Lecture Notes: Week 6a&6b

Topic: SSV: mu and mu-synthesis

ECE/MAE 7360
Optimal and Robust Control
(Fall 2003 Offering)

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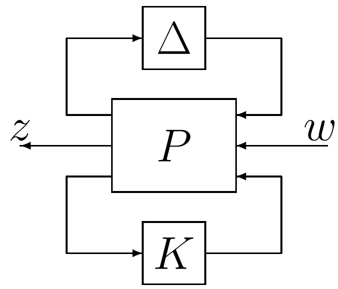
Chapter 10: μ and μ Synthesis

- general framework
- analysis and synthesis methods for unstructured uncertainty
- stability with structured uncertainties
- structured singular value
- structured robust stability
- robust performance
- extension to nonlinear time varying uncertainties
- skewed problem
- overview on μ synthesis

General Framework

General Framework:



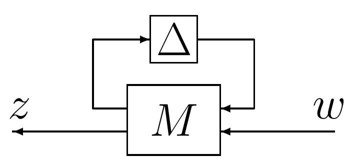
$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) & P_{13}(s) \\ P_{21}(s) & P_{22}(s) & P_{23}(s) \\ P_{31}(s) & P_{32}(s) & P_{33}(s) \end{bmatrix}$$

$$\begin{aligned} z &= \mathcal{F}_u\left(\mathcal{F}_\ell(P, K), \Delta\right) w \\ &= \mathcal{F}_\ell\left(\mathcal{F}_u(P, \Delta), K\right) w. \end{aligned}$$

Analysis Framework

$$M(s) = \mathcal{F}_\ell\left(P(s), K(s)\right) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix},$$

$$z = \mathcal{F}_u(M, \Delta) w = \left[M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12}\right] w.$$



Analysis and Synthesis Methods for Unstructured Uncertainty

<table>
<tr><th>Input Assumptions</th><th>Performance Specifications</th><th>Perturbation Assumptions</th><th>Analysis Tests</th><th>Synthesis Methods</th></tr>
<tr><td>$E(w(t)w(\tau)^*)$
$= \delta(t-\tau)I$</td><td>$E(z(t)^* z(t)) \leq 1$</td><td rowspan="2">$\Delta = 0$</td><td rowspan="2">$\|M_{22}\|_2 \leq 1$</td><td rowspan="2">LQG
Wiener-Hopf
$\mathcal{H}_2$</td></tr>
<tr><td>$w = U_0 \delta(t)$
$E(U_0 U_0^*) = I$</td><td>$E(\|z\|_2^2) \leq 1$</td></tr>
<tr><td>$\|w\|_2 \leq 1$</td><td>$\|z\|_2 \leq 1$</td><td>$\Delta = 0$</td><td>$\|M_{22}\|_\infty \leq 1$</td><td rowspan="2">Singular Value
Loop Shaping
$\mathcal{H}_\infty$</td></tr>
<tr><td>$\|w\|_2 \leq 1$</td><td>Internal Stability</td><td>$\|\Delta\|_\infty < 1$</td><td>$\|M_{11}\|_\infty \leq 1$</td></tr>
</table>

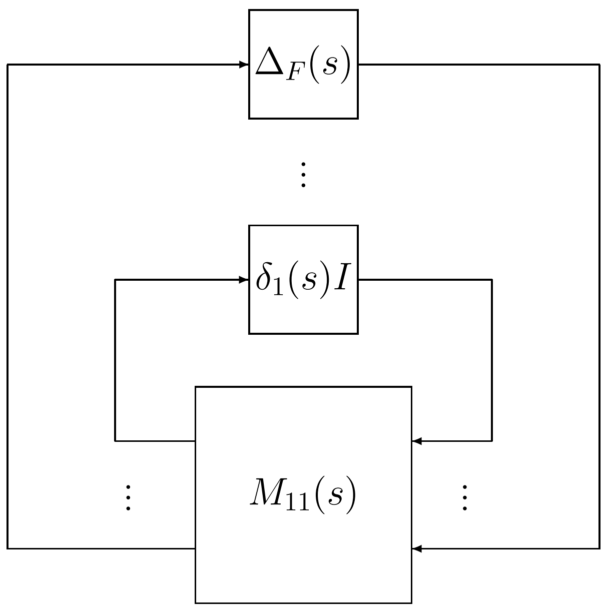
Stability with Structured Uncertainties

Assume

$$\Delta(s) = \text{diag}\ [\delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F] \in \mathcal{RH}_\infty$$

with $\|\delta_i\|_\infty < 1$ and $\|\Delta_j\|_\infty < 1$.

Robust Stability $\Longleftrightarrow$ The following interconnection is stable.



Stability Conditions:

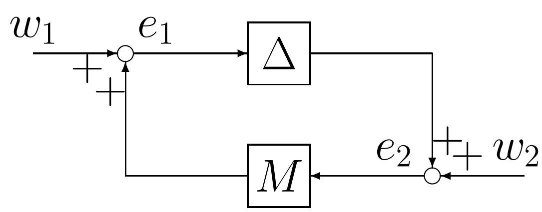
(1) (sufficient conditions) $\|M_{11}\|_\infty \leq 1$.

Conservative, ignoring structure of the uncertainties.

(2) (necessary conditions) Test for each δ_i (Δ_j) individually (assuming no uncertainty in other channels): $\|(M_{11})_{ii}\|_\infty \leq 1$.

Optimistic because it ignores interaction between the δ_i (Δ_j).

Structured Singular Value



Unstructured Δ

Problem: Given $M \in \mathbb{C}^{p\times q}$, find a smallest $\Delta \in \mathbb{C}^{q\times p}$ in the sense of $\overline{\sigma}(\Delta)$ such that

$$\det(I - M\Delta) = 0.$$

It is easy to see that

$$\alpha_{\min} := \inf\left\{\overline{\sigma}(\Delta) :\ \det(I - M\Delta) = 0,\ \Delta \in \mathbb{C}^{q\times p}\right\}$$
$$= \inf\left\{\alpha :\ \det(I - \alpha M\Delta) = 0,\ \overline{\sigma}(\Delta) \le 1,\ \Delta \in \mathbb{C}^{q\times p}\right\}$$

and

$$\max_{\overline{\sigma}(\Delta)\le 1} \rho(M\Delta) = \alpha_{\min}^{-1} = \overline{\sigma}(M)$$

with a smallest "destabilizing" Δ:

$$\Delta_{\text{des}} = \frac{1}{\overline{\sigma}(M)} v_1 u_1^*, \quad \det(I - M\Delta_{\text{des}}) = 0$$

where $M = \overline{\sigma}(M) u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots$

So $\overline{\sigma}(M)$ can be defined as

$$\overline{\sigma}(M) := \frac{1}{\inf\{\overline{\sigma}(\Delta) :\ \det(I - M\Delta) = 0,\ \Delta \in \mathbb{C}^{q\times p}\}}$$

Structured Δ

$$\mathbf{\Delta} = \left\{ \text{diag } \left[\delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F\right] : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \right\}.$$

$$\alpha_{\min} := \inf\left\{\overline{\sigma}(\Delta) : \det(I - M\Delta) = 0, \ \Delta \in \mathbf{\Delta}\right\}$$
$$= \inf\left\{\alpha : \det(I - \alpha M\Delta) = 0, \ \overline{\sigma}(\Delta) \le 1, \ \Delta \in \mathbf{\Delta}\right\}$$

and

$$\max_{\overline{\sigma}(\Delta) \le 1} \rho(M\Delta) = \alpha_{\min}^{-1} \le \overline{\sigma}(M)$$

Definition of SSV

For $M \in \mathbb{C}^{n \times n}$, $\mu_{\mathbf{\Delta}}(M)$ is defined as

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\left\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\right\}} \tag{0.6}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$.

- If $\mathbf{\Delta} = \{\delta I : \delta \in \mathbb{C}\}$ $(S = 1, F = 0, r_1 = n)$, then $\mu_{\mathbf{\Delta}}(M) = \rho(M)$, the spectral radius of M.
- If $\mathbf{\Delta} = \mathbb{C}^{n \times n}$ $(S = 0, F = 1, m_1 = n)$, then $\mu_{\mathbf{\Delta}}(M) = \overline{\sigma}(M)$.

$$\rho(M) \le \mu_{\mathbf{\Delta}}(M) \le \overline{\sigma}(M).$$

(1) $M = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ for any $\beta > 0$. Then $\rho(M) = 0$ and $\overline{\sigma}(M) = \beta$. But $\mu(M) = 0$ since $\det(I - M\Delta) = 1$ for all admissible Δ.

(2) $M = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$. Then $\rho(M) = 0$ and $\overline{\sigma}(M) = 1$. Since

$$\det(I - M\Delta) = 1 + \frac{\delta_1 - \delta_2}{2} = 0$$

if $\delta_1 = -\delta_2 = -1$. so $\mu(M) = 1$.

Thus neither ρ nor $\overline{\sigma}$ provide useful bounds even in these simple cases.

$$\mathcal{U} = \{U \in \mathbf{\Delta} : \; UU^* = I_n\}$$

$$\mathcal{D} = \left\{ \begin{array}{r} \mathrm{diag}\left[D_1, \ldots, D_S, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}}, I_{m_F}\right] : \\ D_i \in \mathbb{C}^{r_i \times r_i}, D_i = D_i^* > 0, d_j \in \mathbb{R}, d_j > 0 \end{array} \right\}.$$

Note that for any $\Delta \in \mathbf{\Delta}, U \in \mathcal{U}$, and $D \in \mathcal{D}$,

$$U^* \in \mathcal{U} \quad U\Delta \in \mathbf{\Delta} \quad \Delta U \in \mathbf{\Delta} \quad \overline{\sigma}(U\Delta) = \overline{\sigma}(\Delta U) = \overline{\sigma}(\Delta)$$

$$D\Delta = \Delta D.$$

For all $U \in \mathcal{U}$ and $D \in \mathcal{D}$

$$\mu_{\mathbf{\Delta}}(MU) = \mu_{\mathbf{\Delta}}(UM) = \mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1}).$$

$$\max_{U \in \mathcal{U}} \rho(UM) \le \max_{\Delta \in \mathbf{B\Delta}} \rho(\Delta M) = \mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right)$$

$$\max_{U \in \mathcal{U}} \rho(UM) \le \mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right).$$

[Doyle, 1982] $\max_{U \in \mathcal{U}} \rho(MU) = \mu_{\mathbf{\Delta}}(M)$. *Not Convex.*

$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1})$ *if* $2S + F \le 3$

F= / S=	0	1	2	3	4
0		yes	yes	yes	no
1	yes	yes	no	no	no
2	no	no	no	no	no

$\gg$ **[bounds,rowd] = mu(M,blk)**

$$\Delta = \begin{bmatrix} \delta_1 I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_5 I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_6 \end{bmatrix}$$

$$\delta_1, \delta_2, \delta_5, \in \mathbb{C},\ \ \Delta_3 \in \mathbb{C}^{2\times 3}, \Delta_4 \in \mathbb{C}^{3\times 3}, \Delta_6 \in \mathbb{C}^{2\times 1}$$

can be specified by

$$\mathbf{blk} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 2 & 3 \\ 3 & 3 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

$\gg [\mathbf{D}_\ell, \mathbf{D_r}] = \mathbf{unwrapd}(\mathbf{rowd}, \mathbf{blk})$

Structured Robust Stability

How large Δ (in the sense of $\|\Delta\|_\infty$) can be without destabilizing the feedback system?

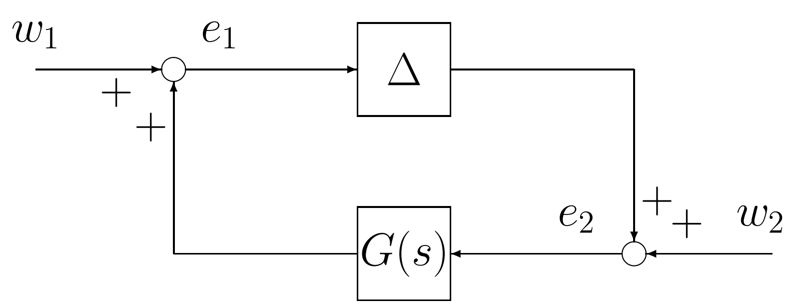
Since the closed-loop poles are given by $\det(I - M\Delta) = 0$, *the feedback system becomes unstable if* $\det(I - M(s)\Delta(s)) = 0$ *for some* $s \in \overline{\mathbb{C}}_+$. *Now let* $\alpha > 0$ *be a sufficiently small number such that the closed-loop system is stable for all stable* $\|\Delta\|_\infty < \alpha$. *Next increase* α *until* α_{max} *so that the closed-loop system becomes unstable. So* α_{max} *is the robust stability margin.*

Define

$$\mathbf{\Delta} := \{\Delta(\cdot) \in \mathcal{RH}_\infty : \Delta(s_o) \in \mathbf{\Delta} \text{ for all } s_o \in \overline{\mathbb{C}}_+\}$$

Let $\beta > 0$. *The system is well-posed and internally stable for all* $\Delta(\cdot) \in \mathbf{\Delta}$ *with* $\|\Delta\|_\infty < \frac{1}{\beta}$ *if and only if*

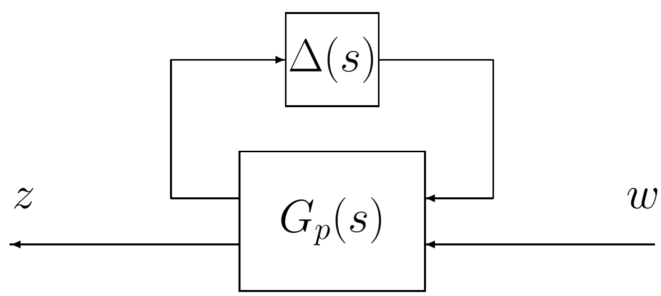
$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}}(G(j\omega)) \leq \beta$$



Robust Performance

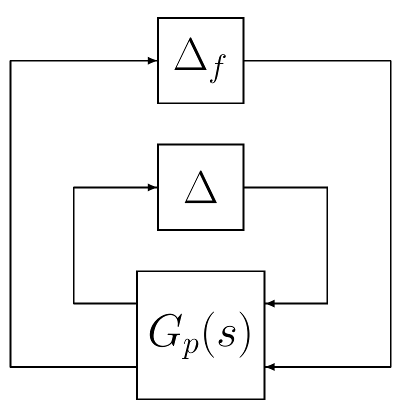
$$G_p(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$\mathbf{\Delta}_P := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_f \end{bmatrix} : \Delta \in \mathbf{\Delta}, \Delta_f \in \mathbb{C}^{q_2 \times p_2} \right\}.$$



Let $\beta > 0$. For all $\Delta(s) \in \mathbf{\Delta}$ with $\|\Delta\|_\infty < \frac{1}{\beta}$, the system is well-posed, internally stable, and $\|F_u(G_p, \Delta)\|_\infty \le \beta$ if and only if

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}_P}(G_p(j\omega)) \le \beta.$$

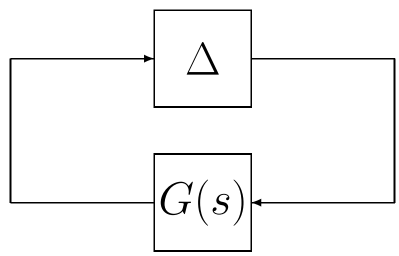


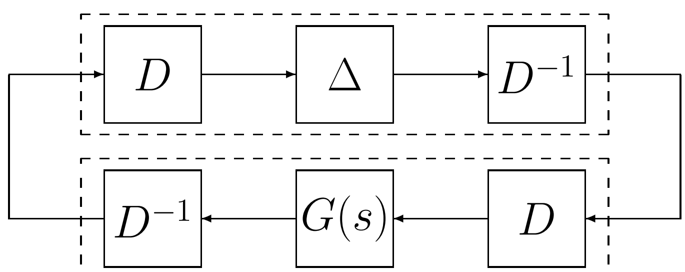
Extension to Nonlinear Time Varying Uncertainty

Suppose $\Delta \in \mathbf{\Delta}_N$ *is a structured Nonlinear (Time-varying) Uncertainty and suppose* D *is constant scaling matrix such that* $D\Delta D^{-1} \in \mathbf{\Delta}_N$.

Then a sufficient condition for stability is (by small gain theorem)

$$\left\| D^{-1}G(s)D \right\|_\infty \leq 1$$





HIMAT Example

$\gg [\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}] = \mathbf{linmod}('\mathbf{aircraft}')$

$\gg \hat{\mathbf{G}} = \mathbf{pck}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D});$

$\gg [\mathbf{K}, \mathbf{G_p}, \gamma] = \mathbf{hinfsyn}(\hat{\mathbf{G}}, \mathbf{2}, \mathbf{2}, \mathbf{0}, \mathbf{10}, \mathbf{0.001}, \mathbf{2});$

which gives $\gamma = 1.8612 = \|G_p\|_\infty$, *a stabilizing controller* K, *and a closed loop transfer matrix* G_p*:*

$$\begin{bmatrix} z_1 \\ z_2 \\ \hdashline e_1 \\ e_2 \end{bmatrix} = G_p(s) \begin{bmatrix} p_1 \\ p_2 \\ \hdashline d_1 \\ d_2 \\ n_1 \\ n_2 \end{bmatrix}, \quad G_p(s) = \begin{bmatrix} G_{p11} & G_{p12} \\ G_{p21} & G_{p22} \end{bmatrix}.$$

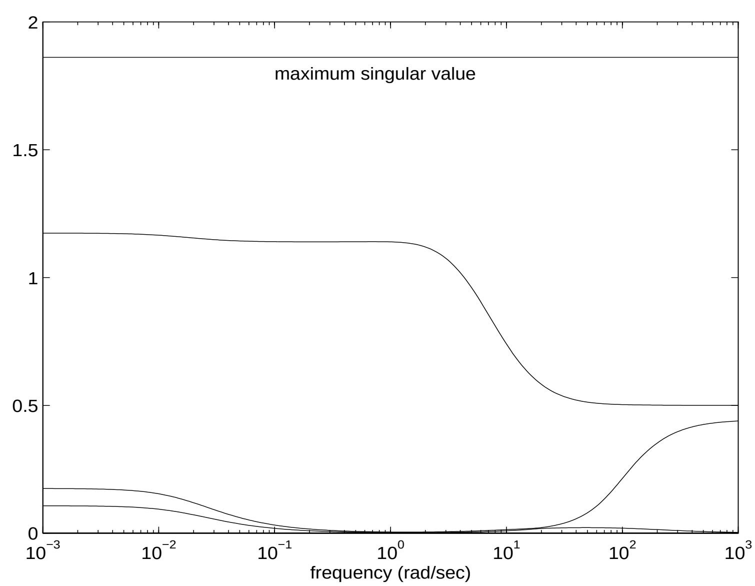


Figure 0.22: Singular values of $G_p(j\omega)$

Now generate the singular value frequency responses of G_p*:*

$\gg$ **w=logspace(-3,3,300);**

$\gg$ $\mathbf{Gpf} = \mathbf{frsp}(\mathbf{G_p}, \mathbf{w})$; % *Gpf is the frequency response of* G_p;

$\gg$ $[\mathbf{u}, \mathbf{s}, \mathbf{v}] = \mathbf{vsvd}(\mathbf{Gpf})$;

$\gg$ $\mathbf{vplot}('\mathbf{liv}, \mathbf{m}', \mathbf{s})$

The singular value frequency responses of G_p *are shown in Figure 0.22. To test the robust stability, we need to compute* $\|G_{p11}\|_\infty$*:*

$\gg$ $\mathbf{G_{p11}} = \mathbf{sel}(\mathbf{G_p}, \mathbf{1} : \mathbf{2}, \mathbf{1} : \mathbf{2})$;

$\gg$ $\mathbf{norm\_of\_G_{p11}} = \mathbf{hinfnorm}(\mathbf{G_{p11}}, \mathbf{0.001})$;

which gives $\|G_{P11}\|_\infty = 0.933 < 1$. *So the system is robustly stable. To check the robust performance, we shall compute the* $\mu_{\Delta_P}(G_p(j\omega))$ *for each frequency with*

$$\Delta_P = \begin{bmatrix} \Delta & \\ & \Delta_f \end{bmatrix}, \ \Delta \in \mathbb{C}^{2\times 2}, \quad \Delta_f \in \mathbb{C}^{4\times 2}.$$

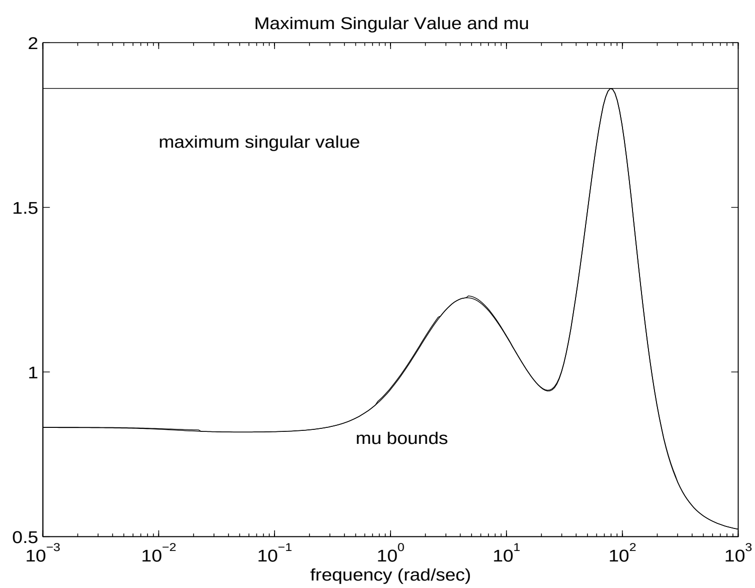


Figure 0.23: $\mu_{\Delta_P}(G_p(j\omega))$ and $\overline{\sigma}(G_p(j\omega))$

$\gg$ **blk=[2,2;4,2];**

```
≫ [bnds,dvec,sens,pvec]=mu(Gpf,blk);
≫ vplot('liv, m', vnorm(Gpf), bnds)
≫ title('Maximum Singular Value and mu')
≫ xlabel('frequency(rad/sec)')
≫ text(0.01, 1.7,' maximum singular value')
≫ text(0.5, 0.8,' mu bounds')
```

The structured singular value $\mu_{\Delta_P}(G_p(j\omega))$ *and* $\overline{\sigma}(G_p(j\omega))$ *are shown in Figure 0.23. It is clear that the robust performance is not satisfied. Note that*

$$\max_{\|\Delta\|_\infty \le 1} \|\mathcal{F}_u(G_p, \Delta)\|_\infty \le \gamma \Longleftrightarrow \sup_\omega \mu_{\Delta_P}\left(\begin{bmatrix} G_{p11} & G_{p12} \\ G_{p21}/\gamma & G_{p22}/\gamma \end{bmatrix}\right) \le 1.$$

Using a bisection algorithm, we can also find the worst performance:

$$\max_{\|\Delta\|_\infty \le 1} \|\mathcal{F}_u(G_p, \Delta)\|_\infty = 12.7824.$$

Skewed Problem

$$G = \begin{bmatrix} -W_2 T_i W_1 & -W_2 K S_o W_d \\ W_e S_o P W_1 & W_e S_o W_d \end{bmatrix}.$$

robust performance condition:

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \inf_{d_\omega \in \mathbb{R}_+} \overline{\sigma}\left(\begin{bmatrix} -W_2 T_i W_1 & -d_\omega W_2 K S_o W_d \\ \frac{1}{d_\omega} W_e S_o P W_1 & W_e S_o W_d \end{bmatrix} \right) \leq 1$$

for all $\omega \geq 0$. An upper bound:

$$\mu_{\mathbf{\Delta}}(G(j\omega)) \leq \sqrt{\kappa(W_d^{-1} P W_1)}(\|W_2 T_i W_1\| + \|W_e S_o W_d\|).$$

μ is proportional to the square root of the plant condition number.

Assumptions:

$$W_e = w_s I,\ W_d = I,\ W_1 = I,\ W_2 = w_t I,$$

and P is stable and has a stable inverse (i.e., minimum phase) and

$$K(s) = P^{-1}(s) l(s)$$

such that $K(s)$ is proper and the closed-loop is stable. Then

$$S_o = S_i = \frac{1}{1 + l(s)} I = \varepsilon(s) I, \quad T_o = T_i = \frac{l(s)}{1 + l(s)} I = \tau(s) I$$

$$G = \begin{bmatrix} -w_t \tau I & -w_t \tau P^{-1} \\ w_s \varepsilon P & w_s \varepsilon I \end{bmatrix}.$$

Then

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \inf_{d \in \mathbb{R}_+} \overline{\sigma}\left(\begin{bmatrix} -w_t \tau I & -w_t \tau (dP)^{-1} \\ w_s \varepsilon d P & w_s \varepsilon I \end{bmatrix} \right).$$

Let the SVD of $P(j\omega)$ be

$$P(j\omega) = U\Sigma V^*, \quad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$$

with $\sigma_1 = \overline{\sigma}$ and $\sigma_m = \underline{\sigma}$ where m is the dimension of P. Then

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \inf_{d\in\mathbb{R}_+} \overline{\sigma}\left(\begin{bmatrix} -w_t\tau I & -w_t\tau(d\Sigma)^{-1} \\ w_s\varepsilon d\Sigma & w_s\varepsilon I \end{bmatrix}\right)$$

$$\begin{bmatrix} -w_t\tau I & -w_t\tau(d\Sigma)^{-1} \\ w_s\varepsilon d\Sigma & w_s\varepsilon I \end{bmatrix} = P_1 \mathrm{diag}(M_1, M_2, \ldots, M_m) P_2$$

where P_1 and P_2 are permutation matrices and where

$$M_i = \begin{bmatrix} -w_t\tau & -w_t\tau(d\sigma_i)^{-1} \\ w_s\varepsilon d\sigma_i & w_s\varepsilon \end{bmatrix}.$$

Hence

$$\begin{aligned}
\mu_{\mathbf{\Delta}}(G(j\omega)) &= \inf_{d\in\mathbb{R}_+} \max_i \overline{\sigma}\left(\begin{bmatrix} -w_t\tau & -w_t\tau(d\sigma_i)^{-1} \\ w_s\varepsilon d\sigma_i & w_s\varepsilon \end{bmatrix}\right) \\
&= \inf_{d\in\mathbb{R}_+} \max_i \overline{\sigma}\left(\begin{bmatrix} -w_t\tau \\ w_s\varepsilon d\sigma_i \end{bmatrix}\begin{bmatrix} 1 & (d\sigma_i)^{-1} \end{bmatrix}\right) \\
&= \inf_{d\in\mathbb{R}_+} \max_i \sqrt{(1+|d\sigma_i|^{-2})(|w_s\varepsilon d\sigma_i|^2 + |w_t\tau|^2)} \\
&= \inf_{d\in\mathbb{R}_+} \max_i \sqrt{|w_s\varepsilon|^2 + |w_t\tau|^2 + |w_s\varepsilon d\sigma_i|^2 + \left|\frac{w_t\tau}{d\sigma_i}\right|^2}.
\end{aligned}$$

The maximum is achieved at

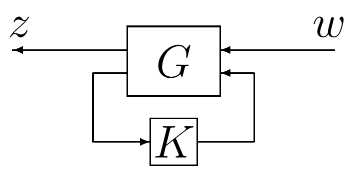
$$d^2 = \frac{|w_t\tau|}{|w_s\varepsilon|\underline{\sigma}\overline{\sigma}},$$

and

$$\mu_{\mathbf{\Delta}}(G(j\omega)) = \sqrt{|w_s\varepsilon|^2 + |w_t\tau|^2 + |w_s\varepsilon||w_t\tau|[\kappa(P) + \frac{1}{\kappa(P)}]}.$$

$$\mu_{\mathbf{\Delta}}(G(j\omega)) \approx \sqrt{|w_s\varepsilon||w_t\tau|\kappa(P)}\ .$$

Overview on μ Synthesis



$$\mathcal{F}_\ell(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

$$\min_K \|\mathcal{F}_\ell(G,K)\|_\mu$$

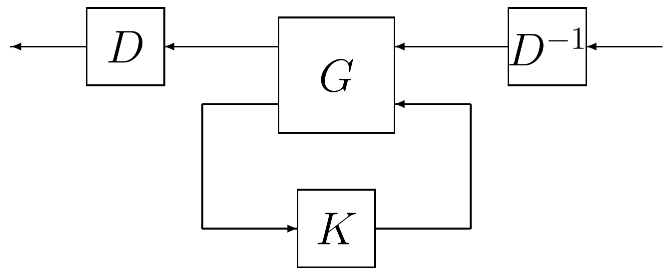
The μ-synthesis is not yet fully solved. But a reasonable approach is to "solve"

$$\min_K \inf_{D,D^{-1}\in\mathcal{H}_\infty} \left\|D\mathcal{F}_\ell(G,K)D^{-1}\right\|_\infty$$

by iteratively solving for K and D, i.e., first minimizing over K with D fixed, then minimizing pointwise over D with K fixed, then again over K, and again over D, etc. This is the so-called *D-K Iteration*.

- Fix D
$$\min_K \left\|D\mathcal{F}_\ell(G,K)D^{-1}\right\|_\infty$$
is a standard $\mathcal{H}_\infty$ optimization problem.

- Fix K
$$\inf_{D,D^{-1}\in\mathcal{H}_\infty} \left\|D\mathcal{F}_\ell(G,K)D^{-1}\right\|_\infty$$
is a standard convex optimization problem and it can be solved pointwise in the frequency domain:
$$\sup_\omega \inf_{D_\omega\in\mathcal{D}} \overline{\sigma}\left[D_\omega\mathcal{F}_\ell(G,K)(j\omega)D_\omega^{-1}\right].$$
Note that when $S=0$, (no scalar blocks)
$$D_\omega = \mathrm{diag}(d_1^\omega I, \ldots, d_{F-1}^\omega I, I) \in \mathcal{D},$$



D-K Iterations:

(i) Fix an initial estimate of the scaling matrix $D_\omega \in \mathcal{D}$ pointwise across frequency.

(ii) Find scalar transfer functions $d_i(s), d_i^{-1}(s) \in \mathcal{RH}_\infty$ for $i = 1, \ldots, (F-1)$ such that $|d_i(j\omega)| \approx d_i^\omega$.

(iii) Let

$$D(s) = \mathrm{diag}(d_1(s)I, \ldots, d_{F-1}(s)I, I).$$

Construct a state space model for system

$$\hat{G}(s) = \begin{bmatrix} D(s) & \\ & I \end{bmatrix} G(s) \begin{bmatrix} D^{-1}(s) & \\ & I \end{bmatrix}.$$

(iv) Solve an $\mathcal{H}_\infty$-optimization problem to minimize

$$\left\| \mathcal{F}_\ell(\hat{G}, K) \right\|_\infty$$

over all stabilizing K's. Denote the minimizing controller by $\hat{K}$.

(v) Minimize $\overline{\sigma}[D_\omega \mathcal{F}_\ell(G, \hat{K}) D_\omega^{-1}]$ over D_ω, pointwise across frequency. The minimization itself produces a new scaling function.

(vi) Compare $\hat{D}_\omega$ with the previous estimate D_ω. Stop if they are close, otherwise, replace D_ω with $\hat{D}_\omega$ and return to step (ii).

The joint optimization of D and K is not convex and the global convergence is not guaranteed, many designs have shown that this approach works very well.