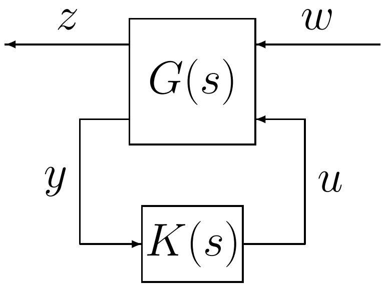
Chapter 15: $\mathcal{H}_\infty$ Controller Reduction

- problem formulation
- additive reduction
- coprime factor reduction

Problem Formulation

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$



All stabilizing controllers satisfying $\|T_{zw}\|_\infty < \gamma$:

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathcal{RH}_\infty, \quad \|Q\|_\infty < \gamma$$

where M_∞ is of the form

$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

such that $\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1$ and $\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$ are both stable, i.e., M_{12}^{-1} and M_{21}^{-1} are both stable.

Find a controller $\hat{K}$ with a minimal order such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$.

$$\Updownarrow$$

Find a stable Q such that $K = \mathcal{F}_\ell(M_\infty, Q)$ has minimal order and $\|Q\|_\infty < \gamma$.

Additive Reduction

Consider the class of (reduced order) controllers:

$$\hat{K} = K_0 + W_2 \Delta W_1, \quad \Delta \in \mathcal{RH}_\infty$$

$$W_1, W_1^{-1}, W_2, W_2^{-1} \in \mathcal{RH}_\infty$$

such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$

$\hat{K}$ and K_0 have the same right half plane poles.

Then

$$\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$$

$$\Updownarrow$$

$$\exists\ Q \in \mathcal{RH}_\infty \text{ with } \|Q\|_\infty < \gamma \text{ such that } \hat{K} = \mathcal{F}_\ell(M_\infty, Q).$$

$$\Downarrow$$

$$Q = \mathcal{F}_\ell(\bar{K}_a^{-1}, \hat{K}), \quad \bar{K}_a^{-1} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M_\infty^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

$$\begin{aligned} \|Q\|_\infty < \gamma \iff & \left\|\mathcal{F}_\ell(\bar{K}_a^{-1}, \hat{K})\right\|_\infty < \gamma \\ \iff & \left\|\mathcal{F}_\ell(\bar{K}_a^{-1}, K_0 + W_2 \Delta W_1)\right\|_\infty < \gamma \\ \iff & \left\|\mathcal{F}_\ell(\tilde{R}, \Delta)\right\|_\infty < 1 \end{aligned}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_2 \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(\bar{K}_a^{-1}, \begin{bmatrix} K_o & I \\ I & 0 \end{bmatrix}).$$

Redheffer's Lemma: $\left\|\tilde{R}\right\|_\infty \le 1$ and $\|\Delta\|_\infty < 1 \Rightarrow \left\|\mathcal{F}_\ell(\tilde{R}, \Delta)\right\|_\infty < 1$.

Suppose W_1 and W_2 are stable, minimum phase and invertible transfer matrices such that $\tilde{R}$ is a contraction. Let K_0 be a stabilizing controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$. Then $\hat{K}$ is also a stabilizing controller such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ if

$$\|\Delta\|_\infty = \left\|W_2^{-1}(\hat{K} - K_0)W_1^{-1}\right\|_\infty < 1.$$

$\tilde{R}$ can always be made contractive for sufficiently small W_1 and W_2. We would like to select the "largest" W_1 and W_2.

Assume $\|R_{22}\|_\infty < \gamma$ and define

$$L = \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix} = \mathcal{F}_\ell(\left[\begin{array}{cc|cc} 0 & -R_{11} & 0 & R_{12} \\ -R_{11}^\sim & 0 & R_{21}^\sim & 0 \\ \hline 0 & R_{21} & 0 & -R_{22} \\ R_{12}^\sim & 0 & -R_{22}^\sim & 0 \end{array}\right], \gamma^{-1}I).$$

Then $\tilde{R}$ is a contraction if W_1 and W_2 satisfy

$$\begin{bmatrix} (W_1^\sim W_1)^{-1} & 0 \\ 0 & (W_2 W_2^\sim)^{-1} \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}.$$

An algorithm that maximizes $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ has been developed by Goddard and Glover [1993].

Coprime Factor Reduction

All controllers such that $\|T_{zw}\|_\infty < \gamma$ can also be written as

$$\begin{aligned} K(s) = \mathcal{F}_\ell(M_\infty, Q) &= (\Theta_{11}Q + \Theta_{12})(\Theta_{21}Q + \Theta_{22})^{-1} := UV^{-1} \\ &= (Q\tilde\Theta_{12} + \tilde\Theta_{22})^{-1}(Q\tilde\Theta_{11} + \tilde\Theta_{21}) := \tilde V^{-1}\tilde U \end{aligned}$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$, and UV^{-1} and $\tilde V^{-1}\tilde U$ are respectively right and left coprime factorizations over $\mathcal{RH}_\infty$, and

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat A - \hat B_1 \hat D_{21}^{-1}\hat C_2 & \hat B_2 - \hat B_1 \hat D_{21}^{-1}\hat D_{22} & \hat B_1 \hat D_{21}^{-1} \\ \hline \hat C_1 - \hat D_{11}\hat D_{21}^{-1}\hat C_2 & \hat D_{12} - \hat D_{11}\hat D_{21}^{-1}\hat D_{22} & \hat D_{11}\hat D_{21}^{-1} \\ -\hat D_{21}^{-1}\hat C_2 & -\hat D_{21}^{-1}\hat D_{22} & \hat D_{21}^{-1} \end{array}\right]$$

$$\tilde\Theta = \begin{bmatrix} \tilde\Theta_{11} & \tilde\Theta_{12} \\ \tilde\Theta_{21} & \tilde\Theta_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat A - \hat B_2 \hat D_{12}^{-1}\hat C_1 & \hat B_1 - \hat B_2 \hat D_{12}^{-1}\hat D_{11} & -\hat B_2 \hat D_{12}^{-1} \\ \hline \hat C_2 - \hat D_{22}\hat D_{12}^{-1}\hat C_1 & \hat D_{21} - \hat D_{22}\hat D_{12}^{-1}\hat D_{11} & -\hat D_{22}\hat D_{12}^{-1} \\ \hat D_{12}^{-1}\hat C_1 & \hat D_{12}^{-1}\hat D_{11} & \hat D_{12}^{-1} \end{array}\right]$$

$$\Theta^{-1} = \left[\begin{array}{c|cc} \hat A - \hat B_2 \hat D_{12}^{-1}\hat C_1 & \hat B_2 \hat D_{12}^{-1} & \hat B_1 - \hat B_2 \hat D_{12}^{-1}\hat D_{11} \\ \hline -\hat D_{12}^{-1}\hat C_1 & \hat D_{12}^{-1} & -\hat D_{12}^{-1}\hat D_{11} \\ \hat C_2 - \hat D_{22}\hat D_{12}^{-1}\hat C_1 & \hat D_{22}\hat D_{12}^{-1} & \hat D_{21} - \hat D_{22}\hat D_{12}^{-1}\hat D_{11} \end{array}\right]$$

$$\tilde\Theta^{-1} = \left[\begin{array}{c|cc} \hat A - \hat B_1 \hat D_{21}^{-1}\hat C_2 & -\hat B_1 \hat D_{21}^{-1} & \hat B_2 - \hat B_1 \hat D_{21}^{-1}\hat D_{22} \\ \hline \hat D_{21}^{-1}\hat C_2 & \hat D_{21}^{-1} & \hat D_{21}^{-1}\hat D_{22} \\ \hat C_1 - \hat D_{11}\hat D_{21}^{-1}\hat C_2 & -\hat D_{11}\hat D_{21}^{-1} & \hat D_{12} - \hat D_{11}\hat D_{21}^{-1}\hat D_{22} \end{array}\right].$$

Let $K_0 = \Theta_{12}\Theta_{22}^{-1}$ be the central $\mathcal{H}_\infty$ controller: $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$
Let $\hat{U}, \hat{V} \in \mathcal{RH}_\infty$ with $\det \hat{V}(\infty) \neq 0$ be such that

$$\left\| \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{U}\hat{V}^{-1}$ is also a stabilizing controller and $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$.

Note that K is a stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if there exists a $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$ such that

$$\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} \Theta_{11}Q + \Theta_{12} \\ \Theta_{21}Q + \Theta_{22} \end{bmatrix} = \Theta \begin{bmatrix} Q \\ I \end{bmatrix} \tag{0.14}$$

and

$$K = UV^{-1}.$$

Define

$$\Delta := \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right)$$

and partition Δ as

$$\Delta := \begin{bmatrix} \Delta_U \\ \Delta_V \end{bmatrix}.$$

Then

$$\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} = \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \Theta \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix} \Delta = \Theta \begin{bmatrix} -\gamma \Delta_U \\ I - \Delta_V \end{bmatrix}$$

and

$$\begin{bmatrix} \hat{U}(I - \Delta_V)^{-1} \\ \hat{V}(I - \Delta_V)^{-1} \end{bmatrix} = \Theta \begin{bmatrix} -\gamma \Delta_U (I - \Delta_V)^{-1} \\ I \end{bmatrix}.$$

Define

$$U := \hat{U}(I - \Delta_V)^{-1}, \quad V := \hat{V}(I - \Delta_V)^{-1}$$

$$Q := -\gamma \Delta_U (I - \Delta_V)^{-1}$$

Then $\hat{K} = \hat{U}\hat{V}^{-1} = UV^{-1}$ and

$$\begin{aligned} Q := -\gamma \Delta_U (I - \Delta_V)^{-1} &= -\gamma \begin{bmatrix} I & 0 \end{bmatrix} \Delta \left(I - \begin{bmatrix} 0 & I \end{bmatrix} \Delta \right)^{-1} \\ &= -\gamma \mathcal{F}_\ell \left(\begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}, \sqrt{2}\Delta \right) \end{aligned}$$

Again by Redheffer's Lemma, $\|\Delta_U (I - \Delta_V)^{-1}\|_\infty < 1$ since

$$\begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}$$

is a contraction and $\left\| \sqrt{2}\Delta \right\|_\infty < 1$.

$$\Longrightarrow \|Q\|_\infty = \left\| \gamma \Delta_U (I - \Delta_V)^{-1} \right\|_\infty < \gamma$$

Therefore $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$.

Let $K_0 = \tilde{\Theta}_{22}^{-1} \tilde{\Theta}_{21}$ be the central $\mathcal{H}_\infty$ controller: $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$
Let $\hat{\tilde{U}}, \hat{\tilde{V}} \in \mathcal{RH}_\infty$ with $\det \hat{\tilde{V}}(\infty) \neq 0$ be such that

$$\left\| \left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \tilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{\tilde{V}}^{-1} \hat{\tilde{U}}$ is also a stabilizing controller and $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$.

sufficient conditions:
$\mathcal{H}_\infty$ controller reduction $\Longrightarrow$ frequency weighted $\mathcal{H}_\infty$ model reduction.