Lecture Notes: Week 4b

Topic: Balanced model reduction

ECE/MAE 7360
Optimal and Robust Control
(Fall 2003 Offering)

Instructor: Dr YangQuan Chen, CSOIS, ECE Dept.,
Tel. : (435)797-0148.
E-mail: yqchen@ieee.org or, yqchen@ece.usu.edu

Chapter 7: Balanced Model Reduction

- Balanced Realization
- Balanced Model Reduction
- Frequency Weighted Balanced Model Reduction
- Relative Reduction

Balanced Realization

Consider the following Lyapunov equation

$$A^*X + XA + Q = 0$$

Assume that A is stable, then the following statements hold:

- $X = \int_0^\infty e^{A^*t} Q e^{At} dt$.
- $X > 0$ if $Q > 0$ and $X \geq 0$ if $Q \geq 0$.
- if $Q \geq 0$, then (Q, A) is observable iff $X > 0$.

Suppose X is the solution of the Lyapunov equation, then

- $\mathrm{Re}\lambda_i(A) \leq 0$ if $X > 0$ and $Q \geq 0$.
- A is stable if $X > 0$ and $Q > 0$.
- A is stable if $X \geq 0$, $Q \geq 0$ and (Q, A) is detectable.

Let A be stable. Then a pair (C, A) is observable iff the *observability Gramian* $Q > 0$

$$A^*Q + QA + C^*C = 0.$$

Similarly, (A, B) is controllable iff the *controllability Gramian* $P > 0$

$$AP + PA^* + BB^* = 0$$

- Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state space realization of a (not necessarily stable) transfer matrix $G(s)$. Suppose that there exists a symmetric matrix

  $$P = P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

  with P_1 nonsingular such that

  $$AP + PA^* + BB^* = 0.$$

Now partition the realization (A, B, C, D) compatibly with P as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then

$$\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is also a realization of G. Moreover, (A_{11}, B_1) is controllable if A_{11} is stable.

Proof Using

$$0 = AP + PA^* + BB^*$$

to get $B_2 = 0$ and $A_{21} = 0$. Hence, part of the realization is not controllable:

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right].$$

- Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state space realization of a (not necessarily stable) transfer matrix $G(s)$. Suppose that there exists a symmetric matrix

$$Q = Q^* = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with Q_1 nonsingular such that

$$QA + A^*Q + C^*C = 0.$$

Now partition the realization (A, B, C, D) compatibly with Q as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then

$$\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is also a realization of G. Moreover, (C_1, A_{11}) is observable if A_{11} is stable.

- Let P and Q be the controllability and observability Gramians,

$$AP + PA^* + BB^* = 0$$

$$A^*Q + QA + C^*C = 0.$$

Suppose

$$P = Q = \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

Then the state space realization is called internally *balanced realization* and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, are called the *Hankel singular values* of the system.

Two other closely related realizations are called *input normal realization* with $P = I$ and $Q = \Sigma^2$, and *output normal realization* with $P = \Sigma^2$ and $Q = I$. Both realizations can be obtained easily from the balanced realization by a suitable scaling on the states.

- Let P and Q be two positive semidefinite matrices. Then there exists a nonsingular matrix T such that

$$TPT^* = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$(T^{-1})^*QT^{-1} = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

respectively, with $\Sigma_1,\ \Sigma_2,\ \Sigma_3$ diagonal and positive definite.

In the special case where $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal realization, a balanced realization can be obtained through the following simplified procedure:

1. Compute $P > 0$ and $Q > 0$.
2. Find a matrix R such that $P = R^*R$.
3. Diagonalize RQR^* to get $RQR^* = U\Sigma^2U^*$.
4. Let $T^{-1} = R^*U\Sigma^{-1/2}$. Then $TPT^* = (T^*)^{-1}QT^{-1} = \Sigma$ and $\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$ is balanced.

Suppose $\sigma_r \gg \sigma_{r+1}$ for some r then the balanced realization implies that those states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ are less controllable and observable than those states corresponding to $\sigma_1, \ldots, \sigma_r$. Therefore, truncating those less controllable and observable states will not lose much information about the system.

input normal realization: $P = I$ and $Q = \Sigma^2$
output normal realization: $P = \Sigma^2$ and $Q = I$.

Suppose

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_\infty$$

is a balanced realization; that is, there exists

$$\Sigma = \mathrm{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N}) \geq 0$$

with $\sigma_1 > \sigma_2 > \ldots > \sigma_N \geq 0$, such that

$$A\Sigma + \Sigma A^* + BB^* = 0 \qquad A^*\Sigma + \Sigma A + C^*C = 0$$

Then

$$\sigma_1 \leq \|G\|_\infty \leq \int_0^\infty \|g(t)\|\, dt \leq 2\sum_{i=1}^N \sigma_i$$

where $g(t) = Ce^{At}B$.

Proof.

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx. \end{aligned}$$

(A, B) is controllable and (C, A) is observable.

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \dot{x}^*\Sigma^{-1}x + x^*\Sigma^{-1}\dot{x} = x^*(A^*\Sigma^{-1} + \Sigma^{-1}A)x + 2\langle w, B^*\Sigma^{-1}x\rangle$$

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \|w\|^2 - \|w - B^*\Sigma^{-1}x\|^2$$

Integration from $t = -\infty$ to $t = 0$ with $x(-\infty) = 0$ and $x(0) = x_0$ gives

$$x_0^*\Sigma^{-1}x_0 = \|w\|_2^2 - \|w - B^*\Sigma^{-1}x\|_2^2 \leq \|w\|_2^2$$

$$\inf_{w \in \mathcal{L}_2[-\infty,0)} \left\{ \|w\|_2^2 \;\middle|\; x(0) = x_0 \right\} = x_0^*\Sigma^{-1}x_0.$$

Given $x(0) = x_0$ and $w = 0$ for $t \geq 0$, the norm of $z(t) = Ce^{At}x_0$ can be found from

$$\int_0^\infty \|z(t)\|^2\, dt = \int_0^\infty x_0^* e^{A^*t}C^*Ce^{At}x_0 dt = x_0^*\Sigma x_0$$

To show $\sigma_1 \leq \|G\|_\infty$, note that

$$\|G\|_\infty = \sup_{w \in \mathcal{L}_2(-\infty,\infty)} \frac{\|g * w\|_2}{\|w\|_2} = \sup_{w \in \mathcal{L}_2(-\infty,\infty)} \frac{\sqrt{\int_{-\infty}^{\infty} \|z(t)\|^2 dt}}{\sqrt{\int_{-\infty}^{\infty} \|w(t)\|^2 dt}}$$

$$\geq \sup_{w \in \mathcal{L}_2(-\infty,0]} \frac{\sqrt{\int_0^{\infty} \|z(t)\|^2 dt}}{\sqrt{\int_{-\infty}^{0} \|w(t)\|^2 dt}} = \sup_{x_0 \neq 0} \sqrt{\frac{x_0^* \Sigma x_0}{x_0^* \Sigma^{-1} x_0}} = \sigma_1$$

We shall now show the other inequalities. Since

$$G(s) := \int_0^\infty g(t) e^{-st} dt, \ \operatorname{Re}(s) > 0,$$

by the definition of $\mathcal{H}_\infty$ norm, we have

$$\begin{aligned} \|G\|_\infty &= \sup_{\operatorname{Re}(s)>0} \left\| \int_0^\infty g(t) e^{-st} dt \right\| \\ &\leq \sup_{\operatorname{Re}(s)>0} \int_0^\infty \left\| g(t) e^{-st} \right\| dt \\ &\leq \int_0^\infty \|g(t)\| \, dt. \end{aligned}$$

To prove the last inequality, let e_i be the ith unit vector and define

$$E_1 = \begin{bmatrix} e_1 & \cdots & e_{s_1} \end{bmatrix}, \quad \ldots,$$

$$E_N = \begin{bmatrix} e_{s_1 + \cdots + s_{N-1} + 1} & \cdots & e_{s_1 + \cdots + s_N} \end{bmatrix}.$$

Then $\sum_{i=1}^{N} E_i E_i^* = I$ and

$$\begin{aligned} \int_0^\infty \|g(t)\| \, dt &= \int_0^\infty \left\| C e^{At/2} \sum_{i=1}^N E_i E_i^* e^{At/2} B \right\| dt \\ &\leq \sum_{i=1}^N \int_0^\infty \left\| C e^{At/2} E_i E_i^* e^{At/2} B \right\| dt \\ &\leq \sum_{i=1}^N \int_0^\infty \left\| C e^{At/2} E_i \right\| \left\| E_i^* e^{At/2} B \right\| dt \\ &\leq \sum_{i=1}^N \sqrt{\int_0^\infty \left\| C e^{At/2} E_i \right\|^2 dt} \sqrt{\int_0^\infty \left\| E_i^* e^{At/2} B \right\|^2 dt} \\ &\leq 2 \sum_{i=1}^N \sigma_i \end{aligned}$$

where we have used Cauchy-Schwarz inequality and the following relations:

$$\int_0^\infty \left\| C e^{At/2} E_i \right\|^2 dt = \int_0^\infty \lambda_{\max}\left(E_i^* e^{A^*t/2} C^* C e^{At/2} E_i\right) dt$$

$$= 2\lambda_{\max}\left(E_i^* \Sigma E_i\right) = 2\sigma_i$$

$$= \int_0^\infty \left\| E_i^* e^{At/2} B \right\|^2 dt = \int_0^\infty \lambda_{\max}\left(E_i^* e^{At/2} B B^* e^{A^*t/2} E_i\right) dt$$

□

$\gg$ **[Ab, Bb, Cb, sig, Tinv]=balreal(A, B, C);** % sig is a vector of Hankel singular values and Tinv $= T^{-1}$;

$\gg [\mathbf{G_b}, \mathbf{sig}] = \mathbf{sysbal}(\mathbf{G});$

$\gg \mathbf{G_r} = \mathbf{strunc}(\mathbf{G_b}, \mathbf{2});$ % truncate to the second-order.

Balanced Model Reduction

$$G = G_r + \Delta_a, \quad \Longrightarrow \quad \inf_{\deg(G_r) \le r} \|G - G_r\|_\infty .$$

- Suppose

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

is a balanced realization with Gramian $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2)$

$$A\Sigma + \Sigma A^* + BB^* = 0 \qquad A^*\Sigma + \Sigma A + C^*C = 0.$$

where

$$\begin{aligned} \Sigma_1 &= \mathrm{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_r I_{s_r}) \\ \Sigma_2 &= \mathrm{diag}(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \ldots, \sigma_N I_{s_N}) \end{aligned}$$

and

$$\sigma_1 > \sigma_2 > \cdots > \sigma_r > \sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_N$$

where σ_i has multiplicity $s_i,\ i = 1, 2, \ldots, N$ and $s_1 + s_2 + \cdots + s_N = n$. Then the truncated system

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is balanced and asymptotically stable. Furthermore

$$\|G(s) - G_r(s)\|_\infty \le 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_N).$$

- $\|G(s) - G(\infty)\|_\infty \le 2(\sigma_1 + \ldots + \sigma_N)$.
- $\|G(s) - G_{n-1}(s)\|_\infty = 2\sigma_N$.

Proof. We shall first show the one step model reduction. Hence we shall assume $\Sigma_2 = \sigma_N I_{s_N}$. Define the approximation error

$$
\begin{aligned}
E_{11} &:= \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] - \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \\
&= \left[\begin{array}{ccc|c} A_{11} & 0 & 0 & B_1 \\ 0 & A_{11} & A_{12} & B_1 \\ 0 & A_{21} & A_{22} & B_2 \\ \hline -C_1 & C_1 & C_2 & 0 \end{array}\right]
\end{aligned}
$$

Apply a similarity transformation T to the preceding state-space realization with

$$
T = \begin{bmatrix} I/2 & I/2 & 0 \\ I/2 & -I/2 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}
$$

to get

$$
E_{11} = \left[\begin{array}{ccc|c} A_{11} & 0 & A_{12}/2 & B_1 \\ 0 & A_{11} & -A_{12}/2 & 0 \\ A_{21} & -A_{21} & A_{22} & B_2 \\ \hline 0 & -2C_1 & C_2 & 0 \end{array}\right]
$$

Consider a dilation of $E_{11}(s)$:

$$
\begin{aligned}
E(s) &= \begin{bmatrix} E_{11}(s) & E_{12}(s) \\ E_{21}(s) & E_{22}(s) \end{bmatrix} \\
&= \left[\begin{array}{ccc|cc} A_{11} & 0 & A_{12}/2 & B_1 & 0 \\ 0 & A_{11} & -A_{12}/2 & 0 & \sigma_N \Sigma_1^{-1} C_1^* \\ A_{21} & -A_{21} & A_{22} & B_2 & -C_2^* \\ \hline 0 & -2C_1 & C_2 & 0 & 2\sigma_N I \\ -2\sigma_N B_1^* \Sigma_1^{-1} & 0 & -B_2^* & 2\sigma_N I & 0 \end{array}\right] \\
&=: \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]
\end{aligned}
$$

Then it is easy to verify that

$$\tilde{P} = \begin{bmatrix} \Sigma_1 & 0 & \\ 0 & \sigma_N^2 \Sigma_1^{-1} & 0 \\ 0 & 0 & 2\sigma_N I_{s_N} \end{bmatrix}$$

satisfies

$$\begin{aligned} \tilde{A}\tilde{P} + \tilde{P}\tilde{A}^* + \tilde{B}\tilde{B}^* &= 0 \\ \tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* &= 0 \end{aligned}$$

Using these two equations, we have

$$\begin{aligned} E(s)E^{\sim}(s) &= \left[\begin{array}{cc|c} \tilde{A} & -\tilde{B}\tilde{B}^* & \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & -\tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right] \\ &= \left[\begin{array}{cc|c} \tilde{A} & -\tilde{A}\tilde{P} - \tilde{P}\tilde{A}^* - \tilde{B}\tilde{B}^* & \tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & -\tilde{C}\tilde{P} - \tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right] \\ &= \left[\begin{array}{cc|c} \tilde{A} & 0 & 0 \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & 0 & \tilde{D}\tilde{D}^* \end{array}\right] \\ &= \tilde{D}\tilde{D}^* = 4\sigma_N^2 I \end{aligned}$$

where the second equality is obtained by applying a similarity transformation

$$T = \begin{bmatrix} I & \tilde{P} \\ 0 & I \end{bmatrix}$$

Hence $\|E_{11}\|_\infty \leq \|E\|_\infty = 2\sigma_N$, which is the desired result.

The remainder of the proof is achieved by using the order reduction by one-step results and by noting that $G_k(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ obtained by the "kth" order partitioning is internally balanced with balanced Gramian given by

$$\Sigma_1 = \mathrm{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_k I_{s_k})$$

Let $E_k(s) = G_{k+1}(s) - G_k(s)$ for $k = 1, 2, \ldots, N-1$ and let $G_N(s) = G(s)$. Then

$$\overline{\sigma}\left[E_k(j\omega)\right] \leq 2\sigma_{k+1}$$

since $G_k(s)$ is a reduced-order model obtained from the internally balanced realization of $G_{k+1}(s)$ and the bound for one-step order reduction holds.

Noting that

$$G(s) - G_r(s) = \sum_{k=r}^{N-1} E_k(s)$$

by the definition of $E_k(s)$, we have

$$\overline{\sigma}\left[G(j\omega) - G_r(j\omega)\right] \leq \sum_{k=r}^{N-1} \overline{\sigma}\left[E_k(j\omega)\right] \leq 2\sum_{k=r}^{N-1} \sigma_{k+1}$$

This is the desired upper bound. $\square$

- bound can be tight. For example,

$$G(s) = \sum_{j=1}^{n} \frac{b_i}{s+a_i} == \left[\begin{array}{cccc|c} -a_1 & & & & \sqrt{b_1} \\ & -a_2 & & & \sqrt{b_2} \\ & & \ddots & & \vdots \\ & & & -a_n & \sqrt{b_n} \\ \hline \sqrt{b_1} & \sqrt{b_2} & \cdots & \sqrt{b_n} & 0 \end{array}\right]$$

with $a_i > 0$ and $b_i > 0$. Then $P = Q = \left[\frac{\sqrt{b_i b_j}}{a_i + a_j}\right]$ and

$$\|G(s)\|_\infty = G(0) = \sum_{i=1}^{n} \frac{b_i}{a_i} = 2\text{trace}(P) = 2\sum_{i=1}^{n} \sigma_i$$

- bound can also be loose for systems with Hankel singular values close to each other. For example,

$$G(s) = \left[\begin{array}{cccc|c} -19.9579 & -5.4682 & 9.6954 & 0.9160 & -6.3180 \\ 5.4682 & 0 & 0 & 0.2378 & 0.0020 \\ -9.6954 & 0 & 0 & -4.0051 & -0.0067 \\ 0.9160 & -0.2378 & 4.0051 & -0.0420 & 0.2893 \\ \hline -6.3180 & -0.0020 & 0.0067 & 0.2893 & 0 \end{array}\right]$$

with Hankel singular values given by

$$\sigma_1 = 1, \quad \sigma_2 = 0.9977, \quad \sigma_3 = 0.9957, \sigma_4 = 0.9952.$$

r	0	1	2	3
$\|G - G_r\|_\infty$	2	1.996	1.991	1.9904
Bounds: $2\sum_{i=r+1}^{4} \sigma_i$	7.9772	5.9772	3.9818	1.9904
$2\sigma_{r+1}$	2	1.9954	1.9914	1.9904

Frequency-Weighted Balanced Model Reduction

$$\text{General Case:} \quad \inf_{\deg(G_r)\le r} \|W_o(G-G_r)W_i\|_\infty$$

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \quad W_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad W_o = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right]$$

$$W_oGW_i = \left[\begin{array}{ccc|c} A & 0 & BC_i & BD_i \\ B_oC & A_o & 0 & 0 \\ 0 & 0 & A_i & B_i \\ \hline D_oC & C_o & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right].$$

Let $\bar{P}$ and $\bar{Q}$ be the solutions to the following Lyapunov equations

$$\begin{aligned} \bar{A}\bar{P} + \bar{P}\bar{A}^* + \bar{B}\bar{B}^* &= 0 \\ \bar{Q}\bar{A} + \bar{A}^*\bar{Q} + \bar{C}^*\bar{C} &= 0. \end{aligned}$$

The input/output weighted Gramians P and Q are defined by

$$P := \begin{bmatrix} I_n & 0 \end{bmatrix} \bar{P} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad Q := \begin{bmatrix} I_n & 0 \end{bmatrix} \bar{Q} \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

P and Q satisfy the following lower order equations

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} BD_i \\ B_i \end{bmatrix} \begin{bmatrix} BD_i \\ B_i \end{bmatrix}^* = 0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix} + \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C^*D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C^*D_o^* \\ C_o^* \end{bmatrix}^* = 0.$$

$W_i = I \Longrightarrow P$ can be obtained from

$$PA^* + AP + BB^* = 0$$

$W_o = I \Longrightarrow Q$ can be obtained from

$$QA + A^*Q + C^*C = 0.$$

Now let T be a nonsingular matrix such that

$$TPT^* = (T^{-1})^* Q T^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

(i.e., balanced) and partition the system accordingly as

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then a reduced order model G_r is obtained as

$$G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right].$$

Works well but with guarantee.

Relative Reduction

$$G_r = G(I + \Delta_{\text{rel}}), \quad \Longrightarrow \quad \inf_{\deg(G_r)\le r} \left\| G^{-1}(G - G_r) \right\|_\infty$$

and a related problem is

$$G = G_r(I + \Delta_{mul})$$

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ be minimum phase and D be nonsingular.

Then $W_o = G^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$.

(a) Then the input/output weighted Gramians P and Q are given by

$$PA^* + AP + BB^* = 0$$

$$Q(A - BD^{-1}C) + (A - BD^{-1}C)^*Q + C^*(D^{-1})^*D^{-1}C = 0.$$

(b) Suppose P and Q are balanced:

$P = Q = \operatorname{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N}) = \operatorname{diag}(\Sigma_1, \Sigma_2)$

and let G be partitioned compatibly with Σ_1 and Σ_2 as

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is stable and minimum phase. Furthermore

$$\|\Delta_{rel}\|_\infty \le \prod_{i=r+1}^{N} \left(1 + 2\sigma_i(\sqrt{1+\sigma_i^2} + \sigma_i)\right) - 1$$

$$\|\Delta_{mul}\|_\infty \le \prod_{i=r+1}^{N} \left(1 + 2\sigma_i(\sqrt{1+\sigma_i^2} + \sigma_i)\right) - 1.$$