

# **Lecture Notes: Week 7b**

**Topic: LQR/Kalman Filter/LQG**

## **ECE/MAE 7360**

### **Optimal and Robust Control**

(Fall 2003 Offering)

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# ***Properties of LQR Control***

## *Linear Quadratic Regulator (LQR)*

Consider a linear system characterized by

$$\dot{x} = A x + B u$$

where  $(A, B)$  is stabilizable. We define the cost index

$$J(x, u, Q, R) = \int_0^{\infty} (x^T Q x + u^T R u) dt, \quad Q \geq 0, R > 0$$

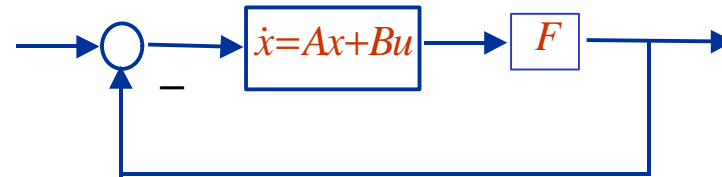
and  $(A, Q^{1/2})$  is detectable. The linear quadratic regulation problem is to find a control law  $u = -F x$  such that  $(A - B F)$  is stable and  $J$  is minimized. It was shown in the first Part of this course that the solution is given by

$$F = R^{-1} B^T P$$

with

$$P A + A^T P - P B R^{-1} B^T P + Q = 0$$

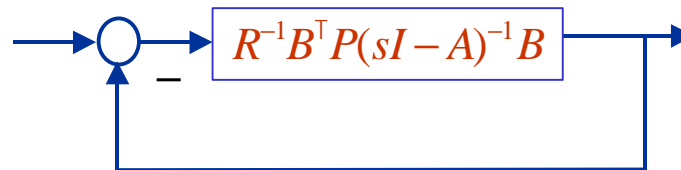
If we arrange the LQR control in the following block diagram,



we can find its gain margin and phase margin as we have done in classical control. It is clear that the open-loop transfer function,

$$\text{Open loop transfer function} = F(sI - A)^{-1}B = R^{-1}B^T P(sI - A)^{-1}B$$

The block diagram can be re-drawn as follows,



## *Return Difference Equality and Inequality*

Consider the LQR control law. The following so-called return difference equality hold:

$$R + B^T(-j\omega I - A^T)^{-1}Q(j\omega I - A)B = [I + B^T(-j\omega I - A^T)^{-1}F^T]R[I + F(j\omega I - A)^{-1}B]$$

The following is called the return difference inequality:

$$[I + B^T(-j\omega I - A^T)^{-1}F^T]R[I + F(j\omega I - A)^{-1}B] \geq R$$

**Proof.** Recall that

$$F = R^{-1}B^T P$$

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

Then we have

$$-Pj\omega I + PA + Pj\omega I + A^T P - (PBR^{-1})R(R^{-1}B^T P) + Q = 0$$



$$P(j\omega I - A) + (-j\omega I - A^T)P + F^T R F = Q$$

Multiplying it on the left by  $B^\top(-j\omega I - A^\top)^{-1}$  and on the right by  $(j\omega I - A)^{-1}B$ , we obtain,

$$B^\top(-j\omega I - A^\top)^{-1}P(j\omega I - A)(j\omega I - A)^{-1}B + B^\top(-j\omega I - A^\top)^{-1}(-j\omega I - A^\top)P(j\omega I - A)^{-1}B \\ + B^\top(-j\omega I - A^\top)^{-1}F^\top RF(j\omega I - A)^{-1}B = B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)^{-1}B$$



$$B^\top(-j\omega I - A^\top)^{-1}PB + B^\top P(j\omega I - A)^{-1}B + B^\top(-j\omega I - A^\top)^{-1}F^\top RF(j\omega I - A)^{-1}B \\ = B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)^{-1}B$$

Noting the fact that

$$F = R^{-1}B^\top P \quad \Rightarrow \quad B^\top P = RF \quad \& \quad PB = F^\top R$$

we have

$$B^\top(-j\omega I - A^\top)^{-1}F^\top R + RF(j\omega I - A)^{-1}B + B^\top(-j\omega I - A^\top)^{-1}F^\top RF(j\omega I - A)^{-1}B \\ = B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)^{-1}B$$



$$R + B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)B = [I + B^\top(-j\omega I - A^\top)^{-1}F^\top]R[I + F(j\omega I - A)^{-1}B]$$

## Single Input Case

In the single input case, the transfer function

$$\text{Open loop transfer function} = f(sI - A)^{-1}b$$

is a scalar function. Let  $Q = h h^T$ . Then, the return difference equation is reduced to

$$r + b^T (-j\omega I - A^T)^{-1} h h^T (j\omega I - A)^{-1} b = r [1 + b^T (-j\omega I - A^T)^{-1} f^T] [1 + f (j\omega I - A)^{-1} b]$$



$$r + \left| h^T (j\omega I - A)^{-1} b \right|^2 = r \left| 1 + f (j\omega I - A)^{-1} b \right|^2$$

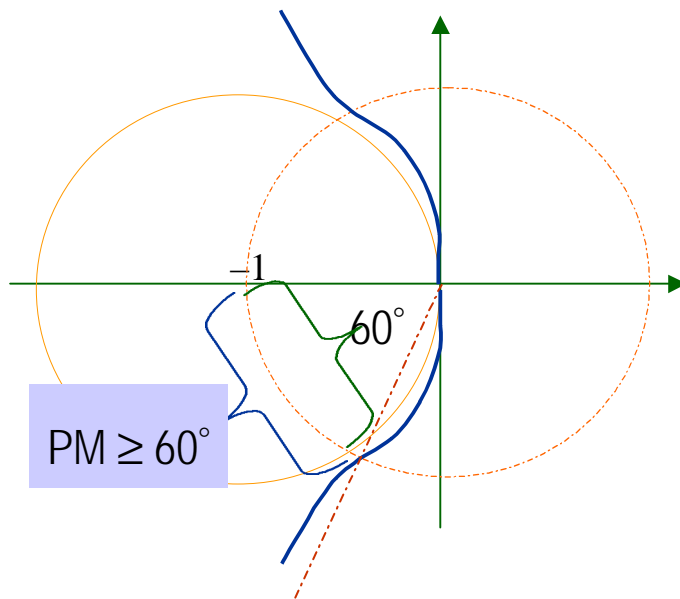


$$r \left| 1 + f (j\omega I - A)^{-1} b \right|^2 \geq r$$

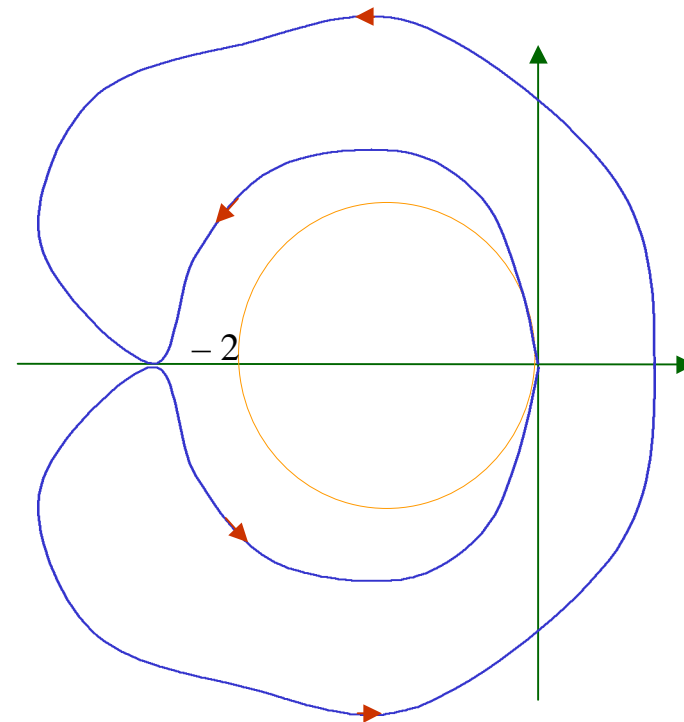


$$\left| 1 + f (j\omega I - A)^{-1} b \right|^2 \geq 1 \quad \text{Return Difference Inequality.}$$

Graphically,  $|1 + f(j\omega I - A)^{-1}b|^2 \geq 1 \Rightarrow |f(j\omega I - A)^{-1}b - (-1 + j0)| \geq 1$  implies that



Clearly, the phase margin resulting from the LQR design is at least 60 degrees.



The gain margin is from [ 0.5, ∞ ).

**Example:** Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solving the LQR problem which minimizes the following cost function

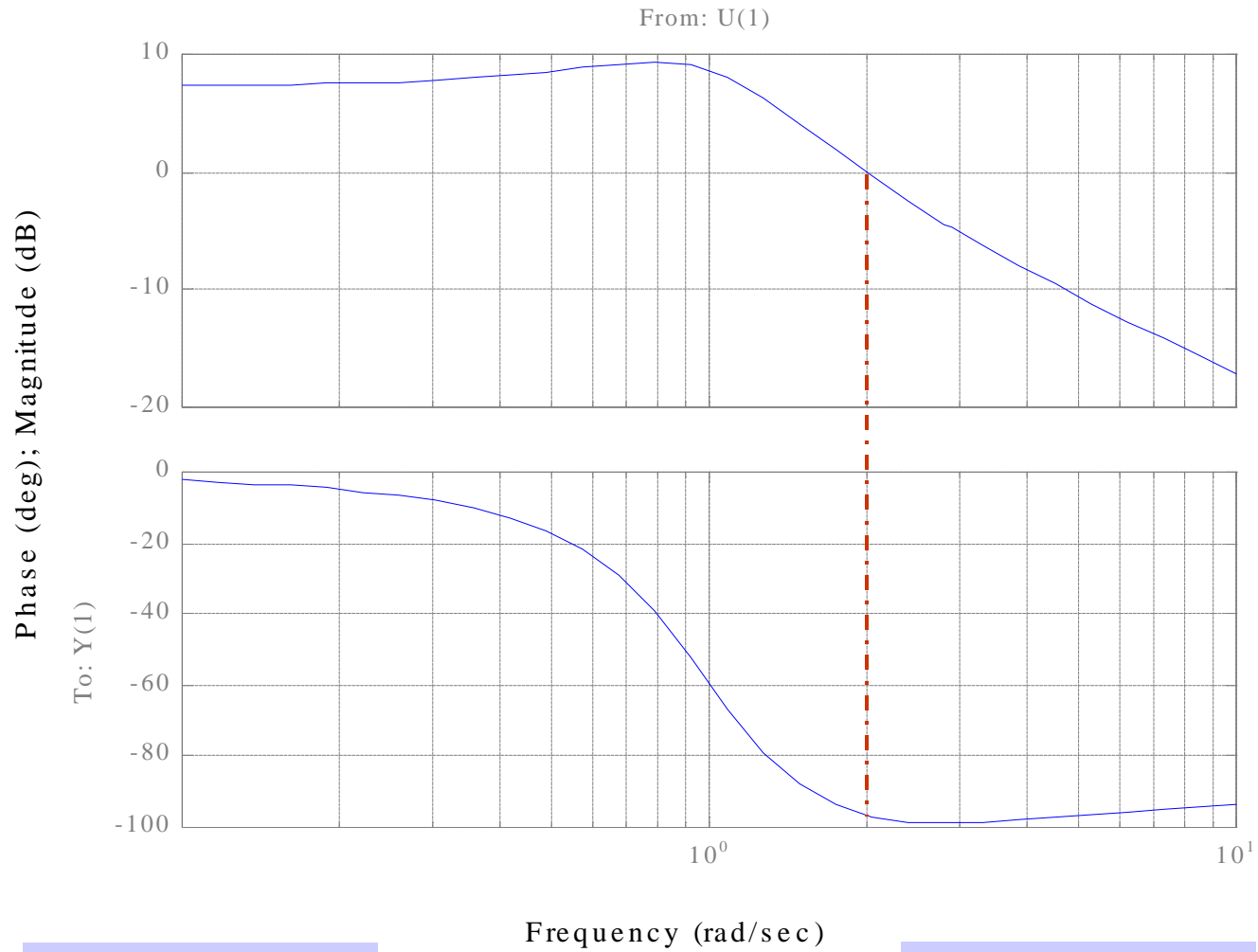
$$J(x, u, Q, R) = \int_0^{\infty} (x^T Q x + u^T R u) dt, \quad \text{with } Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 0.1$$

we obtain

$$P = \begin{bmatrix} 0.6872 & 0.2317 \\ 0.2317 & 0.1373 \end{bmatrix} \quad \text{and} \quad F = [2.3166 \quad 1.3734]$$

which results the closed-loop eigenvalues at  $-1.1867 \pm j1.3814$ . Clearly, the closed-loop system is asymptotically stable.

# Bode Diagrams



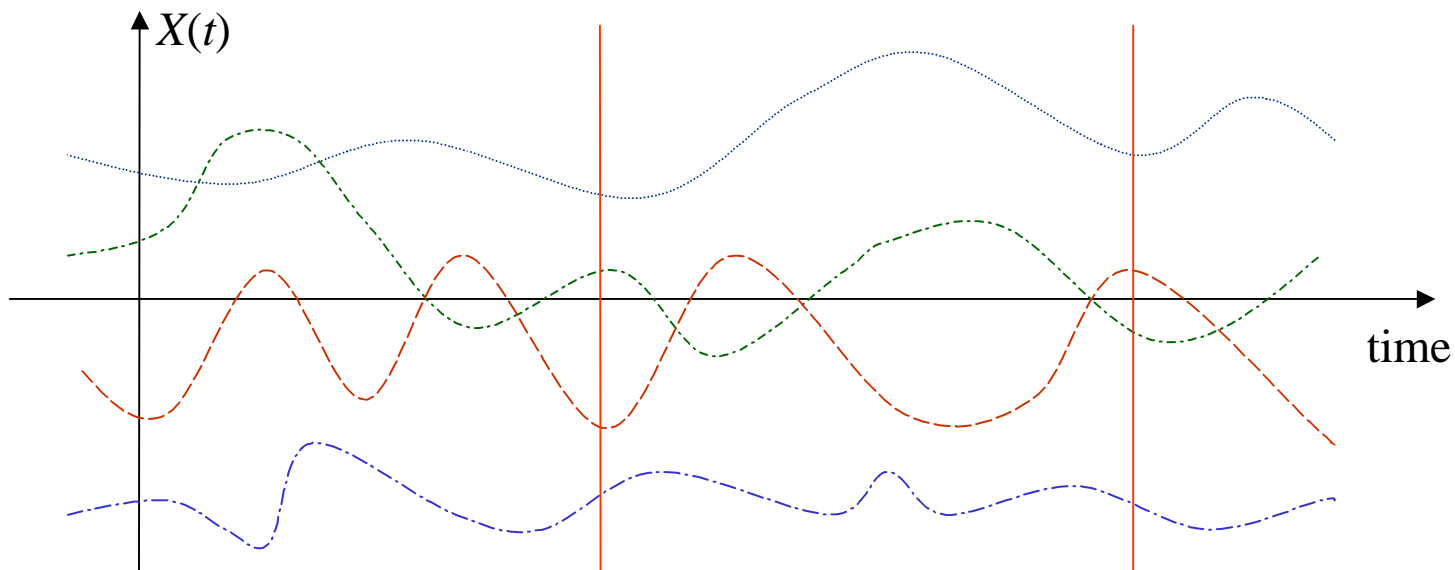
$$PM = 84^\circ$$

$$GM = \infty$$

# ***Kalman Filter***

## *Review: Random Process*

A **random variable**  $X$  is a mapping between the sample space and the real numbers. A **random process** (a.k.a **stochastic process**) is a mapping from the sample space into an ensemble of time functions (known as sample functions). To every member in the sample space, there corresponds a function of time (a sample function)  $X(t)$ .



## Mean, Moment, Variance, Covariance of Stationary Random Process

Let  $f(x, t)$  be the **probability density function** (p.d.f.) associated with a random process  $X(t)$ .

If the p.d.f. is independent of time  $t$ , i.e.,  $f(x, t) = f(x)$ , then the corresponding random process is said to be **stationary**. We will focus our attention only on this class of random processes in this course. For this type of random processes, we define:

1) **mean** (or **expectation**):

$$m = E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

2) **moment** ( $j$ -th order moment)

$$E[X^j] = \int_{-\infty}^{\infty} x^j \cdot f(x) dx$$

3) **variance**

$$s^2 = E[(x - m)^2] = \int_{-\infty}^{\infty} (x - m)^2 f(x) dx$$

4) **covariance of two random processes**

$$\text{con}(v, w) = E[(v - E[v])(w - E[w])]$$

Two random processes  $v$  and  $w$  are said to be **independent** if

$$E[vw] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} vw f(v, w) dv dw = 0, \quad f(v, w) \text{ is the joint p.d.f. of } v \text{ and } w.$$

## ***Autocorrelation Function and Power Spectrum***

Autocorrelation function is used to describe the time domain property of a random process.

Given a random process  $v$ , its **autocorrelation function** is defined as follows:

$$R_x(t_1, t_2) = E[v(t_1)v(t_2)]$$

If  $v$  is a stationary process,

$$R_x(t_1, t_2) = R_x(t_2 - t_1) = R_x(\mathbf{t}) = R_x(t, t + \mathbf{t}) = E[v(t)v(t + \mathbf{t})]$$

Note that  $R_x(0)$  is the time average of the power or energy of the random process.

**Power spectrum** of a random process is the Fourier transform of its autocorrelation function.

It is a frequency domain property of the random process. To be more specific, it is defined as

$$S_x(\mathbf{w}) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} R_x(\mathbf{t}) e^{j\mathbf{w}\mathbf{t}} d\mathbf{t}$$

## *White Noise, Color Noise and Gaussian Random Process*

**White Noise** is a random process with a constant power spectrum, and an autocorrelation function

$$R_x(\mathbf{t}) = q \cdot \mathbf{d}(\mathbf{t})$$

which implies that a white noise has an infinite power and thus it is non-existent in our real life.

However, many noises (or the so-called color noises, or noises with finite energy and finite frequency components) can be modeled as the outputs of linear systems with an injection of white noise into their inputs, i.e., any **color noise** can be generated by a white noise



**Gaussian Process**  $v$  is also known as normal process has a p.d.f.

$$f(v) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(v-m)^2}{2s^2}}, \quad \mathbf{m} = \text{mean}, \quad \mathbf{s}^2 = \text{variance}$$

## *Kalman Filter for a Linear Time Invariant (LTI) System*

Consider a LTI system characterized by

$$\begin{cases} \dot{x} = Ax + Bu + v(t) & v \text{ is the input noise} \\ y = Cx + w(t) & w \text{ is the measurement noise} \end{cases}$$

Assume: 1)  $(A, C)$  is observable

2)  $v(t)$  and  $w(t)$  are independent white noises with the following properties

$$E[v(t)] = 0, \quad E[w(t)] = 0$$

$$E[v(t)v^T(\mathbf{t})] = Q\mathbf{d}(t - \mathbf{t}), \quad Q = Q^T \geq 0, \quad E[w(t)w^T(\mathbf{t})] = R\mathbf{d}(t - \mathbf{t}), \quad R = R^T > 0$$

3)  $\left(A, Q^{1/2}\right)$  is stabilizable (to guarantee closed-loop stability).

The problem of **Kalman Filter** is to design a state estimator to estimate the state  $x(t)$  by  $\hat{x}(t)$

such that the estimation error covariance is minimized, i.e., the following index is minimized:

$$J_e = E[\{x(t) - \hat{x}(t)\}^T \{x(t) - \hat{x}(t)\}]$$

## Construction of Steady State Kalman Filter

Kalman filter is a state observer with a specially selected observer gain (or Kalman filter gain).

It has the dynamic equation:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + K_e(y - \hat{y}), & \hat{x}(0) \text{ is given} \\ \hat{y} &= C\hat{x}\end{aligned}$$

with the Kalman filter gain  $K_e$  being given as

$$K_e = P_e C^T R^{-1}$$

where  $P_e$  is the positive definite solution of the following Riccati equation,

$$P_e A^T + A P_e - P_e C^T R^{-1} C P_e + Q = 0$$

Let  $e = x - \hat{x}$ . We can show (see next) that such a Kalman filter has the following properties:

$$\lim_{t \rightarrow \infty} E[e(t)] = \lim_{t \rightarrow \infty} E[x(t) - \hat{x}(t)] = 0, \quad \lim_{t \rightarrow \infty} J_e = \lim_{t \rightarrow \infty} E[e^T(t) e(t)] = \text{trace } P_e$$

## Kalman Filter and Linear Quadratic Regulator – They Are Dual

Recall the optimal regulator problem,

$$\dot{x} = Ax + Bu \quad x(0) = x_0 \text{ given}$$

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt, \quad Q = Q^T \geq 0 \text{ and } R = R^T > 0$$

The LQR problem is to find a state feedback law  $u = -F x$  such that  $J$  is minimized. It was shown that the solution to the above problem is given by

$$F = R^{-1} B^T P \quad PA + A^T P - PBR^{-1} B^T P + Q = 0, \quad P = P^T > 0$$

and the optimal value of  $J$  is given by  $J = x_0^T P x_0$ . Note that  $x_0$  is arbitrary. Let us consider a special case when  $x_0$  is a random vector with

$$E[x_0] = 0, \quad E[x_0 x_0^T] = I$$

Then, we have

$$E[J] = E[x_0^T P x_0] = E \left[ \sum_{i=1}^n \sum_{j=1}^n p_{ij} x_{0i} x_{0j} \right] = \sum_{i=1}^n \sum_{j=1}^n p_{ij} E[x_{0i} x_{0j}] = \sum_{i=1}^n p_{ii} = \text{trace } P$$

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## The Duality

### § Linear Quadratic Regulator

$$F = R^{-1}B^T P$$

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

$$J_{\text{optimal}} = \text{trace } P$$

### a Kalman Filter

$$K_e = P_e C^T R^{-1}$$

$$P_e A^T + AP_e - P_e C^T R^{-1} C P_e + Q = 0$$

$$J_{\text{optimal}} = \text{trace } P_e$$

These two problems are equivalent (or dual) if we let

$$A^T \longleftrightarrow A$$

$$B^T \longleftrightarrow C$$

$$F^T \longleftrightarrow K_e$$

$$P \longleftrightarrow P_e$$

## *Proof of the Properties of Kalman Filter*

Recall that the dynamics of the given plant and Kalman filter, i.e.,

$$\begin{aligned} \dot{x} &= Ax + Bu + v(t) & \& & \dot{\hat{x}} &= A\hat{x} + Bu + K_e(y - \hat{y}) \\ y &= Cx + w(t) & & & \hat{y} &= C\hat{x} \end{aligned}$$

We have

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = Ax + Bu + v(t) - A\hat{x} - Bu - K_e[Cx + w(t) - C\hat{x}] \\ &= (A - K_e C)(x - \hat{x}) + v(t) - K_e w(t) \\ &= (A - K_e C)e + [I \quad -K_e] \begin{pmatrix} v \\ w \end{pmatrix} = \bar{A}e + d(t) \end{aligned}$$

with

$$E[d(t)] = E \left[ [I \quad -K_e] \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right] = [I \quad -K_e] \begin{pmatrix} E[v(t)] \\ E[w(t)] \end{pmatrix} = [I \quad -K_e] \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

Next, it is reasonable to assume that initial error  $e(0)$  and  $d(t)$  are independent, i.e.,

$$E[e(0) d^T(t)] = 0$$

Furthermore,

$$\begin{aligned}
 E[d(t)d^T(\mathbf{t})] &= [I \quad -K_e] \begin{pmatrix} E[v(t)v^T(\mathbf{t})] & E[v(t)w^T(\mathbf{t})] \\ E[w(t)v^T(\mathbf{t})] & E[w(t)w^T(\mathbf{t})] \end{pmatrix} \begin{bmatrix} I \\ -K_e^T \end{bmatrix} \\
 &= [I \quad -K_e] \begin{pmatrix} Q\mathbf{d}(t-\mathbf{t}) & 0 \\ 0 & R\mathbf{d}(t-\mathbf{t}) \end{pmatrix} \begin{bmatrix} I \\ -K_e^T \end{bmatrix} \\
 &= (Q + K_e R K_e^T) \mathbf{d}(t-\mathbf{t}) = \nabla \mathbf{d}(t-\mathbf{t})
 \end{aligned}$$

We will next show  $\bar{A}$  is asymptotically stable and

$$\lim_{t \rightarrow \infty} E[e(t) e^T(t)] = P_e$$

and  $P_e \bar{A}^T + \bar{A} P_e = -\nabla$ . Recall that the solution to the following state equation from your linear systems course notes:

$$\dot{e} = \bar{A}e + d(t)$$

i.e.,

$$e(t) = e^{\bar{A}t} \cdot e(0) + \int_0^t e^{\bar{A}(t-\tau)} d(\tau) \cdot d\tau$$

Also, recall that  $K_e = P_e C^T R^{-1}$  and

$$P_e A^T + A P_e - P_e C^T R^{-1} C P_e + Q = 0$$

We have

$$P_e A^T - P_e C^T R^{-1} C P_e + A P_e - P_e C^T R^{-1} C P_e + P_e C^T R^{-1} C P_e + Q = 0$$

$$\Rightarrow P_e (A^T - C^T R^{-1} C P_e) + (A - P_e C^T R^{-1} C) P_e + P_e C^T R^{-1} C P_e + Q = 0$$

$$\Rightarrow P_e (A^T - C^T K_e^T) + (A - K_e C) P_e + P_e C^T R^{-1} C P_e + Q = 0$$

$$\Rightarrow P_e \bar{A}^T + \bar{A} P_e = -P_e C^T R^{-1} C P_e - Q = -\nabla \leq 0$$

Since  $Q = Q^T \geq 0$  and  $\left( A, Q^{1/2} \right)$  is assumed to be stabilizable, it follows from Lyapunov stability theory that matrix  $\bar{A} = (A - K_e C)$  is asymptotically stable.

Noting that  $e^{\bar{A}t}$  is deterministic, we have

$$\begin{aligned}
P(t) &= E[e(t)e^T(t)] = E\left[\left(e^{\bar{A}t} \cdot e(0) + \int_0^t e^{\bar{A}(t-t)} d(\mathbf{t}) \cdot dt\right) \cdot \left(e^{\bar{A}t} \cdot e(0) + \int_0^t e^{\bar{A}(t-t)} d(\mathbf{t}) \cdot dt\right)^T\right] \\
&= e^{\bar{A}t} E[e(0)e^T(0)]e^{\bar{A}^T t} + \int_0^t e^{\bar{A}(t-t)} E[d(\mathbf{t})e^T(0)]e^{\bar{A}^T t} \cdot dt \\
&\quad + \int_0^t e^{\bar{A}t} E[e(0)d^T(\mathbf{t})]e^{\bar{A}^T(t-t)} \cdot dt + \int_0^t e^{\bar{A}(t-t)} dt \int_0^t E[d(\mathbf{t})d^T(\mathbf{s})]e^{\bar{A}^T(t-s)} \cdot ds \\
&= e^{\bar{A}t} E[e(0)e^T(0)]e^{\bar{A}^T t} + \int_0^t e^{\bar{A}(t-t)} dt \int_0^t \nabla d(\mathbf{t}-\mathbf{s})e^{\bar{A}^T(t-s)} \cdot ds \\
&= e^{\bar{A}t} E[e(0)e^T(0)]e^{\bar{A}^T t} + \int_0^t e^{\bar{A}(t-t)} \nabla e^{\bar{A}^T(t-t)} \cdot dt = e^{\bar{A}t} E[e(0)e^T(0)]e^{\bar{A}^T t} + \int_0^t e^{\bar{A}h} \nabla e^{\bar{A}^T h} \cdot dh
\end{aligned}$$

Since  $\bar{A}$  is stable, we have  $e^{\bar{A}t} \rightarrow 0$ , as  $t \rightarrow \infty$ . Thus,

$$P(\infty) = \int_0^{\infty} e^{\bar{A}h} \nabla e^{\bar{A}^T h} \cdot dh$$

We next show that  $P(\infty) = P_e$ , i.e., the solution associated with the Kalman filter ARE. Let

$$\dot{z} = \bar{A}^T z, \quad z(0) \text{ given} \Rightarrow z(t) = e^{\bar{A}t} z(0), \quad z(\infty) = 0$$

In view of  $P_e \bar{A}^T + \bar{A} P_e = -\nabla$ , we have

$$\begin{aligned} z^T [P_e \bar{A}^T + \bar{A} P_e] z &= -z^T \nabla z \Rightarrow z^T P_e \bar{A}^T z + z^T \bar{A} P_e z = -z^T \nabla z \\ \Rightarrow z^T P_e \dot{z} + \dot{z}^T P_e z &= -z^T \nabla z \Rightarrow \frac{d}{dt} (z^T P_e z) = -z^T \nabla z \end{aligned}$$

Next, we have

$$\begin{aligned} \int_0^{\infty} z^T \nabla z dt &= \int_0^{\infty} z^T(0) e^{\bar{A}^T t} \nabla e^{\bar{A}t} z(0) dt = z^T(0) \left[ \int_0^{\infty} e^{\bar{A}^T t} \nabla e^{\bar{A}t} dt \right] z(0) = z^T(0) P(\infty) z(0) \\ \int_0^{\infty} \frac{d}{dt} (z^T P_e z) dt &= z^T(t) P_e z(t) \Big|_0^{\infty} = z^T(\infty) P_e z(\infty) - z^T(0) P_e z(0) = 0 - z^T(0) P_e z(0) \end{aligned}$$

Thus, we have for every given  $z(0)$ ,

$$z^T(0) P_e z(0) = z^T(0) P(\infty) z(0) \Rightarrow P_e = P(\infty) = \int_0^{\infty} e^{\bar{A}^T h} \nabla e^{\bar{A}h} dh$$

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It is now simple to see that

$$\lim_{t \rightarrow \infty} E[e(t)e^T(t)] = P(\infty) = P_e \quad \Rightarrow \quad \lim_{t \rightarrow \infty} E[e^T(t)e(t)] = \text{trace } P_e$$

Finally, we have

$$E[e(t)] = e^{\bar{A}t} \cdot E[e(0)] + \int_0^t e^{\bar{A}(t-t)} E[d(t)] \cdot dt = 0$$

**Example:** Consider a given plant characterized by the following state space model,

$$\left\{ \begin{array}{l} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + v(t), \quad E[v(t)v^T(t)] = Qd(t-t) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix} d(t-t) \\ y = [1 \quad 0]x + w(t), \quad E[w(t)w^T(t)] = Rd(t-t) = 0.2d(t-t) \end{array} \right.$$

Solving the Kalman filter ARE, we obtain

$$P_e = \begin{bmatrix} 0.0792 & -0.0343 \\ -0.0343 & 0.0314 \end{bmatrix}, \quad K_e = \begin{bmatrix} 0.3962 \\ -0.1715 \end{bmatrix}$$

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

# ***Linear Quadratic Gaussian (LQG)***

## ***Problem Statement***

It is very often in control system design for a real life problem that one cannot measure all the state variables of the given plant. Thus, the linear quadratic regulator, although it has a very impressive gain and phase margins ( $GM = \infty$  and  $PM = 60$  degrees), is impractical as it utilizes all state variables in the feedback, i.e.,  $u = -F x$ . In most of practical situations, only partial information of the state of the given plant is accessible or can be measured for feedback. The natural questions one would ask:

- Can we recover or estimate the state variables of the plant through the partially measurable information? The answer is yes. The solution is Kalman filter.
- Can we replace  $x$  the control law in LQR, i.e.,  $u = -F x$ , by the estimated state to carry out a meaningful control system design? The answer is yes. The solution is called LQG.
- Do we still have impressive properties associated with LQG? The answer is no. Any solution? Yes. It is called **loop transfer recovery** (LTR) technique (to be covered later).

## Linear Quadratic Gaussian Design

Consider a given plant characterized by

$$\begin{cases} \dot{x} = Ax + Bu + v(t) & v \text{ is the input noise} \\ y = Cx + w(t) & w \text{ is the measurement noise} \end{cases}$$

where  $v(t)$  and  $w(t)$  are white with zero means.  $v(t)$ ,  $w(t)$  and  $x(0)$  are independent, and

$$E[v(t)v^T(\mathbf{t})] = Q_e \mathbf{d}(t - \mathbf{t}), \quad Q_e \geq 0, \quad E[w(t)w^T(\mathbf{t})] = R_e \mathbf{d}(t - \mathbf{t}), \quad R_e > 0, \quad E[x(0)] = x_0$$

The performance index has to be modified as follows:

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} E \left[ \int_0^T (x^T Q x + u^T R u) dt \right], \quad Q \geq 0, \quad R > 0$$

The **Linear Quadratic Gaussian** (LQG) control is to design a control law that only requires the measurable information such that when it is applied to the given plant, the overall system is stable and the performance index is minimized.

## Solution to the LQG Problem – Separation Principle

**Step 1.** Design an LQR control law  $u = -F x$  which solves the following problem,

$$\dot{x} = A x + B u \quad J(x, u, Q, R) = \int_0^{\infty} (x^T Q x + u^T R u) dt, \quad Q \geq 0, R > 0$$

i.e., compute

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \quad P > 0, \quad F = R^{-1}B^T P.$$

**Step 2.** Design a Kalman filter for the given plant, i.e.,

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_e(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned}$$

where

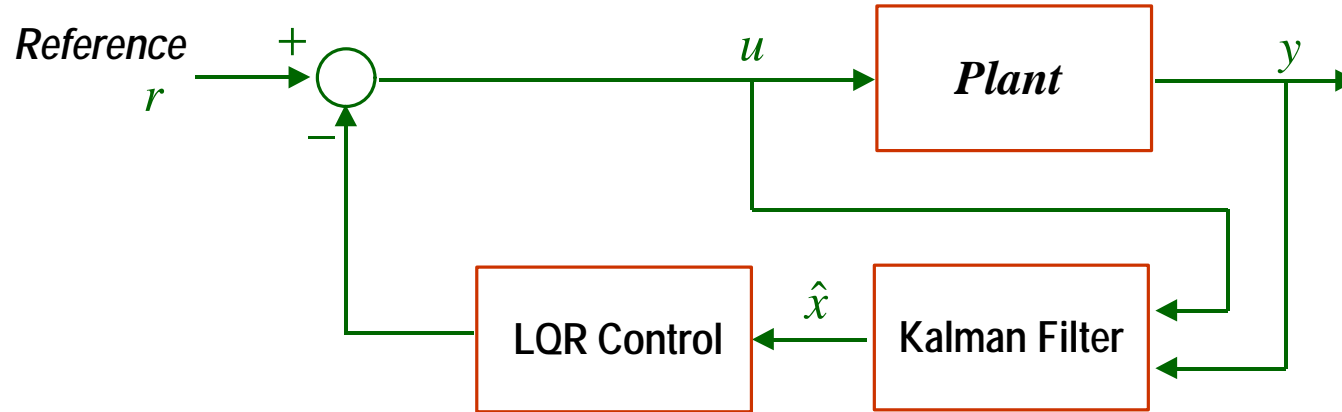
$$K_e = P_e C^T R^{-1}, \quad P_e A^T + A P_e - P_e C^T R^{-1} C P_e + Q_e = 0, \quad P_e > 0.$$

**Step 3.** The LQG control law is given by  $u = -F \hat{x}$ , i.e.,

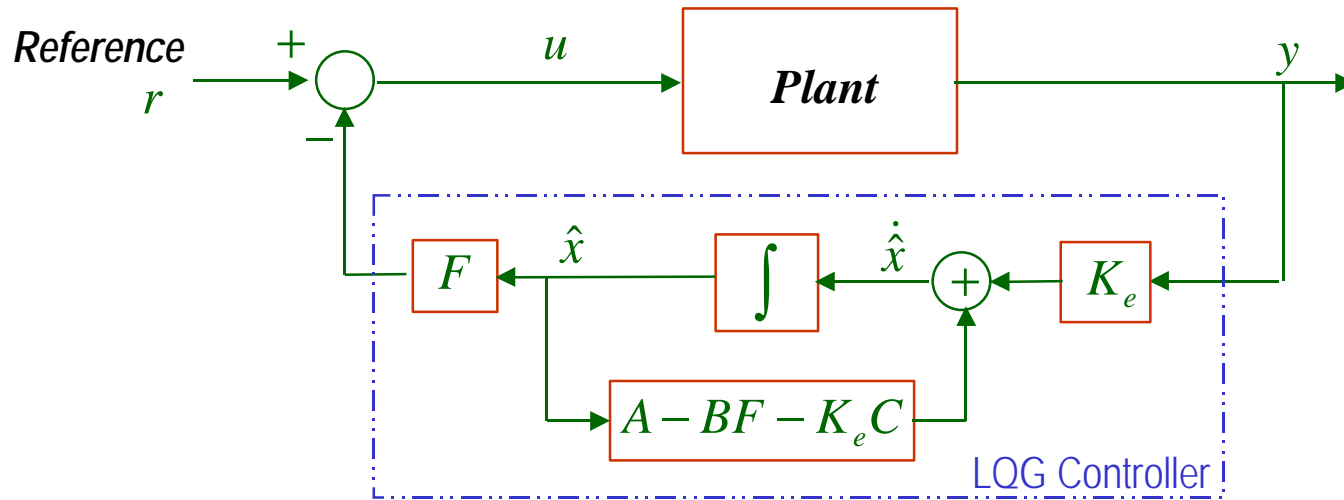
$$\begin{cases} \dot{\hat{x}} = A \hat{x} + B u + K_e (y - C \hat{x}) \\ u = -F \hat{x} \end{cases} \Rightarrow \begin{cases} \dot{\hat{x}} = (A - BF - K_e C) \hat{x} + K_e y \\ u = -F \hat{x} \end{cases}$$

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## Block Diagram Implementation of LQG Control Law



## More Detailed Block Diagram



## Closed-Loop Dynamics of the Given Plant together with LQG Controller

Recall the plant: 
$$\begin{cases} \dot{x} = Ax + Bu + v(t) \\ y = Cx + w(t) \end{cases}$$
 and the controller 
$$\begin{cases} \dot{\hat{x}} = (A - BF - K_e C) \hat{x} + K_e y \\ u = -F \hat{x} + r \end{cases}$$

We define a new variable  $e = x - \hat{x}$  and thus

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = Ax - BF\hat{x} + Br + v(t) - A\hat{x} + BF\hat{x} + K_e C\hat{x} - K_e Cx - K_e w(t) \\ &= A(x - \hat{x}) - K_e C(x - \hat{x}) + Br + v(t) - K_e w(t) = (A - K_e C)e + Br + v(t) - K_e w(t) \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= Ax + Bu + v(t) = Ax - BF\hat{x} + Br + v(t) = Ax - BF(x - e) + Br + v(t) \\ &= (A - BF)x + BFe + Br + v(t) \end{aligned}$$

Clearly, the closed-loop system is characterized by the following state space equation,

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - K_e C \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r + \tilde{v}, & \tilde{v} = \begin{pmatrix} v \\ v - K_e w \end{pmatrix} \\ y = [C \quad 0] \begin{pmatrix} x \\ e \end{pmatrix} + w \end{cases}$$

The closed-loop poles are given by  $\mathbf{I}(A - BF) \cup \mathbf{I}(A - K_e C)$ , which are stable. 74