Lecture Notes: Weeks 8b, 9a, 9b

Topic: H-infinity control

ECE/MAE 7360
Optimal and Robust Control
(Fall 2003 Offering)

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Chapter 14a: Understanding H_∞ Control

Objective: Derivation of H_∞ controller

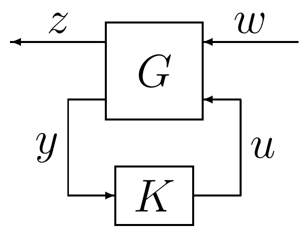
Methods: Intuition and handwaving

Background: State Feedback and Observer

- Problem Formulation and Solutions
- An intuitive Derivation

Problem Formulation and Solutions

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$



(i) (A, B_1) is stabilizable and (C_1, A) is detectable

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable

(iii) $D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$

(i) Together with (ii) guarantees that the two AREs have nonnegative definite stabilizing solutions.

(ii) Necessary and sufficient for G to be internally stabilizable.

(iii) The penalty on $z = C_1x + D_{12}u$ includes a nonsingular, normalized penalty on the control u. In the conventional $\mathcal{H}_2$ setting this means that there is no cross weighting between the state and control and that the control weight matrix is the identity.

(iv) w includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is normalized and nonsingular.

These assumptions can be relaxed.

Output Feedback $\mathcal{H}_\infty$ Control

$\exists K$ such that $\|T_{zw}\|_\infty < \gamma$ if and only if

(i) $\exists\ X_\infty \geq 0$

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

(ii) $\exists\ Y_\infty \geq 0$

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$
$$F_\infty := -B_2^* X_\infty, \quad L_\infty := -Y_\infty C_2^*$$
$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

Bounded Real Lemma

$$z = G(s)w, \quad G(s) = C(sI - A)^{-1}B \in \mathcal{H}_\infty$$

$$\|G\|_\infty = \sup_w \frac{\|z\|_2}{\|w\|_2} := \sup_w \frac{\sqrt{\int_0^\infty \|z\|^2 \, dt}}{\sqrt{\int_0^\infty \|w\|^2 \, dt}}$$

$$\|G\|_\infty < \gamma$$

$$\Updownarrow$$

$$\int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2\right) dt < 0, \quad \forall w \neq 0$$

$$\Updownarrow$$

$\exists\ X = X^* \geq 0$ such that
$XA + A^*X + XBB^*X/\gamma^2 + C^*C = 0$
and $A + BB^*X/\gamma^2$ is stable

$$\Updownarrow$$

$\exists\ Y = Y^* \geq 0$ such that
$YA^* + AY + YC^*CY/\gamma^2 + BB^* = 0$
and $A + YC^*C/\gamma^2$ is stable

Let $\Phi(s) = \gamma^2 I - G^{\sim}(s)G(s)$. Then

$$\|G\|_\infty < \gamma \Longleftrightarrow \Phi(j\omega) > 0,\ \forall \omega \in \mathbb{R} \quad \Longleftrightarrow \det \Phi(j\omega) \neq 0$$

since $\Phi(\infty) = \gamma^2 I > 0$ and $\Phi(j\omega)$ is continuous

$\Longleftrightarrow$ $\Phi(s)$ has no imaginary axis zero.

$\Longleftrightarrow$ $\Phi^{-1}(s)$ has no imaginary axis pole.

$$\Phi(s) = \left[\begin{array}{cc|c} A & 0 & -B \\ -C^*C & -A^* & 0 \\ \hline 0 & B^* & \gamma^2 I \end{array}\right]$$

$$\Phi^{-1} = \left[\begin{array}{cc|c} A & BB^*/\gamma^2 & B/\gamma^2 \\ -C^*C & -A^* & 0 \\ \hline 0 & B^*/\gamma^2 & \gamma^{-2} I \end{array}\right]$$

$$\Longleftrightarrow \begin{bmatrix} A & BB^*/\gamma^2 \\ -C^*C & -A^* \end{bmatrix} \text{ has no } j\omega \text{ axis eigenvalues}$$

Apply the following similarity transformation to Φ^{-1}

$$T = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$$

$$\Phi^{-1} = \left[\begin{array}{cc|c} A + BB^*X/\gamma^2 & BB^*/\gamma^2 & B/\gamma^2 \\ M(X) & -A^* - XBB^*/\gamma^2 & -XB/\gamma^2 \\ \hline B^*X/\gamma^2 & B^*/\gamma^2 & \gamma^{-2} I \end{array}\right]$$

$$M(X) := -XA - A^*X - XBB^*X/\gamma^2 - C^*C$$

If $M(X) = 0$, we have

$$\Phi^{-1} = \gamma^2 \left[\begin{array}{c|c} A + BB^*X/\gamma^2 & B/\gamma^2 \\ \hline B^*X/\gamma^2 & I/\gamma^2 \end{array}\right] \left[\begin{array}{c|c} -(A + BB^*X/\gamma^2)^* & -XB/\gamma^2 \\ \hline B/\gamma^2 & I/\gamma^2 \end{array}\right]$$

$\Phi(j\omega) > 0$ if $A + BB^*X/\gamma^2$ has no $j\omega$ eigenvalue

System Equations:

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u\\ z &= C_1 x + D_{12} u\\ y &= C_2 x + D_{21} w\end{aligned}$$

State feedback $u = Fx$:

$$\begin{aligned}\dot{x} &= (A + B_2 F)x + B_1 w\\ z &= (C_1 + D_{12} F)x\end{aligned}$$

By Bounded Real Lemma, $\|T_{zw}\|_\infty < \gamma$

$$\Updownarrow$$

$\exists\ X = X^* \geq 0$ such that

$$X(A + B_2F) + (A + B_2F)^*X + XB_1B_1^*X/\gamma^2 + (C_1 + D_{12}F)^*(C_1 + D_{12}F) = 0$$

and $A + B_2F + B_1B_1^*X/\gamma^2$ is stable

complete $\Updownarrow$ square

$\exists\ X = X^* \geq 0$ such that

$$XA + A^*X + XB_1B_1^*X/\gamma^2 - XB_2B_2^*X + C_1^*C_1 + (F + B_2^*X)^*(F + B_2^*X) = 0$$

and $A + B_2F + B_1B_1^*X/\gamma^2$ is stable

Intuition $\Longrightarrow \quad F = -B_2^*X$

$$\Updownarrow$$

$\exists\ X = X^* \geq 0$ such that

$$XA + A^*X + XB_1B_1^*X/\gamma^2 - XB_2B_2^*X + C_1^*C_1 = 0$$

and $A + B_1B_1^*X/\gamma^2 - B_2B_2^*X$ is stable

$$\Longrightarrow \quad F = F_\infty, \quad X = X_\infty$$

Output Feedback: Converting to State Estimation
Suppose $\exists$ a K such that

$$\|T_{zw}\|_\infty < \gamma$$

Then $x(\infty) = 0$ by stability (note also $x(0) = 0$)

$$\int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2\right) dt$$

$$= \int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2 + \frac{d}{dt}(x^* X_\infty x)\right) dt$$

$$= \int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2 + \dot{x}^* X_\infty x + x^* X_\infty \dot{x}\right) dt$$

Substituting $\dot{x} = Ax + B_1 w + B_2 u$ and $z = C_1 x + D_{12} u$

$$= \int_0^\infty \left(\|C_1 x\|^2 + \|u\|^2 - \gamma^2 \|w\|^2\right.$$

$$\left. + 2x^* X_\infty A x + 2x^* X_\infty B_1 w + 2x^* X_\infty B_2 u\right) dt$$

$$= \int_0^\infty \left(x^*(C_1^* C_1 + X_\infty A + A^* X_\infty) x + \|u\|^2\right.$$

$$\left. - \gamma^2 \|w\|^2 + 2x^* X_\infty B_1 w + 2x^* X_\infty B_2 u\right) dt$$

using X_∞ equation

$$= \int_0^\infty \left(x^*(-X_\infty B_1 B_1^* X_\infty/\gamma^2 + X_\infty B_2 B_2^* X_\infty) x + \|u\|^2\right.$$

$$\left. - \gamma^2 \|w\|^2 + 2x^* X_\infty B_1 w + 2x^* X_\infty B_2 u\right) dt$$

$$= \int_0^\infty \left(-\|B_1^* X_\infty x/\gamma\|^2 - \gamma^2 \|w\|^2 + 2x^* X_\infty B_1 w\right.$$

$$\left. + \|B_2^* X_\infty x\|^2 + \|u\|^2 + 2x^* X_\infty B_2 u\right) dt$$

completing the squares with respect to u and w

$$= \int_0^\infty \left(\|u + B_2^* X_\infty x\|^2 - \gamma^2 \left\|w - \gamma^{-2} B_1^* X_\infty x\right\|^2\right) dt$$

Summary:

$$\int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2\right) dt = \int_0^\infty \left(\|v\|^2 - \gamma^2 \|r\|^2\right) dt$$

$$v = u + B_2^* X_\infty x = u - F_\infty x, \quad r = w - \gamma^{-2} B_1^* X_\infty x$$

Rewrite the system equation with: $w = r + \gamma^{-2} B_1^* X_\infty x$

$$\begin{aligned} \dot{x} &= (A + B_1 B_1^* X_\infty/\gamma^2)x + B_1 r + B_2 u \\ v &= -F_\infty x + u \\ y &= C_2 x + D_{21} r \end{aligned}$$

$$\|T_{zw}\|_\infty < \gamma \Longleftrightarrow \|T_{vr}\|_\infty < \gamma$$

$$\Longleftrightarrow \quad \int_0^\infty \left(\|u - F_\infty x\|^2 - \gamma^2 \|r\|^2\right) dt < 0$$

If state is available: $u = F_\infty x$

worst disturbance: $w_* = \gamma^{-2} B_1^* X_\infty x$

State is not available: using estimated state

$$u = F_\infty \hat{x}$$

A standard observer:

$$\dot{\hat{x}} = (A + B_1 B_1^* X_\infty/\gamma^2)\hat{x} + B_2 u + L(C_2 \hat{x} - y)$$

where L is the observer gain to be determined.

Let $e := x - \hat{x}$. Then

$$\begin{aligned}\dot{e} &= (A + B_1 B_1^* X_\infty/\gamma^2 + LC_2)e + (B_1 + LD_{21})r \\ v &= -F_\infty e\end{aligned}$$

$\|T_{vr}\|_\infty < \gamma \Longrightarrow \exists$ a $Y \geq 0$ by bounded real lemma

$$Y(A+B_1B_1^*X_\infty/\gamma^2+LC_2)^*+(A+B_1B_1^*X_\infty/\gamma^2+LC_2)Y+YF_\infty^*F_\infty Y/\gamma^2$$
$$+(B_1 + LD_{21})(B_1 + LD_{21})^* = 0$$

Complete square w.r.t. L

$$(A+B_1B_1^*X_\infty/\gamma^2)^*+(A+B_1B_1^*X_\infty/\gamma^2)Y+YF_\infty^*F_\infty Y/\gamma^2+B_1B_1^*-YC_2^*C_2Y$$
$$+(L + YC_2^*)(L + YC_2^*)^* = 0$$

Again, intuition suggests that we can take

$$L = -YC_2^*$$

which gives

$$Y(A + B_1B_1^*X_\infty/\gamma^2)^* + (A + B_1B_1^*X_\infty/\gamma^2)Y$$
$$+YF_\infty^*F_\infty Y/\gamma^2 - YC_2^*C_2Y + B_1B_1^* = 0$$

It is easy to verify that

$$Y = Y_\infty(I - \gamma^{-2}X_\infty Y_\infty)^{-1}$$

Since $Y \geq 0$, we must have

$$\rho(X_\infty Y_\infty) < \gamma^2$$

Hence $L = Z_\infty L_\infty$ and the controller is give by

$$\begin{aligned}\dot{\hat{x}} &= (A + B_1B_1^*X_\infty/\gamma^2)\hat{x} + B_2u + Z_\infty L_\infty(C_2\hat{x} - y) \\ u &= F_\infty \hat{x}\end{aligned}$$

Chapter 14: $\mathcal{H}_\infty$ Control

- $\mathcal{H}_\infty$ background
- $\mathcal{H}_\infty$: 1984 workshop approach
- Assumptions
- output feedback $\mathcal{H}_\infty$ control
- a matrix fact
- inequality characterization
- connection between ARE and ARI (LMI)
- proof for necessity
- proof for sufficiency
- comments
- optimality and dependence on γ
- $\mathcal{H}_\infty$ controller structure
- example
- an optimal controller
- $\mathcal{H}_\infty$ control: general case
- relaxing assumptions
- $\mathcal{H}_2$ and $\mathcal{H}_\infty$ integral control
- $\mathcal{H}_\infty$ filtering

$\mathcal{H}_\infty$ Background

- Initial theory was SISO (Zames, Helton, Tannenbaum)
- Nevanlinna-Pick interpolation
- Operator-theoretic methods (Sarason, Adamjan *et al*, Ball-Helton)
- Initial work handled restricted problems
  ("1-block" and "2-block")
- Solution to "2×2-block" problem
  (1984 Honeywell-ONR Workshop)

$\mathcal{H}_\infty$: 1984 H/ONR Workshop Approach

Solution approach:

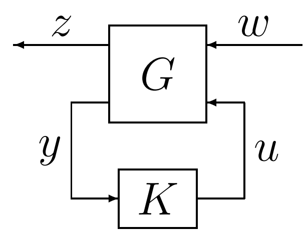
- Parameterize all stabilizing controllers via [Youla *et al*]
- Obtain realizations of the closed-loop transfer matrix
- Transform to "2×2-block" general distance problem
- Reduce to the Nehari problem and solve via Glover

Properties of the solution:

- State-space using standard operations
- Computationally intensive (many Ric. eqns.)
- Potentially high-order controllers
- Find solution $< \gamma$, iterate for optimal

Assumptions

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$



(i) (A, B_1) is Controllable and (C_1, A) is observable

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable

(iii) $D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$

(i) Together with (ii) guarantees that the two AREs have positive definite stabilizing solution.

(ii) Necessary and sufficient for G to be internally stabilizable.

(iii) The penalty on $z = C_1 x + D_{12} u$ includes a nonsingular, normalized penalty on the control u. In the conventional $\mathcal{H}_2$ setting this means that there is no cross weighting between the state and control input, and that the control weight matrix is the identity.

(iv) w includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is normalized and nonsingular.

These assumptions simplify the theorem statements and proofs, and can be relaxed.

Output Feedback $\mathcal{H}_\infty$ Control

$\exists K$ such that $\|T_{zw}\|_\infty < \gamma$ iff

(i) $\exists\ X_\infty > 0$

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

(ii) $\exists\ Y_\infty > 0$

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$
$$F_\infty := -B_2^* X_\infty, \quad L_\infty := -Y_\infty C_2^*$$
$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

A Matrix Fact

[Packard, 1994] Suppose $X, Y \in \mathbb{R}^{n\times n}$ and $X = X^* > 0, Y = Y^* > 0$. Let r be a positive integer. Then there exists matrices $X_{12} \in \mathbb{R}^{n\times r}$, $X_2 \in \mathbb{R}^{r\times r}$ such that $X_2 = X_2^*$, and

$$\begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0 \qquad \& \qquad \begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix}^{-1} = \begin{bmatrix} Y & \star \\ \star & \star \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \qquad \& \qquad \text{rank} \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r.$$

Proof. ($\Leftarrow$) By assumption, there is a matrix $X_{12} \in \mathbb{R}^{n\times r}$ such that $X - Y^{-1} = X_{12}X_{12}^*$. Defining $X_2 := I_r$ completes the construction.

($\Rightarrow$) Using Schur complements,

$$Y = X^{-1} + X^{-1}X_{12}(X_2 - X_{12}^* X^{-1} X_{12})^{-1} X_{12}^* X^{-1}.$$

Inverting, using the matrix inversion lemma, gives

$$Y^{-1} = X - X_{12}X_2^{-1}X_{12}^*.$$

Hence, $X - Y^{-1} = X_{12}X_2^{-1}X_{12}^* \geq 0$, and indeed,

rank$(X - Y^{-1})$ =rank$(X_{12}X_2^{-1}X_{12}^*) \leq r$. □

Inequality Characterization

Lemma IC: $\exists$ r-th order K such that $\|T_{zw}\|_\infty < \gamma$ only if

(i) $\exists\ Y_1 > 0$

$$AY_1 + Y_1A^* + Y_1C_1^*C_1Y_1/\gamma^2 + B_1B_1^* - \gamma^2 B_2B_2^* < 0$$

(ii) $\exists X_1 > 0$

$$X_1A + A^*X_1 + X_1B_1B_1^*X_1/\gamma^2 + C_1^*C_1 - \gamma^2 C_2^*C_2 < 0$$

(iii) $\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0$ and rank $\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \leq n + r.$

Proof. Suppose that there exists an r-th order controller $K(s)$ such that $\|T_{zw}\|_\infty < \gamma$. Let $K(s)$ have a state space realization

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

then

$$T_{zw} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] := \left[\begin{array}{cc|c} A + B_2\hat{D}C_2 & B_2\hat{C} & B_1 + B_2\hat{D}D_{21} \\ \hat{B}C_2 & \hat{A} & \hat{B}D_{21} \\ \hline C_1 + D_{12}\hat{D}C_2 & D_{12}\hat{C} & D_{12}\hat{D}D_{21} \end{array}\right].$$

Denote

$$R = \gamma^2 I - D_c^*D_c, \qquad \tilde{R} = \gamma^2 I - D_cD_c^*.$$

By Bounded Real Lemma, $\exists \tilde{X} = \begin{bmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0$ such that

$$\tilde{X}(A_c + B_cR^{-1}D_c^*C_c) + (A_c + B_cR^{-1}D_c^*C_c)^*\tilde{X}$$

$$+\tilde{X}B_cR^{-1}B_c^*\tilde{X}+C_c^*\tilde{R}^{-1}C_c<0$$

This gives after much algebraic manipulation

$$X_1A+A^*X_1+X_1B_1B_1^*X_1/\gamma^2+C_1^*C_1-\gamma^2C_2^*C_2$$

$$+(X_1B_1\hat{D}+X_{12}\hat{B}+\gamma^2C_2^*)(\gamma^2I-\hat{D}^*\hat{D})^{-1}(X_1B_1\hat{D}+X_{12}\hat{B}+\gamma^2C_2^*)^*<0$$

which implies that

$$X_1A+A^*X_1+X_1B_1B_1^*X_1/\gamma^2+C_1^*C_1-\gamma^2C_2^*C_2<0.$$

Let $\tilde{Y}=\gamma^2\tilde{X}^{-1}$ and partition $\tilde{Y}$ as $\tilde{Y}=\begin{bmatrix}Y_1 & Y_{12}\\ Y_{12}^* & Y_2\end{bmatrix}>0$ then

$$(A_c+B_cR^{-1}D_c^*C_c)\tilde{Y}+\tilde{Y}(A_c+B_cR^{-1}D_c^*C_c)^*$$

$$+\tilde{Y}C_c^*\tilde{R}^{-1}C_c\tilde{Y}+B_cR^{-1}B_c^*<0$$

This gives

$$AY_1+Y_1A^*+B_1B_1^*-\gamma^2B_2B_2^*+Y_1C_1^*C_1Y_1/\gamma^2$$

$$+(Y_1C_1^*\hat{D}^*+Y_{12}\hat{C}^*+\gamma^2B_2)(\gamma^2I-\hat{D}\hat{D}^*)^{-1}(Y_1C_1^*\hat{D}^*+Y_{12}\hat{C}^*+\gamma^2B_2)^*<0$$

which implies that

$$AY_1+Y_1A^*+B_1B_1^*-\gamma^2B_2B_2^*+Y_1C_1^*C_1Y_1/\gamma^2<0.$$

By the matrix fact, given $X_1>0$ and $Y_1>0$, there exists X_{12} and X_2 such that $\tilde{Y}=\gamma^2\tilde{X}^{-1}$ or $\tilde{Y}/\gamma=(\tilde{X}/\gamma)^{-1}$:

$$\begin{bmatrix}X_1/\gamma & X_{12}/\gamma\\ X_{12}^*/\gamma & X_2/\gamma\end{bmatrix}^{-1}=\begin{bmatrix}Y_1/\gamma & \star\\ \star & \star\end{bmatrix}$$

$$\Longleftrightarrow\begin{bmatrix}X_1/\gamma & I_n\\ I_n & Y_1/\gamma\end{bmatrix}\geq 0 \text{ and rank}\begin{bmatrix}X_1/\gamma & I_n\\ I_n & Y_1/\gamma\end{bmatrix}\leq n+r. \qquad \square$$

Connection between ARE and ARI (LMI)

Lemma ARE: [Ran and Vreugdenhil, 1988] Suppose (A, B) is controllable and there is an $X = X^*$ such that

$$\mathcal{Q}(X) := XA + A^*X + XBB^*X + Q < 0.$$

Then there exists a solution $X_+ > X$ to the Riccati equation

$$XA + A^*X + XBB^*X + Q = 0 \tag{0.7}$$

such that $A + BB^*X_+$ is antistable.

Proof. Let X be such that $\mathcal{Q}(X) < 0$.

Choose F_0 such that $A_0 := A - BF_0$ is antistable.

Let $X_0 = X_0^*$ solve

$$X_0A_0 + A_0^*X_0 - F_0^*F_0 + Q = 0.$$

Define $\hat{F}_0 := F_0 + B^*X$. Then

$$(X_0 - X)A_0 + A_0^*(X_0 - X) = \hat{F}_0^*\hat{F}_0 - \mathcal{Q}(X) > 0.$$

and $X_0 > X$ (by anti-stability of A_0)

Define a non-increasing sequence of hermitian matrices $\{X_i\}$:

$$X_0 \geq X_1 \geq \cdots \geq X_{n-1} > X,$$

$$A_i = A - BF_i, \text{ is antistable, } i = 0, \ldots, n-1;$$

$$F_i = -B^*X_{i-1}, \ i = 1, \ldots, n-1;$$

$$X_iA_i + A_i^*X_i = F_i^*F_i - Q, \ i = 0, 1, \ldots, n-1. \tag{0.8}$$

By Induction: We show this sequence can indeed be defined:

Introduce

$$F_n = -B^*X_{n-1}, \quad A_n = A - BF_n.$$

We show that A_n is antistable. Using (0.8), with $i = n-1$, we get

$$X_{n-1}A_n + A_n^* X_{n-1} + Q - F_n^* F_n - (F_n - F_{n-1})^*(F_n - F_{n-1}) = 0.$$

Let $\hat{F}_n := F_n + B^* X$; then

$$(X_{n-1} - X)A_n + A_n^*(X_{n-1} - X) = -\mathcal{Q}(X)$$
$$+\hat{F}_n^* \hat{F}_n + (F_n - F_{n-1})^*(F_n - F_{n-1}) > 0$$

$\Rightarrow A_n$ is antistable by Lyapunov theorem since $X_{n-1} - X > 0$.

Let X_n be the unique solution of

$$X_n A_n + A_n^* X_n = F_n^* F_n - Q. \tag{0.9}$$

Then X_n is hermitian. Next, we have

$$(X_n - X)A_n + A_n^*(X_n - X) = -\mathcal{Q}(X) + \hat{F}_n^* \hat{F}_n > 0,$$

$$(X_{n-1} - X_n)A_n + A_n^*(X_{n-1} - X_n) = (F_n - F_{n-1})^*(F_n - F_{n-1}) \geq 0.$$

Since A_n is antistable, we have $X_{n-1} \geq X_n > X$.

We have a non-increasing sequence $\{X_i\}$.

Since the sequence is bounded below by $X_i > X$. Hence the limit

$$X_+ := \lim_{n\to\infty} X_n$$

exists and is hermitian, and we have $X_+ \geq X$. Passing the limit $n \to \infty$ in (0.9), we get $\mathcal{Q}(X_+) = 0$. So X_+ is a solution of (0.7).

Note that $X_+ - X \geq 0$ and

$$(X_+ - X)A_+ + A_+^*(X_+ - X) = -\mathcal{Q}(X)$$
$$+(X_+ - X)BB^*(X_+ - X) > 0 \tag{0.10}$$

hence, $X_+ - X > 0$ and $A_+ = A + BB^* X_+$ is antistable.

□

Proof for Necessary

There exists a controller such that $\|T_{zw}\|_\infty < \gamma$ only if the following three conditions hold:

(i) there exists a stabilizing solution $X_\infty > 0$ to

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

(ii) there exists a stabilizing solution $Y_\infty > 0$ to

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

(iii)

$$\begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} > 0 \quad \text{or} \quad \rho(X_\infty Y_\infty) < \gamma^2.$$

Proof. Applying Lemma ARE to part (i) of Lemma IC, we conclude that there exists a $Y > Y_1 > 0$ such that

$$AY + YA^* + YC_1^* C_1 Y/\gamma^2 + B_1 B_1^* - \gamma^2 B_2 B_2^* = 0$$

and $A + C_1^* C_1 Y/\gamma^2$ is antistable. Let $X_\infty := \gamma^2 Y^{-1}$, we have

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

and

$$\begin{aligned} A + (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty &= -X_\infty^{-1}(A + C_1^* C_1 X_\infty^{-1}) X_\infty \\ &= -X_\infty^{-1}(A + C_1^* C_1 Y/\gamma^2) X_\infty \end{aligned}$$

is stable.

Similarly, applying Lemma ARE to part (ii) of Lemma IC, we conclude that there exists an $X > X_1 > 0$ such that

$$XA + A^* X + XB_1 B_1^* X/\gamma^2 + C_1^* C_1 - \gamma^2 C_2^* C_2 = 0$$

and $A + B_1B_1^*X/\gamma^2$ is antistable. Let $Y_\infty := \gamma^2 X^{-1}$, we have

$$AY_\infty + Y_\infty A^* + Y_\infty(C_1^*C_1/\gamma^2 - C_2^*C_2)Y_\infty + B_1B_1^* = 0 \qquad (0.11)$$

and $A + (C_1^*C_1/\gamma^2 - C_2^*C_2)Y_\infty$ is stable.

Finally, note that the rank condition in part (iii) of Lemma IC is automatically satisfied by $r \geq n$, and

$$\begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} = \begin{bmatrix} X/\gamma & I_n \\ I_n & Y/\gamma \end{bmatrix}$$

$$> \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0.$$

or $\rho(X_\infty Y_\infty) < \gamma^2$. □

Proof for Sufficiency

Show K_{sub} renders $\|T_{zw}\|_\infty < \gamma$.

The closed-loop transfer function with K_{sub}:

$$T_{zw} = \left[\begin{array}{cc|c} A & B_2F_\infty & B_1 \\ -Z_\infty L_\infty C_2 & \hat{A}_\infty & -Z_\infty L_\infty D_{21} \\ \hline C_1 & D_{12}F_\infty & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

Define

$$P = \begin{bmatrix} \gamma^2 Y_\infty^{-1} & -\gamma^2 Y_\infty^{-1} Z_\infty^{-1} \\ -\gamma^2 (Z_\infty^*)^{-1} Y_\infty^{-1} & \gamma^2 Y_\infty^{-1} Z_\infty^{-1} \end{bmatrix}$$

Then $P > 0$ and

$$PA_c + A_c^* P + PB_cB_c^*P/\gamma^2 + C_c^*C_c = 0.$$

Moreover

$$A_c + B_cB_c^*P/\gamma^2 = \begin{bmatrix} A + B_1B_1^*Y_\infty^{-1} & B_2F_\infty - B_1B_1^*Y_\infty^{-1}Z_\infty^{-1} \\ 0 & A + B_1B_1^*X_\infty/\gamma^2 + B_2F_\infty \end{bmatrix}$$

has no eigenvalues on the imaginary axis since

$$A + B_1B_1^*Y_\infty^{-1} \quad \text{is antistable}$$

and

$$A + B_1B_1^*X_\infty/\gamma^2 + B_2F_\infty \quad \text{is stable}$$

By Bounded Real Lemma, $\|T_{zw}\|_\infty < \gamma$.

Comments

The conditions in Lemma IC are in fact necessary and sufficient.

But the three conditions have to be checked simultaneously. This is because if one finds an $X_1 > 0$ and a $Y_1 > 0$ satisfying conditions (i) and (ii) but not condition (iii), this does not imply that there is no admissible $\mathcal{H}_\infty$ controller since there might be other $X_1 > 0$ and $Y_1 > 0$ that satisfy all three conditions.

For example, consider $\gamma = 1$ and

$$G(s) = \left[\begin{array}{c|cc} -1 & \begin{bmatrix} 1 & 0 \end{bmatrix} & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

It is easy to check that $X_1 = Y_1 = 0.5$ satisfy (i) and (ii) but not (iii). Nevertheless, we can show that $\gamma_{opt} = 0.7321$ and thus a suboptimal controller exists for $\gamma = 1$. In fact, we can check that $1 < X_1 < 2,\ 1 < Y_1 < 2$ also satisfy $(i), (ii)$ and (iii). For this reason, Riccati equation approach is usually preferred over the Riccati inequality and LMI approaches whenever possible.

Example

Consider the feedback system shown in Figure 0.4 with

$$P = \frac{50(s+1.4)}{(s+1)(s+2)}, \quad W_e = \frac{2}{s+0.2}, \quad W_u = \frac{s+1}{s+10}.$$

Design a K to minimize the $\mathcal{H}_\infty$ norm from $w = \begin{bmatrix} d \\ d_i \end{bmatrix}$ to $z = \begin{bmatrix} e \\ \tilde{u} \end{bmatrix}$:

$$\begin{bmatrix} e \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} W_e(I+PK)^{-1} & W_e(I+PK)^{-1}P \\ -W_uK(I+PK)^{-1} & -W_uK(I+PK)^{-1}P \end{bmatrix} \begin{bmatrix} d \\ d_i \end{bmatrix} =: T_{zw} \begin{bmatrix} d \\ d_i \end{bmatrix}.$$

LFT framework:

$$G(s) = \left[\begin{array}{cc:c} W_e & W_eP & -W_eP \\ 0 & 0 & -W_u \\ \hdashline I & P & -P \end{array}\right] = \left[\begin{array}{cccc|cc:c} -0.2 & 2 & 2 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 20 & -20 \\ 0 & 0 & -2 & 0 & 0 & 30 & -30 \\ 0 & 0 & 0 & -10 & 0 & 0 & -3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & -1 \\ \hdashline 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right].$$

$\gg$ $[\mathbf{K}, \mathbf{T_{zw}}, \gamma_{\mathbf{subopt}}] = \mathbf{hinfsyn}(\mathbf{G}, \mathbf{n_y}, \mathbf{n_u}, \gamma_{\min}, \gamma_{\max}, \mathbf{tol})$

where n_y = dimensions of y, n_u = dimensions of u; $\gamma_{\min}$ = a lower bound, $\gamma_{\max}$ = an upper bound for γ_{opt}; and tol is a tolerance to the optimal value. Set $n_y = 1, n_u = 1$, $\gamma_{\min} = 0$, $\gamma_{\max} = 10$, tol $= 0.0001$; we get $\gamma_{\text{subopt}} = 0.7849$ and a suboptimal controller

$$K = \frac{12.82(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/32449447.67+1)(s/22.19+1)(s/1.4+1)(s/0.2+1)}.$$

If we set tol $= 0.01$, we would get $\gamma_{\text{subopt}} = 0.7875$ and a suboptimal controller

$$\tilde{K} = \frac{12.78(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/2335.59+1)(s/21.97+1)(s/1.4+1)(s/0.2+1)}.$$

The only significant difference between K and $\tilde{K}$ is the exact location of the far-away stable controller pole. Figure 0.25 shows the closed-loop frequency response of $\overline{\sigma}(T_{zw})$ and Figure 0.26 shows the frequency responses of S, T, KS, and SP.

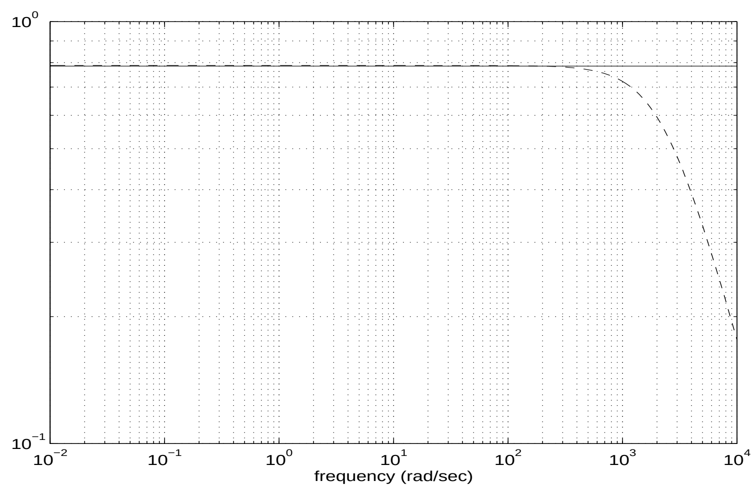


Figure 0.25: The closed-loop frequency responses of $\overline{\sigma}(T_{zw})$ with K (solid line) and $\tilde{K}$ (dashed line)

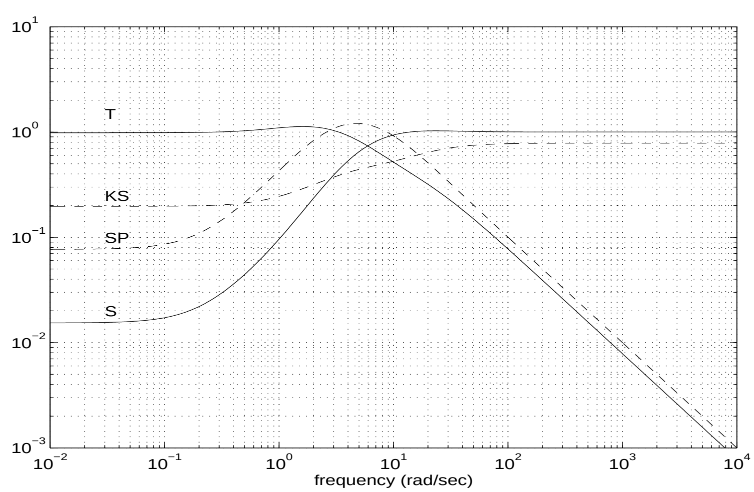


Figure 0.26: The frequency responses of S, T, KS, and SP with K

Consider again the two-mass/spring/damper system shown in Figure 0.1. Assume that F_1 is the control force, F_2 is the disturbance force, and the measurements of x_1 and x_2 are corrupted by measurement noise:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + W_n \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \quad W_n = \begin{bmatrix} \frac{0.01(s+10)}{s+100} & 0 \\ 0 & \frac{0.01(s+10)}{s+100} \end{bmatrix}.$$

Our objective is to design a control law so that the effect of the disturbance force F_2 on the positions of the two masses, x_1 and x_2, are reduced in a frequency range $0 \leq \omega \leq 2$.

The problem can be set up as shown in Figure 0.27, where $W_e = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$ is the performance weight and W_u is the control weight. In order to limit the control force, we shall choose

$$W_u = \frac{s+5}{s+50}.$$

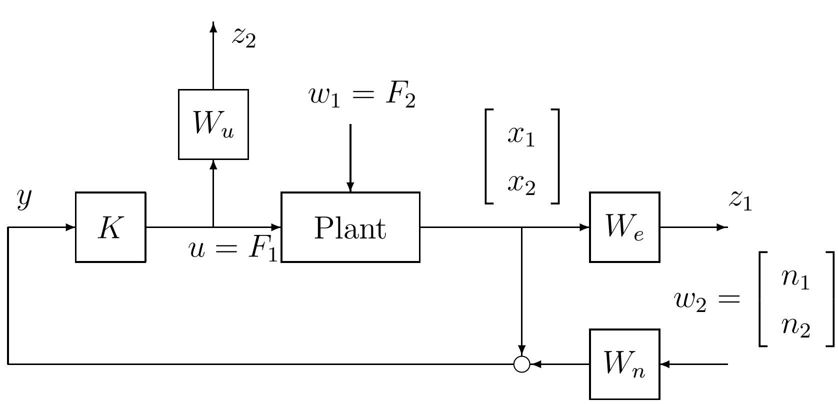


Figure 0.27: Rejecting the disturbance force F_2 by a feedback control

Let $u = F_1$, $w = \begin{bmatrix} F_2 \\ n_1 \\ n_2 \end{bmatrix}$:

$$G(s) = \begin{bmatrix} \begin{bmatrix} W_e P_1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} W_e P_2 \\ W_u \end{bmatrix} \\ \begin{bmatrix} P_1 & W_n \end{bmatrix} & P_2 \end{bmatrix}$$

where P_1 and P_2 denote the transfer matrices from F_1 and F_2 to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, respectively.

- $W_1 = \frac{5}{s/2+1}$, $W_2 = 0$: only reject the effect of the disturbance force F_2 on the position x_1.

  $\|\mathcal{F}_\ell(G, K_2)\|_2 = 2.6584$

  $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 2.6079$

  $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 1.6101.$

  This means that the effect of the disturbance force F_2 in the desired frequency rang $0 \leq \omega \leq 2$ will be effectively reduced with the $\mathcal{H}_\infty$ controller K_∞ by $5/1.6101 = 3.1054$ times at x_1.

- $W_1 = 0$, $W_2 = \frac{5}{s/2+1}$: only reject the effect of the disturbance force F_2 on the position x_2.

  $\|\mathcal{F}_\ell(G, K_2)\|_2 = 0.1659$

  $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 0.5202$

  $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 0.5189.$

  This means that the effect of the disturbance force F_2 in the desired frequency rang $0 \leq \omega \leq 2$ will be effectively reduced with the $\mathcal{H}_\infty$ controller K_∞ by $5/0.5189 = 9.6358$ times at x_2.

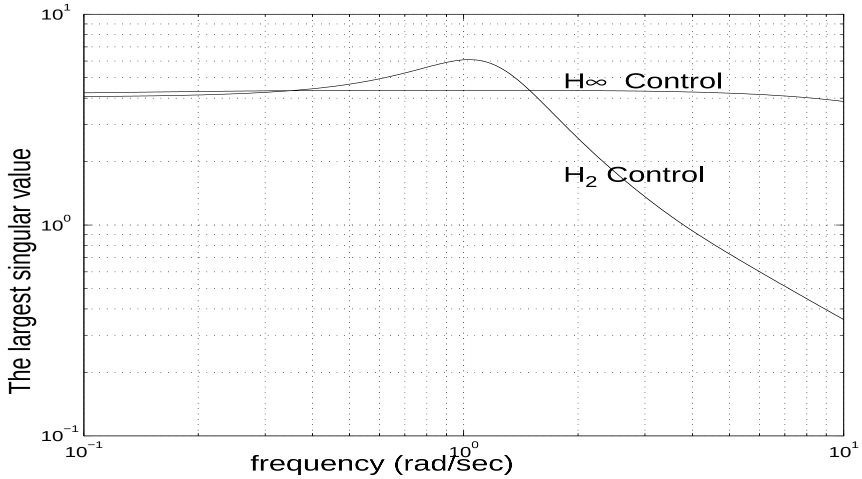


Figure 0.28: The largest singular value plot of the closed-loop system T_{zw} with an $\mathcal{H}_2$ controller and an $\mathcal{H}_\infty$ controller

- $W_1 = W_2 = \frac{5}{s/2+1}$: want to reject the effect of the disturbance force F_2 on both x_1 and x_2.

  $\|\mathcal{F}_\ell(G, K_2)\|_2 = 4.087$

  $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 6.0921$

  $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 4.3611.$

  This means that the effect of the disturbance force F_2 in the desired frequency rang $0 \leq \omega \leq 2$ will only be effectively reduced with the $\mathcal{H}_\infty$ controller K_∞ by $5/4.3611 = 1.1465$ times at both x_1 and x_2.

  This result shows clearly that it is very hard to reject the disturbance effect on both positions at the same time. The largest singular value Bode plots of the closed-loop system are shown in Figure 0.28. We note that the $\mathcal{H}_\infty$ controller typically gives a relatively flat frequency response since it tries to minimize the peak of the frequency response. On the other hand, the $\mathcal{H}_2$ controller would typically produce a frequency response that rolls off fast in the high-frequency range but with a large peak in the low-frequency range.

Optimality and dependence on γ

There exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold:

(i) $\exists$ a stabilizing $X_\infty > 0$

(ii) $\exists$ a stabilizing $Y_\infty > 0$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

- Denote by γ_o the infimum over all γ such that conditions (i)-(iii) are satisfied.
- Descriptor formulae can be obtained for $\gamma = \gamma_o$.
- As $\gamma \to \infty$, $H_\infty \to H_2$, $X_\infty \to X_2$, etc., and $K_{sub} \to K_2$.
- If $\gamma_2 \geq \gamma_1 > \gamma_0$ then $X_\infty(\gamma_1) \geq X_\infty(\gamma_2)$ and $Y_\infty(\gamma_1) \geq Y_\infty(\gamma_2)$.
- Thus X_∞ and Y_∞ are decreasing functions of γ, as is $\rho(X_\infty Y_\infty)$.
- At $\gamma = \gamma_o$, any one of the 3 conditions can fail.
- It is most likely that condition (iii) will fail first.
- To understand this, consider (i) and let γ_1 be the largest γ for which H_∞ fails to be in $dom(Ric)$, because it fails to have either the stability property or the complementarity property. The same remarks will apply to (ii) by duality.
- If the stability property fails at $\gamma = \gamma_1$, then $H_\infty \notin dom(Ric)$ but Ric can be extended to obtain X_∞ and the controller $u = -B_2^* X_\infty x$ is stabilizing and makes $\|T_{zw}\|_\infty = \gamma_1$. The stability property will also not hold for any $\gamma \leq \gamma_1$, and no controller whatsoever exists which makes $\|T_{zw}\|_\infty < \gamma_1$.

- In other words, if stability breaks down first then the infimum over stabilizing controllers equals the infimum over all controllers, stabilizing or otherwise.
- In view of this, we would expect that typically complementarity would fail first.

- Complementarity failing at $\gamma = \gamma_1$ means $\rho(X_\infty) \to \infty$ as $\gamma \to \gamma_1$ so condition (iii) would fail at even larger values of γ, unless the eigenvectors associated with $\rho(X_\infty)$ as $\gamma \to \gamma_1$ are in the null space of Y_∞.
- Thus condition (iii) is the most likely of all to fail first.

$\mathcal{H}_\infty$ Controller Structure

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2^* X_\infty, \quad L_\infty := -Y_\infty C_2^*, \quad Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$

$$\dot{\hat{x}} = A\hat{x} + B_1 \hat{w}_{worst} + B_2 u + Z_\infty L_\infty (C_2 \hat{x} - y)$$

$$u = F_\infty \hat{x}, \qquad \hat{w}_{worst} = \gamma^{-2} B_1^* X_\infty \hat{x}$$

---

1) $\hat{w}_{worst}$ is the estimate of w_{worst}

2) $Z_\infty L_\infty$ is the filter gain for the OE problem of estimating $F_\infty x$ in the presence of the "worst-case" w, w_{worst}

3) The $\mathcal{H}_\infty$ controller has a separation interpretation

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Optimal Controller:

$$(I - \gamma_{opt}^{-2} Y_\infty X_\infty)\dot{\hat{x}} = A_s \hat{x} - L_\infty y \qquad (0.12)$$

$$u = F_\infty \hat{x} \qquad (0.13)$$

$$A_s := A + B_2 F_\infty + L_\infty C_2$$
$$+\gamma_{opt}^{-2} Y_\infty A^* X_\infty + \gamma_{opt}^{-2} B_1 B_1^* X_\infty + \gamma_{opt}^{-2} Y_\infty C_1^* C_1$$

See the example below.

Example

$$G(s) = \left[\begin{array}{c|cc} a & \begin{bmatrix} 1 & 0 \end{bmatrix} & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

Then all assumptions for output feedback problem are satisfied and

$$H_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}, \quad J_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}.$$

The eigenvalues of H_∞ and J_∞ are given, respectively, by

$$\sigma(H_\infty) = \left\{\pm\frac{\sqrt{(a^2+1)\gamma^2-1}}{\gamma}\right\}, \quad \sigma(J_\infty) = \left\{\pm\frac{\sqrt{(a^2+1)\gamma^2-1}}{\gamma}\right\}.$$

If $\gamma > \dfrac{1}{\sqrt{a^2+1}}$, then $\mathcal{X}_-(H_\infty)$ and $\mathcal{X}_-(J_\infty)$ exist and

$$\mathcal{X}_-(H_\infty) = \mathrm{Im}\begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1}-a\gamma}{\gamma} \\ 1 \end{bmatrix}$$

$$\mathcal{X}_-(J_\infty) = \mathrm{Im}\begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1}-a\gamma}{\gamma} \\ 1 \end{bmatrix}.$$

Note that if $\gamma > 1$, then $H_\infty \in dom(Ric)$, $J_\infty \in dom(Ric)$, and

$$X_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1}-a\gamma} > 0$$

$$Y_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1}-a\gamma} > 0.$$

It can be shown that

$$\rho(X_\infty Y_\infty) = \frac{\gamma^2}{(\sqrt{(a^2+1)\gamma^2 - 1} - a\gamma)^2} < \gamma^2$$

is satisfied if and only if

$$\gamma > \sqrt{a^2+2} + a.$$

So condition (iii) will fail before either (i) or (ii) fails.

The optimal γ for the output feedback is given by

$$\gamma_{opt} = \sqrt{a^2+2} + a$$

and the optimal controller given by the descriptor formula in equations (0.12) and (0.13) is a constant. In fact,

$$u_{opt} = -\frac{\gamma_{opt}}{\sqrt{(a^2+1)\gamma_{opt}^2 - 1} - a\gamma_{opt}}\, y.$$

For instance, let $a = -1$ then $\gamma_{opt} = \sqrt{3} - 1 = 0.7321$ and $u_{opt} = -0.7321\, y$. Further,

$$T_{zw} = \left[\begin{array}{c|cc} -1.7321 & 1 & -0.7321 \\ \hline 1 & 0 & 0 \\ -0.7321 & 0 & -0.7321 \end{array}\right].$$

It is easy to check that $\|T_{zw}\|_\infty = 0.7321$.

An Optimal Controller

There exists an admissible controller such that $\|T_{zw}\|_\infty \le \gamma$ iff the following three conditions hold:

(i) there exists a full column rank matrix $\begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} \in \mathbb{R}^{2n\times n}$ such that

$$H_\infty \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} = \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} T_X, \quad \mathrm{Re}\ \lambda_i(T_X) \le 0\ \forall i$$

and

$$X_{\infty 1}^* X_{\infty 2} = X_{\infty 2}^* X_{\infty 1};$$

(ii) there exists a full column rank matrix $\begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} \in \mathbb{R}^{2n\times n}$ such that

$$J_\infty \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} = \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} T_Y, \quad \mathrm{Re}\ \lambda_i(T_Y) \le 0\ \forall i$$

and

$$Y_{\infty 1}^* Y_{\infty 2} = Y_{\infty 2}^* Y_{\infty 1};$$

(iii) $\begin{bmatrix} X_{\infty 2}^* X_{\infty 1} & \gamma^{-1} X_{\infty 2}^* Y_{\infty 2} \\ \gamma^{-1} Y_{\infty 2}^* X_{\infty 2} & Y_{\infty 2}^* Y_{\infty 1} \end{bmatrix} \ge 0.$

Moreover, when these conditions hold, one such controller is

$$K_{opt}(s) := C_K(sE_K - A_K)^+ B_K$$

where

$$\begin{aligned} E_K &:= Y_{\infty 1}^* X_{\infty 1} - \gamma^{-2} Y_{\infty 2}^* X_{\infty 2} \\ B_K &:= Y_{\infty 2}^* C_2^* \\ C_K &:= -B_2^* X_{\infty 2} \\ A_K &:= E_K T_X - B_K C_2 X_{\infty 1} = T_Y^* E_K + Y_{\infty 1}^* B_2 C_K. \end{aligned}$$

$\mathcal{H}_\infty$ Control: General Case

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Assumptions:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

$$R := D_{1\bullet}^* D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{1\bullet} := [D_{11} \quad D_{12}]$$

$$\tilde{R} := D_{\bullet 1} D_{\bullet 1}^* - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

$$H_\infty := \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix}$$

$$J_\infty := \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1 D_{\bullet 1}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^* & C \end{bmatrix}$$

$$X_\infty := Ric(H_\infty) \qquad Y_\infty := Ric(J_\infty)$$

$$F := \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1}\left[D_{1\bullet}^* C_1 + B^* X_\infty\right]$$

$$L := \begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} := -[B_1 D_{\bullet 1}^* + Y_\infty C^*]\tilde{R}^{-1}$$

D, $F_{1\infty}$, and $L_{1\infty}$ are Partitioned as follows:

$$\left[\begin{array}{c|c} & F' \\ \hline L' & D \end{array}\right] = \left[\begin{array}{c|ccc} & F_{11\infty}^* & F_{12\infty}^* & F_{2\infty}^* \\ \hline L_{11\infty}^* & D_{1111} & D_{1112} & 0 \\ L_{12\infty}^* & D_{1121} & D_{1122} & I \\ L_{2\infty}^* & 0 & I & 0 \end{array}\right].$$

There exists a stabilizing controller $K(s)$ such that

$$\|\mathcal{F}_\ell(G, K)\|_\infty < \gamma$$

if and only if

(i) $\gamma > max(\overline{\sigma}[D_{1111}, D_{1112},], \overline{\sigma}[D_{1111}^*, D_{1121}^*]);$

(ii) $H_\infty \in dom(Ric)$ with $X_\infty = Ric(H_\infty) \geq 0;$

(iii) $J_\infty \in dom(Ric)$ with $Y_\infty = Ric(J_\infty) \geq 0;$

(iv) $\rho(X_\infty Y_\infty) < \gamma^2.$

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathcal{RH}_\infty, \quad \|Q\|_\infty < \gamma$$

where

$$M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

$$\hat{D}_{11} = -D_{1121} D_{1111}^* (\gamma^2 I - D_{1111} D_{1111}^*)^{-1} D_{1112} - D_{1122},$$

$\hat{D}_{12} \in \mathbb{R}^{m_2 \times m_2}$ and $\hat{D}_{21} \in \mathbb{R}^{p_2 \times p_2}$ are any matrices satisfying

$$\begin{aligned}
\hat{D}_{12}\hat{D}_{12}^* &= I - D_{1121}(\gamma^2 I - D_{1111}^* D_{1111})^{-1} D_{1121}^*, \\
\hat{D}_{21}^*\hat{D}_{21} &= I - D_{1112}^*(\gamma^2 I - D_{1111} D_{1111}^*)^{-1} D_{1112},
\end{aligned}$$

and

$$\begin{aligned}
\hat{B}_2 &= Z_\infty(B_2 + L_{12\infty})\hat{D}_{12}, \\
\hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12\infty}), \\
\hat{B}_1 &= -Z_\infty L_{2\infty} + \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11}, \\
\hat{C}_1 &= F_{2\infty} + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2, \\
\hat{A} &= A + BF + \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2
\end{aligned}$$

where

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

Some Special Cases:

- $D_{12} = I$. Then (i) becomes $\gamma > \overline{\sigma}(D_{1121})$ and

$$\hat{D}_{11} = -D_{1122}, \quad \hat{D}_{12}\hat{D}_{12}^* = I - \gamma^{-2} D_{1121} D_{1121}^*, \quad \hat{D}_{21}^*\hat{D}_{21} = I.$$

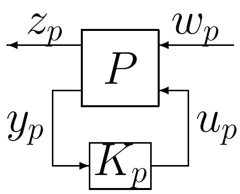
- $D_{21} = I$. Then (i) becomes $\gamma > \overline{\sigma}(D_{1112})$ and

$$\hat{D}_{11} = -D_{1122}, \quad \hat{D}_{12}\hat{D}_{12}^* = I, \quad \hat{D}_{21}^*\hat{D}_{21} = I - \gamma^{-2} D_{1112}^* D_{1112}.$$

- $D_{12} = I$ & $D_{21} = I$. Then (i) drops out and

$$\hat{D}_{11} = -D_{1122}, \quad \hat{D}_{12}\hat{D}_{12}^* = I, \quad \hat{D}_{21}^*\hat{D}_{21} = I.$$

Relaxing Assumptions



$$P(s) = \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array}\right]$$

Assume D_{p12} *has full column rank and* D_{p21} *has full row rank:*

Normalize D_{12} and D_{21}

Perform SVD

$$D_{p12} = U_p \begin{bmatrix} 0 \\ I \end{bmatrix} R_p, \quad D_{p21} = \tilde{R}_p \begin{bmatrix} 0 & I \end{bmatrix} \tilde{U}_p$$

such that U_p and $\tilde{U}_p$ are square and unitary. Now let

$$z_p = U_p z, \quad w_p = \tilde{U}_p^* w, \quad y_p = \tilde{R}_p y, \quad u_p = R_p u$$

$$K(s) = R_p K_p(s) \tilde{R}_p$$

$$\begin{aligned} G(s) &= \begin{bmatrix} U_p^* & 0 \\ 0 & \tilde{R}_p^{-1} \end{bmatrix} P(s) \begin{bmatrix} \tilde{U}_p^* & 0 \\ 0 & R_p^{-1} \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A_p & B_{p1}\tilde{U}_p^* & B_{p2}R_p^{-1} \\ \hline U_p^* C_{p1} & U_p^* D_{p11} \tilde{U}_p^* & U_p^* D_{p12} R_p^{-1} \\ \tilde{R}_p^{-1} C_{p2} & \tilde{R}_p^{-1} D_{p21} \tilde{U}_p^* & \tilde{R}_p^{-1} D_{p22} R_p^{-1} \end{array}\right] \\ &=: \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \end{aligned}$$

Then

$$D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad D_{21} = \begin{bmatrix} 0 & I \end{bmatrix},$$

$$\|\mathcal{F}_\ell(P, K_p)\|_\infty = \|\mathcal{F}_\ell(G, K)\|_\infty$$

Remove the Assumption $D_{22} = 0$

Suppose $K(s)$ is a controller for G with D_{22} set to zero. Then the controller for $D_{22} \neq 0$ is $K(I + D_{22}K)^{-1}$.

Relaxing A3 and A4

Complicated. Suppose that

$$G = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

which violates both A3 and A4 and corresponds to the robust stabilization of an integrator. If the controller $u = -\epsilon x$ where $\epsilon > 0$ is used, then

$$T_{zw} = \frac{-\epsilon s}{s + \epsilon}, \quad \text{with } \|T_{zw}\|_\infty = \epsilon.$$

Hence the norm can be made arbitrarily small as $\epsilon \to 0$, but $\epsilon = 0$ is not stabilizing.

Relaxing A1

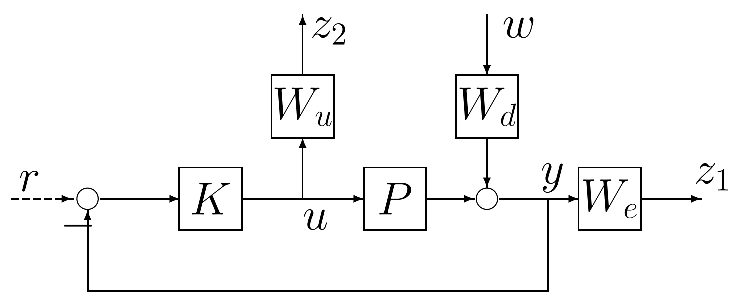
Complicated.

Relaxing A2

Singular Problem: reduced ARE or LMI, ...

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Integral Control

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ design frameworks do not in general produce integral control.



Ways to achieve the integral control:

1. introduce an integral in the performance weight W_e:

$$z_1 = W_e(I + PK)^{-1} W_d w.$$

Now if the norm (2-norm or ∞-norm) between w and z_1 is finite, then K must have a pole at $s = 0$ which is the zero of the sensitivity function.

The standard $\mathcal{H}_2$ (or $\mathcal{H}_\infty$) control theory can not be applied to this problem formulation directly because the pole $s = 0$ of W_e becomes an uncontrollable pole of the feedback system (A1 is violated).

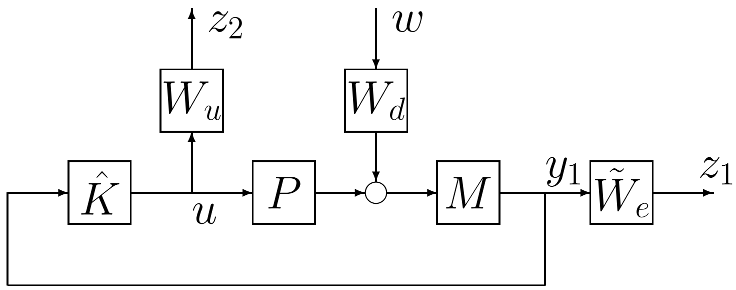
Suppose W_e can be factorized as follows

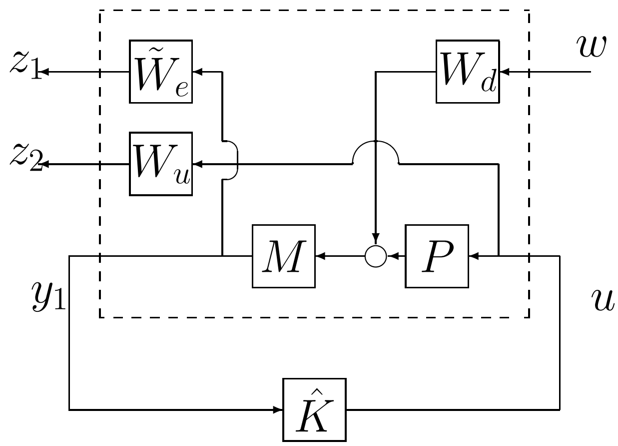
$$W_e = \tilde{W}_e(s) M(s)$$

where $M(s)$ is proper, containing all the imaginary axis poles of W_e, and $M^{-1}(s) \in \mathcal{RH}_\infty$, $\tilde{W}_e(s)$ is stable and minimum phase. Now suppose there exists a controller $K(s)$ which contains the same imaginary axis poles that achieves the performance. Then without loss of generality, K can be factorized as

$$K(s) = -\hat{K}(s) M(s)$$

Now the problem can be reformulated as





A simple numerical example:

$$P = \frac{s-2}{(s+1)(s-3)} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 3 & 2 & 1 \\ \hline -2 & 1 & 0 \end{array}\right], \quad W_d = 1,$$

$$W_u = \frac{s+10}{s+100} = \left[\begin{array}{c|c} -100 & -90 \\ \hline 1 & 1 \end{array}\right], \quad W_e = \frac{1}{s}.$$

Then we can choose without loss of generality that

$$M = \frac{s+\alpha}{s}, \quad \tilde{W}_e = \frac{1}{s+\alpha}, \quad \alpha > 0.$$

This gives the following generalized system

$$G(s) = \left[\begin{array}{ccccc|cc} -\alpha & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -100 & 0 & 0 & 0 & 0 & -90 \\ 0 & 0 & 0 & -2\alpha & \alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 & 0 \end{array}\right]$$

suboptimal $\mathcal{H}_\infty$ controller $\hat{K}_\infty$:

$$\hat{K}_\infty = \frac{-2060381.4(s+1)(s+\alpha)(s+100)(s-0.1557)}{(s+\alpha)^2(s+32.17)(s+262343)(s-19.89)}$$

which gives the closed-loop ∞ norm 7.854.

$$K_\infty = -\hat{K}_\infty(s)M(s) = \frac{2060381.4(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s+262343)(s-19.89)}$$

$$\approx \frac{7.85(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s-19.89)}$$

An optimal $\mathcal{H}_2$ controller

$$\hat{K}_2 = \frac{-43.487(s+1)(s+\alpha)(s+100)(s-0.069)}{(s+\alpha)^2(s^2+30.94s+411.81)(s-7.964)}$$

and

$$K_2(s) = -\hat{K}_2(s)M(s) = \frac{43.487(s+1)(s+100)(s-0.069)}{s(s^2+30.94s+411.81)(s-7.964)}.$$

---

2. An approximate integral control:

$$W_e = \tilde{W}_e = \frac{1}{s+\epsilon}, \quad M(s) = 1$$

for a sufficiently small $\epsilon > 0$. For example, a controller for $\epsilon = 0.001$ is given by

$$\begin{aligned} K_\infty &= \frac{316880(s+1)(s+100)(s-0.1545)}{(s+0.001)(s+32)(s+40370)(s-20)} \\ &\approx \frac{7.85(s+1)(s+100)(s-0.1545)}{s(s+32)(s-20)} \end{aligned}$$

which gives the closed-loop $\mathcal{H}_\infty$ norm of 7.85.

$$K_2 = \frac{43.47(s+1)(s+100)(s-0.0679)}{(s+0.001)(s^2+30.93s+411.7)(s-7.9718)}.$$

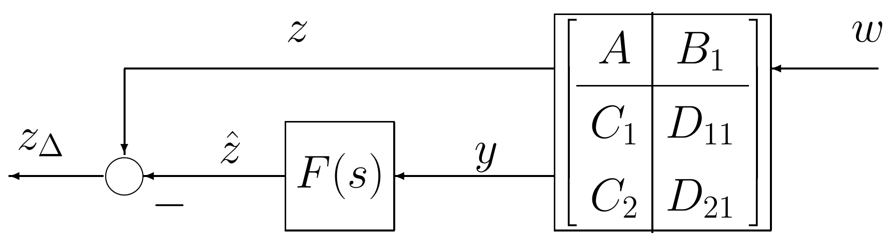
$\mathcal{H}_\infty$ Filtering

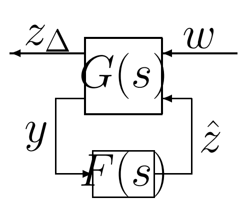
$$
\begin{aligned}
\dot{x} &= Ax + B_1 w(t), \qquad x(0) = 0 \\
y &= C_2 x + D_{21} w(t) \\
z &= C_1 x, \quad B_1 D_{21}^* = 0, \; D_{21} D_{21}^* = I
\end{aligned}
$$

$\mathcal{H}_\infty$ **Filtering**: Given a $\gamma > 0$, find a causal filter $F(s) \in \mathcal{RH}_\infty$ if it exists such that

$$J := \sup_{w \in \mathcal{L}_2[0,\infty)} \frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2} < \gamma^2$$

with $\hat{z} = F(s)y$.





$$G(s) = \left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{array}\right]$$

This can be regarded as a $\mathcal{H}_\infty$ problem without internal stability.

There exists a causal filter $F(s) \in \mathcal{RH}_\infty$ such that $J < \gamma^2$ if and only if $J_\infty \in dom(Ric)$ and $Y_\infty = Ric(J_\infty) \geq 0$

$$\hat{z} = F(s)y = \left[\begin{array}{c|c} A - Y_\infty C_2^* C_2 & Y_\infty C_2^* \\ \hline C_1 & 0 \end{array}\right] y$$

where Y_∞ is the stabilizing solution to

$$Y_\infty A^* + AY_\infty + Y_\infty(\gamma^{-2} C_1^* C_1 - C_2^* C_2)Y_\infty + B_1 B_1^* = 0.$$