\ y_p \sin(\omega_0 t + \theta_p) \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

for some $y_i,\ \theta_i,\ i = 1, 2, \ldots, p$, and furthermore,

$$\|G\|_\infty = \sup_{\phi_i, \omega_o, \hat{u}} \frac{\|\hat{y}\|}{\|\hat{u}\|}$$

where $\|\cdot\|$ is the Euclidean norm.

Examples

Consider a mass/spring/damper system as shown in Figure 0.1.

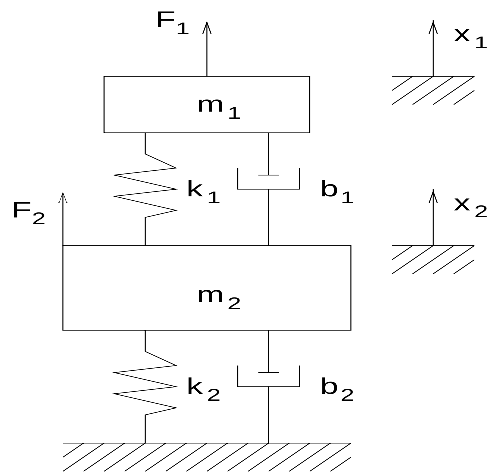


Figure 0.1: A two-mass/spring/damper system

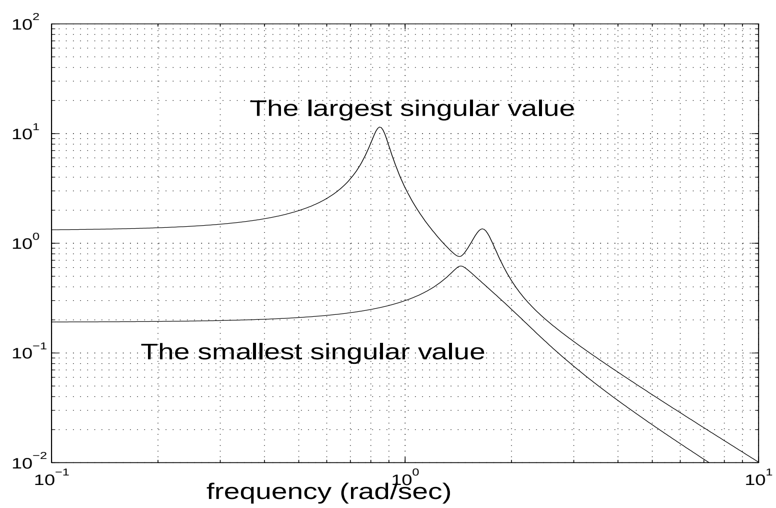


Figure 0.2: $\|G\|_\infty$ is the peak of the largest singular value of $G(j\omega)$

The dynamical system can be described by the following differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + B \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}.$$

Suppose that $G(s)$ is the transfer matrix from (F_1, F_2) to (x_1, x_2); that is,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0,$$

and suppose $k_1 = 1$, $k_2 = 4$, $b_1 = 0.2$, $b_2 = 0.1$, $m_1 = 1$, and $m_2 = 2$ with appropriate units.

$\gg$ **G=pck(A,B,C,D);**

$\gg$ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error ≤ 0.0001

$\gg$ **w=logspace(-1,1,200);** % **200** points between $1 = 10^{-\mathbf{1}}$ and $10 = 10^{\mathbf{1}}$;

$\gg$ **Gf=frsp(G,w);** % computing frequency response;

$\gg$ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

$\gg$ **vplot('liv, lm', s),** **grid** % plot both singular values and grid.

$\|G(s)\|_\infty = 11.47 =$ the peak of the largest singular value Bode plot in Figure 0.2.

Since the peak is achieved at $\omega_{\max} = 0.8483$, exciting the system using the following sinusoidal input

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0.9614 \sin(0.8483t) \\ 0.2753 \sin(0.8483t - 0.12) \end{bmatrix}$$

gives the steady-state response of the system as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11.47 \times 0.9614 \sin(0.8483t - 1.5483) \\ 11.47 \times 0.2753 \sin(0.8483t - 1.4283) \end{bmatrix}.$$

This shows that the system response will be amplified 11.47 times for an input signal at the frequency $\omega_{\max}$, which could be undesirable if F_1 and F_2 are disturbance force and x_1 and x_2 are the positions to be kept steady.

Consider a two-by-two transfer matrix

$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}.$$

A state-space realization of G can be obtained using the following MATLAB commands:

$\gg$ **G11=nd2sys([10,10],[1,0.2,100]);**

$\gg$ **G12=nd2sys(1,[1,1]);**

$\gg$ **G21=nd2sys([1,2],[1,0.1,10]);**

$\gg$ **G22=nd2sys([5,5],[1,5,6]);**

$\gg$ **G=sbs(abv(G11,G21),abv(G12,G22));**

Next, we set up a frequency grid to compute the frequency response of G and the singular values of $G(j\omega)$ over a suitable range of frequency.

$\gg$ **w=logspace(0,2,200);** % **200** points between $1 = 10^{\mathbf{0}}$ and $100 = 10^{\mathbf{2}}$;

$\gg$ **Gf=frsp(G,w);** % computing frequency response;

$\gg$ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

$\gg$ **vplot**('**liv, lm**', **s**), **grid** % plot both singular values and grid;

$\gg$ **pkvnorm(s)** % find the norm from the frequency response of the singular values.

The singular values of $G(j\omega)$ are plotted in Figure 0.3, which gives an estimate of $\|G\|_\infty \approx 32.861$. The state-space bisection algorithm described previously leads to $\|G\|_\infty = 50.25 \pm 0.01$ and the corresponding MATLAB command is

$\gg$ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error ≤ 0.0001.

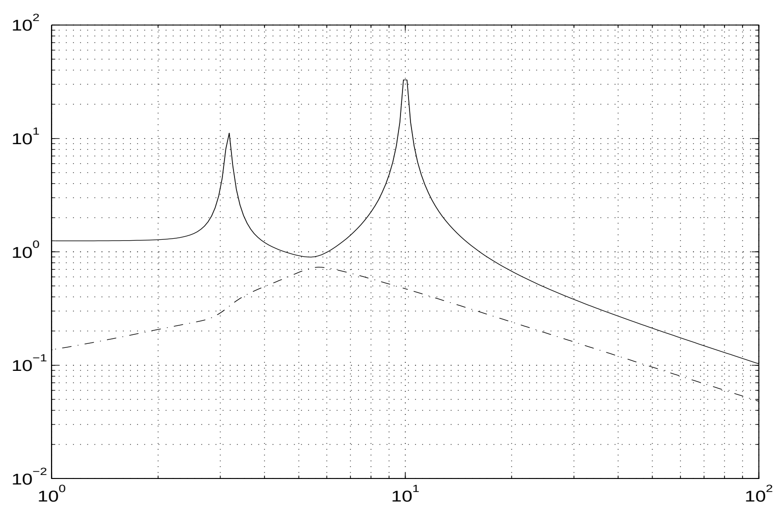


Figure 0.3: The largest and the smallest singular values of $G(j\omega)$

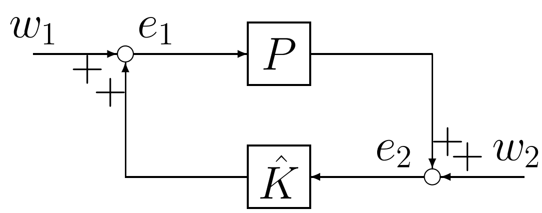
The preceding computational results show clearly that the graphical method can lead to a wrong answer for a lightly damped system if the frequency grid is not sufficiently dense. Indeed, we would get $\|G\|_\infty \approx 43.525, 48.286$ and 49.737 from the graphical method if 400, 800, and 1600 frequency points are used, respectively.

Chapter 5: Internal Stability

- internal stability
- coprime factorization over $\mathcal{RH}_\infty$
- performance

Internal Stability

Consider the following feedback system:



- well-posed if $I - \hat{K}(\infty)P(\infty)$ is invertible.
- **Internal Stability:** if

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-\hat{K}P)^{-1} & \hat{K}(I-P\hat{K})^{-1} \\ P(I-\hat{K}P)^{-1} & (I-P\hat{K})^{-1} \end{bmatrix} \in \mathcal{RH}_\infty$$

- Need to check all **Four** transfer matrices. For example,

$$P = \frac{s-1}{s+1}, \qquad \hat{K} = -\frac{1}{s-1}.$$

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{s+1}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & \frac{s+1}{s+2} \end{bmatrix}$$

- Suppose $\hat{K} \in \mathcal{H}_\infty$. Internal stability $\Longleftrightarrow P(I-\hat{K}P)^{-1} \in \mathcal{H}_\infty$.
- Suppose $P \in \mathcal{H}_\infty$. Internal stability $\Longleftrightarrow \hat{K}(I-P\hat{K})^{-1} \in \mathcal{H}_\infty$.
- Suppose $P, \hat{K} \in \mathcal{H}_\infty$. Internal stability $\Longleftrightarrow (I-P\hat{K})^{-1} \in \mathcal{H}_\infty$.
- Suppose no unstable pole-zero cancellation in PK.
  Internal stability $\Longleftrightarrow (I - P(s)\hat{K}(s))^{-1} \in \mathcal{H}_\infty$

Example

Let P and $\hat{K}$ be two-by-two transfer matrices

$$P = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} \frac{1-s}{s+1} & -1 \\ 0 & -1 \end{bmatrix}.$$

Then

$$P\hat{K} = \begin{bmatrix} \frac{-1}{s+1} & \frac{-1}{s-1} \\ 0 & \frac{-1}{s+1} \end{bmatrix}, \quad (I - P\hat{K})^{-1} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{(s+1)^2}{(s+2)^2(s-1)} \\ 0 & \frac{s+1}{s+2} \end{bmatrix}.$$

So the closed-loop system is not stable even though

$$\det(I - P\hat{K}) = \frac{(s+2)^2}{(s+1)^2}$$

has no zero in the closed right-half plane and the number of unstable poles of $P\hat{K} = n_k + n_p = 1$. Hence, in general, $\det(I - P\hat{K})$ having no zeros in the closed right-half plane does not necessarily imply $(I - P\hat{K})^{-1} \in \mathcal{RH}_\infty$.

Coprime Factorization over $\mathcal{RH}_\infty$

- two polynomials $m(s)$ and $n(s)$ are coprime if the only common factors are constants.

- two transfer functions $m(s)$ and $n(s)$ in $\mathcal{RH}_\infty$ are *coprime over* $\mathcal{RH}_\infty$ if the only common factors are stable and invertible transfer functions (units):

$$h, mh^{-1}, nh^{-1} \in \mathcal{RH}_\infty \Longrightarrow h^{-1} \in \mathcal{RH}_\infty.$$

  Equivalent, there exists $x, y \in \mathcal{RH}_\infty$ such that

$$xm + yn = 1.$$

- Matrices M and N in $\mathcal{RH}_\infty$ are *right coprime over* $\mathcal{RH}_\infty$ if there exist matrices X_r and Y_r in $\mathcal{RH}_\infty$ such that

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = X_r M + Y_r N = I.$$

- Matrices $\tilde{M}$ and $\tilde{N}$ in $\mathcal{RH}_\infty$ are *left coprime over* $\mathcal{RH}_\infty$ if there exist matrices X_l and Y_l in $\mathcal{RH}_\infty$ such that

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} X_l \\ Y_l \end{bmatrix} = \tilde{M} X_l + \tilde{N} Y_l = I.$$

Let $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $\hat{K} = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ be *rcf* and *lcf*, respectively. Then the following conditions are equivalent:

1. The feedback system is internally stable.
2. $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ is invertible in $\mathcal{RH}_\infty$.
3. $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ is invertible in $\mathcal{RH}_\infty$.
4. $\tilde{M}V - \tilde{N}U$ is invertible in $\mathcal{RH}_\infty$.
5. $\tilde{V}M - \tilde{U}N$ is invertible in $\mathcal{RH}_\infty$.

Let $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stabilizable and detectable realization, and let F and L be such that $A + BF$ and $A + LC$ are both stable.

Define

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & -L \\ \hline F & I & 0 \\ C + DF & D & I \end{array}\right]$$

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & -(B + LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right].$$

Then

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I.$$

Example

Let $P(s) = \dfrac{s-2}{s(s+3)}$ and $\alpha = (s+1)(s+3)$. Then $P(s) = n(s)/m(s)$ with $n(s) = \dfrac{s-2}{(s+1)(s+3)}$ and $m(s) = \dfrac{s}{s+1}$ forms a coprime factorization. To find an $x(s) \in \mathcal{H}_\infty$ and a $y(s) \in \mathcal{H}_\infty$ such that $x(s)n(s) + y(s)m(s) = 1$, consider a stabilizing controller for P: $\hat{K} = -\dfrac{s-1}{s+10}$. Then $\hat{K} = u/v$ with $u = \hat{K}$ and $v = 1$ is a coprime factorization and

$$m(s)v(s) - n(s)u(s) = \frac{(s+11.7085)(s+2.214)(s+0.077)}{(s+1)(s+3)(s+10)} =: \beta(s)$$

Then we can take

$$x(s) = -u(s)/\beta(s) = \frac{(s-1)(s+1)(s+3)}{(s+11.7085)(s+2.214)(s+0.077)}$$

$$y(s) = v(s)/\beta(s) = \frac{(s+1)(s+3)(s+10)}{(s+11.7085)(s+2.214)(s+0.077)}$$

MATLAB programs can be used to find the appropriate F and L matrices in state-space so that the desired coprime factorization can be obtained. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Then an F and an L can be obtained from

≫ **F=-lqr(A, B, eye(n), eye(m));** % or

≫ **F=-place(A, B, Pf);** % Pf= poles of A+BF

≫ $\mathbf{L} = -\mathbf{lqr}(\mathbf{A}', \mathbf{C}', \mathbf{eye}(\mathbf{n}), \mathbf{eye}(\mathbf{p}));$ % or

≫ $\mathbf{L} = -\mathbf{place}(\mathbf{A}', \mathbf{C}', \mathbf{Pl});$ % Pl=poles of A+LC.