Chapter 16: $\mathcal{H}_\infty$ Loop Shaping

- Robust Stabilization of Coprime factors
- Robust Stabilization of Normalized Coprime Factors
- $\mathcal{H}_\infty$ Loop Shaping Design
- Weighted $\mathcal{H}_\infty$ Control Interpretation
- Further Guidelines for Loop Shaping
Robust Stabilization of Coprime Factors

Robust Stabilization Condition:
Let $P = \tilde{M}^{-1}\tilde{N}$ be the nominal model and

$$P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$$

with $\tilde{M}, \tilde{N}, \tilde{\Delta}_M, \tilde{\Delta}_N \in \mathcal{RH}_\infty$ and $
\left\| \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix} \right\|_\infty < \epsilon$.

The perturbed system is robustly stable iff

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} \left( I + PK \right)^{-1} \tilde{M}^{-1} \right\|_\infty \leq 1/\epsilon.$$

State Space Coprime Factorization:
Let

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and let $L$ be such that $A + LC$ is stable. Then

$$P = \tilde{M}^{-1}\tilde{N}, \quad \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} A + LC & B + LD & L \\ C & D & I \end{bmatrix}.$$

Denote

$$\hat{K} = -K$$
**LFT framework:**

\[
G(s) = \begin{bmatrix}
0 & I \\
\tilde{M}^{-1} & P \\
\end{bmatrix} = \begin{bmatrix}
A & -L & B \\
0 & I & D \\
C & I & D \\
\end{bmatrix} =: \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22} \\
\end{bmatrix}.
\]

Controller for a Special Case: \( D = 0 \).

\[
\left\| \begin{bmatrix}
K \\
I \\
\end{bmatrix} (I + PK)^{-1} \tilde{M}^{-1} \right\|_{\infty} < \gamma
\]

iff \( \gamma > 1 \) and there exists a stabilizing solution \( X_\infty \geq 0 \) solving

\[
X_\infty \left( A - \frac{LC}{\gamma^2 - 1} \right) + \left( A - \frac{LC}{\gamma^2 - 1} \right)^* X_\infty - X_\infty \left( BB^* - \frac{LL^*}{\gamma^2 - 1} \right) X_\infty + \frac{\gamma^2 C^* C}{\gamma^2 - 1} = 0.
\]

a central controller:

\[
K = \begin{bmatrix}
A - BB^* X_\infty + LC & L \\
- B^* X_\infty & 0 \\
\end{bmatrix}.
\]
Suppose \( \tilde{M} \) and \( \tilde{N} \) are normalized coprime factors
\[
\tilde{M}(j\omega)\tilde{M}^*(j\omega) + \tilde{N}(j\omega)\tilde{N}^*(j\omega) = I
\]
Then \( \tilde{M} \) and \( \tilde{N} \) can be obtained as
\[
\begin{bmatrix}
\tilde{N} & \tilde{M}
\end{bmatrix} = \begin{bmatrix}
A - YC^*C & B - YC^* \\
C & 0 & I
\end{bmatrix}
\]
where \( L = -YC^* \) and \( Y \geq 0 \) is the stabilizing solution to
\[
AY + YA^* - YC^*CY + BB^* = 0
\]
Moreover, for any \( \gamma > \gamma_{\text{min}} \) a controller achieving
\[
\gamma_{\text{min}} := \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \frac{1}{\sqrt{1 - \lambda_{\text{max}}(YQ)}}
\]
where \( \lambda_{\text{max}}(YQ) \) is the solution to
\[
Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0.
\]

Moreover, for any \( \gamma > \gamma_{\text{min}} \) a controller achieving
\[
\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma
\]
is given by
\[
K(s) = \begin{bmatrix}
A - BB^*X_\infty - YC^*C & -YC^* \\
-B^*X_\infty & 0
\end{bmatrix}
\]
where
\[
X_\infty = \frac{\gamma^2}{\gamma^2 - 1}Q \left( I - \frac{\gamma^2}{\gamma^2 - 1}YQ \right)^{-1}
\]
\[ P = \tilde{M}^{-1}\tilde{N} \]
be a normalized left coprime factorization and
\[ P_{\Delta} = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N) \]
with
\[ \left\| \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix} \right\|_\infty < \epsilon. \]
Then there is a robustly stabilizing controller for \( P_{\Delta} \) if and only if
\[ \epsilon \leq \sqrt{1 - \lambda_{\text{max}}(YQ)} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2}. \]

Let \( X \geq 0 \) be the stabilizing solution to
\[ XA + A^*X - XBB^*X + C^*C = 0 \]
then
\[ Q = (I + XY)^{-1}X \]
and
\[ \gamma_{\text{min}} = \frac{1}{\sqrt{1 - \lambda_{\text{max}}(YQ)}} = \left(1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2 \right)^{-1/2} = \sqrt{1 + \lambda_{\text{max}}(XY)}. \]

Let \( P = \tilde{M}^{-1}\tilde{N} \) be a normalized left coprime factorization. Then
\[ \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I \\ P \end{bmatrix} \right\|_\infty. \]

\[ \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I \\ P \end{bmatrix} \right\| = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I + KP)^{-1} \begin{bmatrix} I \\ K \end{bmatrix} \right\|. \]

Let \( P = \tilde{M}^{-1}\tilde{N} = NM^{-1} \) be respectively the normalized left and right coprime factorizations. Then
\[ \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \left\| M^{-1}(I + KP)^{-1} \begin{bmatrix} I \\ K \end{bmatrix} \right\|_\infty. \]
Loop Shaping Design Procedure

(1) Loop Shaping: Using a precompensator $W_1$ and/or a postcompensator $W_2$, the singular values of the nominal plant are shaped to give a desired open-loop shape.

$$P_s = W_2PW_1$$

Assume that $W_1$ and $W_2$ are such that $P_s$ contains no hidden modes.

(2) Robust Stabilization: a) Calculate $\epsilon_{max}$, where

$$\epsilon_{max} = \left( \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} I & K \end{bmatrix} (I + P_sK)^{-1}\tilde{M}_s^{-1}  \right\|_{\infty} \right)^{-1}$$

$$= \sqrt{1 - \left\| \begin{bmatrix} \tilde{N}_s & \tilde{M}_s \end{bmatrix} \right\|_H^2} < 1$$
\[ P_s = \tilde{M}_s^{-1} \tilde{N}_s \text{ and} \]
\[ \tilde{M}_s(j\omega)\tilde{M}_s^*(j\omega) + \tilde{N}_s(j\omega)\tilde{N}_s^*(j\omega) = I. \]

If \( \epsilon_{max} \ll 1 \) return to (1) and adjust \( W_1 \) and \( W_2 \).

b) Select \( \epsilon \leq \epsilon_{max} \), then synthesize a stabilizing controller \( K_\infty \), which satisfies
\[
\left\| \begin{bmatrix} I & (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \end{bmatrix} \right\|_\infty \leq \epsilon^{-1}.
\]

(3) The final controller \( K \)
\[
K = W_1 K_\infty W_2.
\]

A typical design works as follows: the designer inspects the open-loop singular values of the nominal plant, and shapes these by pre- and/or postcompensation until nominal performance (and possibly robust stability) specifications are met. (Recall that the open-loop shape is related to closed-loop objectives.) A feedback controller \( K_\infty \) with associated stability margin (for the shaped plant) \( \epsilon \leq \epsilon_{max} \), is then synthesized. If \( \epsilon_{max} \) is small, then the specified loop shape is incompatible with robust stability requirements, and should be adjusted accordingly, then \( K_\infty \) is reevaluated.
Weighted $\mathcal{H}_\infty$ Control Interpretation

\[
\begin{bmatrix}
I \\
K_\infty
\end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1}
= \begin{bmatrix}
I \\
K_\infty
\end{bmatrix} (I + P_s K_\infty)^{-1}
\begin{bmatrix}
I & P_s
\end{bmatrix}
\]

\[
= \begin{bmatrix}
W_2 \\
W_1^{-1}
\end{bmatrix} \begin{bmatrix}
I \\
K
\end{bmatrix} (I + P K)^{-1}
\begin{bmatrix}
I & P
\end{bmatrix}
\begin{bmatrix}
W_2^{-1} \\
W_1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
W_1^{-1} \\
W_2
\end{bmatrix} \begin{bmatrix}
I \\
P
\end{bmatrix} (I + K P)^{-1}
\begin{bmatrix}
I & P
\end{bmatrix}
\begin{bmatrix}
W_1 \\
W_2^{-1}
\end{bmatrix}
\]

This shows how all the closed-loop objective are incorporated.

\[
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
W_2 \\
W_1^{-1}
\end{bmatrix} \begin{bmatrix}
I \\
K
\end{bmatrix} (I + P K)^{-1}
\begin{bmatrix}
I & P
\end{bmatrix}
\begin{bmatrix}
W_2^{-1} \\
W_1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\]
Define
\[ b_{P,K} := \begin{cases} \left( \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty \right)^{-1} & \text{if } K \text{ stabilizes } P \\ 0 & \text{otherwise} \end{cases} \]
and
\[ b_{opt} := \sup_K b_{P,K}. \]
Then \( b_{P,K} = b_{K,P} \) and
\[ b_{opt} = \sqrt{1 - \lambda_{\max}(YQ)} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2}. \]

SISO \( P \):

\[
\text{gain margin} \geq \frac{1 + b_{P,K}}{1 - b_{P,K}}
\]
and
\[
\text{phase margin} \geq 2 \arcsin(b_{P,K}).
\]

**Proof.** Note that for SISO system
\[
b_{P,K} \leq \frac{|1 + P(j\omega)K(j\omega)|}{\sqrt{1 + |P(j\omega)|^2} \sqrt{1 + |K(j\omega)|^2}}, \quad \forall \omega.
\]
So, at frequencies where \( k := -P(j\omega)K(j\omega) \in \mathbb{R}^+, \)
\[
b_{P,K} \leq \frac{|1 - k|}{\sqrt{(1 + |P|^2)(1 + \frac{k^2}{|P|^2})}} \leq \frac{|1 - k|}{\min_P \left\{ (1 + |P|^2)(1 + \frac{k^2}{|P|^2}) \right\}} = \frac{1 - k}{1 + k},
\]
which implies that
\[
k \leq \frac{1 - b_{P,K}}{1 + b_{P,K}}, \quad \text{or} \quad k \geq \frac{1 + b_{P,K}}{1 - b_{P,K}}.
\]
from which the gain margin result follows. Similarly, at frequencies where
\( P(j\omega)K(j\omega) = -e^{j\theta} \),

\[
b_{P,K} \leq \frac{|1 - e^{j\theta}|}{\sqrt{(1 + |P|^2)(1 + \frac{1}{|P|^2})}} \leq \frac{|2 \sin \frac{\theta}{2}|}{\sqrt{\min_P \left\{ (1 + |P|^2)(1 + \frac{1}{|P|^2}) \right\}}} = \frac{|2 \sin \frac{\theta}{2}|}{2},
\]

which implies \( \theta \geq 2 \arcsin b_{P,K} \). \( \square \)

For example, \( b_{P,K} = 1/2 \) guarantees a gain margin of 3 and a phase
margin of 60°.

\[
\gg b_{p,k} = \text{emargin}(P, K); \quad \% \text{given } P \text{ and } K, \text{ compute } b_{P,K}.
\]
\[
\gg [K_{opt}, b_{p,k}] = \text{ncfsyn}(P, 1); \quad \% \text{find the optimal controller } K_{opt}.
\]
\[
\gg [K_{sub}, b_{p,k}] = \text{ncfsyn}(P, 2); \quad \% \text{find a suboptimal controller } K_{sub}.
\]
Further Guidelines for Loop Shaping

\[ P = NM^{-1} \]: normalized right coprime factorization.

\[ b_{\text{opt}}(P) \leq \lambda(P) := \inf_{\Re s > 0} \sigma \left( \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} \right). \]

small \( \lambda(P) \implies \) small \( b_{\text{opt}}(P) \).

open right-half plane zeros and poles of \( P \):

\[ z_1, z_2, \ldots, z_m, \quad p_1, p_2, \ldots, p_k \]

Define

\[
N_z(s) = \frac{z_1 - s}{z_1 + s} \frac{z_2 - s}{z_2 + s} \cdots \frac{z_m - s}{z_m + s}, \quad
N_p(s) = \frac{p_1 - s}{p_1 + s} \frac{p_2 - s}{p_2 + s} \cdots \frac{p_k - s}{p_k + s}.
\]

Then

\[ P(s) = P_0(s) \frac{N_z(s)}{N_p(s)} \]

where \( P_0(s) \) has no open right-half plane poles or zeros.

Let \( N_0(s) \) and \( M_0(s) \) be stable and minimum phase spectral factors:

\[
N_0(s) N_0^\sim(s) = \left(1 + \frac{1}{P(s) P^\sim(s)}\right)^{-1}, \quad
M_0(s) M_0^\sim(s) = (1 + P(s) P^\sim(s))^{-1}.
\]

Then \( P_0 = N_0/M_0 \) is a normalized coprime factorization and \((N_0 N_z)\) and \((M_0 N_p)\) form a pair of normalized coprime factorizations of \( P \). Thus

\[
b_{\text{opt}}(P) \leq \sqrt{|N_0(s) N_z(s)|^2 + |M_0(s) N_p(s)|^2}, \quad \forall \Re(s) > 0.
\]

\[
\ln |N_0(re^{j\theta})| = \int_{-\infty}^{\infty} \ln \left( \frac{1}{\sqrt{1 + 1/|P(j\omega)|^2}} \right) K_\theta(\omega/r) \, d(\ln \omega)
\]

\[
\ln |M_0(re^{j\theta})| = \int_{-\infty}^{\infty} \ln \left( \frac{1}{\sqrt{1 + |P(j\omega)|^2}} \right) K_\theta(\omega/r) \, d(\ln \omega)
\]
where $r > 0$, $-\pi/2 < \theta < \pi/2$, and

$$K_\theta(\omega/r) = \frac{1}{\pi} \frac{2(\omega/r)[1 + (\omega/r)^2] \cos \theta}{[1 - (\omega/r)^2]^2 + 4(\omega/r)^2 \cos^2 \theta}$$

$K_\theta(\omega/r)$ large near $\omega = r$: $|N_0(re^{j\theta})|$ will be small if $|P(j\omega)|$ is small near $\omega = r$ and $|M_0(re^{j\theta})|$ will be small if $|P(j\omega)|$ is large near $\omega = r$.

Large $\theta$: $K_\theta(\omega/r)$ very near $\omega = r$ and small otherwise. Hence $|N_0(re^{j\theta})|$ and $|M_0(re^{j\theta})|$ will essentially be determined by $|P(j\omega)|$ in a very narrow frequency range near $\omega = r$ when $\theta$ is large. On the other hand, when $\theta$ is small, a larger range of frequency response $|P(j\omega)|$ around $\omega = r$ will have affect on the value $|N_0(re^{j\theta})|$ and $|M_0(re^{j\theta})|$. (This, in fact, will imply that a right-plane zero (pole) with a much larger real part than the imaginary part will have much worse effect on the performance than a right-plane zero (pole) with a much larger imaginary part than the real part.)
When can $b_{\text{opt}}(P)$ be small

Let $s = re^{j\theta}$ and note that $N_z(z_i) = 0$ and $N_p(p_j) = 0$. Then the bound can be small if

- $|N_z(s)|$ and $|N_p(s)|$ are both small for some $s$. That is, $|N_z(s)| \approx 0$ (i.e., $s$ is close to a right-half plane zero of $P$) and $|N_p(s)| \approx 0$ (i.e., $s$ is close to a right-half plane pole of $P$). This is only possible if $P(s)$ has a right-half plane zero near a right-half plane pole. (See Example 0.1.)

- $|N_z(s)|$ and $|M_0(s)|$ are both small for some $s$. That is, $|N_z(s)| \approx 0$ (i.e., $s$ is close to a right-half plane zero of $P$) and $|M_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is large around $\omega = |s| = r$). This is only possible if $|P(j\omega)|$ is large around $\omega = r$, where $r$ is the modulus of a right-half plane zero of $P$. (See Example 0.2.)

- $|N_p(s)|$ and $|N_0(s)|$ are both small for some $s$. That is, $|N_p(s)| \approx 0$ (i.e., $s$ is close to a right-half plane pole of $P$) and $|N_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is small around $\omega = |s| = r$). This is only possible if $|P(j\omega)|$ is small around $\omega = r$, where $r$ is the modulus of a right-half plane pole of $P$. (See Example 0.3.)

- $|N_0(s)|$ and $|M_0(s)|$ are both small for some $s$. That is, $|N_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is small around $\omega = |s| = r$) and $|M_0(s)| \approx 0$ (i.e., $|P(j\omega)|$ is large around $\omega = |s| = r$). The only way in which $|P(j\omega)|$ can be both small and large at frequencies near $\omega = r$ is that $|P(j\omega)|$ is approximately equal to 1 and the absolute value of the slope of $|P(j\omega)|$ is large. (See Example 0.4.)
Example 0.1

\[ P_1(s) = \frac{K(s - r)}{(s + 1)(s - 1)}. \]

\( b_{opt}(P_1) \) will be very small for all \( K \) whenever \( r \) is close to 1 (i.e., whenever there is an unstable pole close to an unstable zero).

<table>
<thead>
<tr>
<th>( K = 0.1 )</th>
<th>( r )</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{opt}(P_1) )</td>
<td>0.0125</td>
<td>0.0075</td>
<td>0.0025</td>
<td>0.0025</td>
<td>0.0074</td>
<td>0.0124</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( K = 1 )</th>
<th>( r )</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{opt}(P_1) )</td>
<td>0.1036</td>
<td>0.0579</td>
<td>0.0179</td>
<td>0.0165</td>
<td>0.0457</td>
<td>0.0706</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( K = 10 )</th>
<th>( r )</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{opt}(P_1) )</td>
<td>0.0658</td>
<td>0.0312</td>
<td>0.0088</td>
<td>0.0077</td>
<td>0.0208</td>
<td>0.0318</td>
<td></td>
</tr>
</tbody>
</table>

Figure 0.30: Frequency responses of \( P_1 \) for \( r = 0.9 \) and \( K = 0.1, 1, \) and 10
Nonminimum Phase

Example 0.2

\[ P_2(s) = \frac{K(s - 1)}{s(s + 1)}. \]

\( b_{\text{opt}}(P_2) \) will be small if \(|P_2(j\omega)|\) is large around \( \omega = 1 \), the modulus of the right-half plane zero.

<table>
<thead>
<tr>
<th>( K )</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{\text{opt}}(P_2) )</td>
<td>0.7001</td>
<td>0.6451</td>
<td>0.3827</td>
<td>0.0841</td>
<td>0.0098</td>
</tr>
</tbody>
</table>

Figure 0.31: Frequency responses of \( P_2 \) and \( P_3 \) for \( K = 0.1, 1, \) and 10

Note that \( b_{\text{opt}}(L/s) = 0.707 \) for any \( L \) and \( b_{\text{opt}}(P_2) \to 0.707 \) as \( K \to 0 \). This is because \(|P_2(j\omega)|\) around the frequency of the right-half plane zero is very small as \( K \to 0 \).
Complex Nonminimum Phase Zeros

\[ P_3(s) = \frac{K[(s - \cos \theta)^2 + \sin^2 \theta]}{s[(s + \cos \theta)^2 + \sin^2 \theta]}. \]

<table>
<thead>
<tr>
<th>$K = 0.1$</th>
<th>$\theta$ (degree)</th>
<th>0</th>
<th>45</th>
<th>60</th>
<th>80</th>
<th>85</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$b_{\text{opt}}(P_3)$</td>
<td>0.5952</td>
<td>0.6230</td>
<td>0.6447</td>
<td>0.6835</td>
<td>0.6950</td>
</tr>
<tr>
<td>$K = 1$</td>
<td>$\theta$ (degree)</td>
<td>0</td>
<td>45</td>
<td>60</td>
<td>80</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>$b_{\text{opt}}(P_3)$</td>
<td>0.2588</td>
<td>0.3078</td>
<td>0.3568</td>
<td>0.4881</td>
<td>0.5512</td>
</tr>
<tr>
<td>$K = 10$</td>
<td>$\theta$ (degree)</td>
<td>0</td>
<td>45</td>
<td>60</td>
<td>80</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>$b_{\text{opt}}(P_3)$</td>
<td>0.0391</td>
<td>0.0488</td>
<td>0.0584</td>
<td>0.0813</td>
<td>0.0897</td>
</tr>
</tbody>
</table>

- $b_{\text{opt}}(P_3)$ will be small if $|P_3(j\omega)|$ is large around the frequency of $\omega = 1$ (the modulus of the right-half plane zero).

- for zeros with the same modulus, $b_{\text{opt}}(P_3)$ will be smaller for a plant with relatively larger real part zeros than for a plant with relatively larger imaginary part zeros (i.e., a pair of real right-half plane zeros has a much worse effect on the performance than a pair of almost pure imaginary axis right-half plane zeros of the same modulus).
Example 0.3

\[ P_4(s) = \frac{K(s + 1)}{s(s - 1)}. \]

\( b_{\text{opt}}(P_4) \) will be small if \( |P_4(j\omega)| \) is small around \( \omega = 1 \) (the modulus of the right-half plane pole).

<table>
<thead>
<tr>
<th>( K )</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{\text{opt}}(P_4) )</td>
<td>0.0098</td>
<td>0.0841</td>
<td>0.3827</td>
<td>0.6451</td>
<td>0.7001</td>
</tr>
</tbody>
</table>

Note that \( b_{\text{opt}}(P_4) \to 0.707 \) as \( K \to \infty \). This is because \( |P_4(j\omega)| \) is very large around the frequency of the modulus of the right-half plane pole as \( K \to \infty \).

\[ P_5(s) = \frac{K[(s + \cos \theta)^2 + \sin^2 \theta]}{s[(s - \cos \theta)^2 + \sin^2 \theta]}. \]

The optimal \( b_{\text{opt}}(P_5) \) for various \( \theta \)'s are listed in the following table:

<table>
<thead>
<tr>
<th>( K ) = 0.1</th>
<th>( \theta ) (degree)</th>
<th>0</th>
<th>45</th>
<th>60</th>
<th>80</th>
<th>85</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{\text{opt}}(P_5) )</td>
<td>0.0391</td>
<td>0.0488</td>
<td>0.0584</td>
<td>0.0813</td>
<td>0.0897</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( K ) = 1</th>
<th>( \theta ) (degree)</th>
<th>0</th>
<th>45</th>
<th>60</th>
<th>80</th>
<th>85</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{\text{opt}}(P_5) )</td>
<td>0.2588</td>
<td>0.3078</td>
<td>0.3568</td>
<td>0.4881</td>
<td>0.5512</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( K ) = 10</th>
<th>( \theta ) (degree)</th>
<th>0</th>
<th>45</th>
<th>60</th>
<th>80</th>
<th>85</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{\text{opt}}(P_5) )</td>
<td>0.5952</td>
<td>0.6230</td>
<td>0.6447</td>
<td>0.6835</td>
<td>0.6950</td>
<td></td>
</tr>
</tbody>
</table>

- \( b_{\text{opt}}(P_5) \) will be small if \( |P_5(j\omega)| \) is small around the frequency of the modulus of the right-half plane pole.
• for poles with the same modulus, $b_{\text{opt}}(P_5)$ will be smaller for a plant with relatively larger real part poles than for a plant with relatively larger imaginary part poles (i.e., a pair of real right-half plane poles has a much worse effect on the performance than a pair of almost pure imaginary axis right-half plane poles of the same modulus).
Large Slope near Crossover

Example 0.4

\[ P_6(s) = \frac{K(0.2s + 1)^4}{s(s + 1)^4}. \]

![Graph showing frequency response of \( P_6 \) for various values of \( K \).]

Figure 0.32: Frequency response of \( P_6 \) for \( K = 10^{-5}, 10^{-1} \) and \( 10^4 \)

- \( K = 10^{-5} \): slope near crossover is not too large \( \implies b_{\text{opt}}(P_6) \) not too small.
- \( K = 10^4 \): Similar.
- \( K = 0.1 \): slope near crossover is quite large \( \implies b_{\text{opt}}(P_6) \) quite small.

<table>
<thead>
<tr>
<th>( K )</th>
<th>( 10^{-5} )</th>
<th>( 10^{-3} )</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>( 10^2 )</th>
<th>( 10^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{\text{opt}}(P_6) )</td>
<td>0.3566</td>
<td>0.0938</td>
<td>0.0569</td>
<td>0.0597</td>
<td>0.0765</td>
<td>0.1226</td>
<td>0.4933</td>
</tr>
</tbody>
</table>
Guidelines

Based on the preceding discussion, we can give some guidelines for the loop-shaping design.

✧ The loop transfer function should be shaped in such a way that it has low gain around the frequency of the modulus of any right-half plane zero $z$. Typically, it requires that the crossover frequency be much smaller than the modulus of the right-half plane zero; say, $\omega_c < |z|/2$ for any real zero and $\omega_c < |z|$ for any complex zero with a much larger imaginary part than the real part (see Figure 0.29).

✧ The loop transfer function should have a large gain around the frequency of the modulus of any right-half plane pole.

✧ The loop transfer function should not have a large slope near the crossover frequencies.

These guidelines are consistent with the rules used in classical control theory (see Bode [1945] and Horowitz [1963]).