Collective motion and mobile sensor networks

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Outline

- Motivation of the investigation of collective motion
- Oscillator model and collective motion
- Laplacian quadratic forms and Laplacian potentials
- Synchronized and balanced circular formations, symmetric circular formations and Symmetry breaking
Engineering motivation

- Study of multi-vehicle and mobile sensor networks
Engineering motivation

- Different of Mobile sensor networks and Static sensor networks
  - Monitor large areas
  - Intelligent data-gathering

- Application of mobile sensor networks
  - Search, mapping and environmental monitoring
  - Land, air, space and underwater
Engineering motivation

- Central objective
  Optimal data collection with limit resources and uncertain environment

- The key in optimal data collection
  The coordination of the sensors in the network to get better trajectories
Scientific motivation

- Analysis of emergent and self-organized swarming behaviors in biological groups.
Oscillator model

- Particles with coupled oscillator dynamics (PCOD)

Parties: with unit mass and unit speed, identified with their positions and directions of motion.

For $k = 1, \ldots, N$, the position of the $k$th particle is $r_k = x_k + iy_k \in \mathbb{C}$

the velocity of the $k$th particle is $e^{i\theta_k} = \cos \theta_k + i \sin \theta_k$,

$\theta_k$ is the phase of the $k$th particle, $u_k$ is the steering control.

And the particle model is

- $r_k = e^{i\theta_k}$,
- $\theta_k = u_k (r, \theta), k = 1, \ldots, N.$
Particle model:

- Particles with coupled oscillator dynamics
- If $u_k = 0$ for all the particles, then the particles travel in a straight line in its initial direction
- If $u_k = \omega_0$ for all the partials, then each particle travels around a circle
- The center of the kth circle is $c_k = r_k + \omega_0^{-1} i e^{i\theta}$
- The center of mass and the velocity of the particle group are

\[
R = \frac{1}{N} \sum_{j=1}^{N} r_j, \quad \dot{R} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}
\]
Particle model:

\[ r_k = e^{i\theta_k}, \]
\[ \theta_k = u_k (r, \theta), k = 1, ..., N. \]

Two types of relative equilibria formations:

- Parallel formation: particles travels in a constant and identical direction, which means
  \[ \dot{\theta}_k = 0 \quad \text{and} \quad \theta_k = \theta_j \quad \text{for all pairs } j \text{ and } k \]
- Circular formation: particles travels around the same circle
  \[ \dot{\theta}_{kj} = 0 \quad \text{and} \quad c_k = c_j \quad \text{for all pairs } j \text{ and } k \]

Compare to synchronization and balanced state

- Synchronized phase arrangement: all of the particles have the same direction of motion.
  \[ \dot{\theta}_{kj} = 0 \quad \theta_k = \theta_j \]
- Balanced phase arrangement: the group have a fixed center of mass.
Phase model

- We split the control $u_k$ into three terms

$$u_k = \omega_0 + u_k^{\text{spac}}(r, \theta) + u_k^{\text{ori}}(\theta), \ k = 1, \ldots, N,$$

where $\omega_0 \in \mathbb{R}$ is a constant, $u_k^{\text{spac}}(r, \theta)$ is the spacing control, and $u_k^{\text{ori}}(\theta)$ is the orientation control. By ignoring the particle positions and setting $u_k^{\text{spac}} = 0$, we obtain the phase model

$$\dot{\theta}_k = \omega_0 + u_k^{\text{ori}}(\theta), \ k = 1, \ldots, N,$$
Phase model

- Phase order parameter
  \[ p_\theta = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \]

- The magnitude
  \[ 0 \leq |p_\theta| \leq 1 \]

- Synchronized phase
  \[ |p_\theta| = 1 \]

- Balanced phase
  \[ |p_\theta| = 0 \]

\[
\dot{R} = \frac{1}{N} \sum_{j=1}^{N} r_j, \quad \ddot{R} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}, \quad p_\theta = \dot{R}
\]
Phase model

- Define phase potential
  \[ U_1(\theta) = \frac{N}{2} |p_\theta|^2 \]

- The gradient of
  \[ \frac{\partial U_1}{\partial \theta_k} = <ie^{i\theta_k}, p_\theta> \]
  where \(<x, y> = \text{Re}\{x^*y\}\) , * denotes the conjugate transpose, \(\text{Re}\{\bullet\}\) is the real part of a complex number

- The orientation control
  \[ u_{k}^{ori} = -K_1 \frac{\partial U_1}{\partial \theta_k} = -K_1 <ie^{i\theta_k}, p_\theta>, k = 1, \ldots, N. \]
Phase model

- From the orientation control in last page, it follows that

\[ \dot{U}_1(\theta) = \frac{\partial U_1^T}{\partial \theta} \dot{\theta} = -K_1 \sum_{k=1}^{N} < ie^{i\theta_k}, p_\theta >^2 \]

- The phase model

\[ \dot{\theta}_k = \omega_0 + u_{ori}^k(\theta), k = 1,..., N. \]

with the orientation control

\[ u_{ori}^k = -K_1 < e^{i\theta_k}, p_\theta >, k = 1,..., N. \]

is equivalent to

\[ \dot{\theta}_k = \omega_0 + \frac{K_1}{N} \sum_{j=1}^{N} \sin \theta_{kj} \]
Phase model: \[ \dot{\theta}_k = \omega_0 + \frac{K_1}{N} \sum_{j=1}^{N} \sin \theta_{kj} \]

- When \( K_1 < 0 \), the phase model is a Kuramoto model with identical natural frequencies, and it stabilizes synchronization phase arrangement.
- When \( K_1 > 0 \), the phase model stabilizes balanced phase arrangement.
- The position of particles: red circle
- The velocity: black arrow
- The center of mass: black circle marked by X

(a) \( \omega_0 = 0 \), and \( K_1 < 0 \)
(b) \( \omega_0 = 0 \), and \( K_1 > 0 \)
(c) \( \omega_0 \neq 0 \), and \( K_1 < 0 \)
(d) \( \omega_0 \neq 0 \), and \( K_1 > 0 \)
Laplacian quadratic forms

- All-to-all communication \[\rightarrow\] Limited interaction
- G: a graph that describes the interaction network (orientation or spacing interaction network)
- L: the Laplacian matrix of G

Let \[
\sum_{j=1}^{N} \langle x_j, y_j \rangle
\]

where \(x = (x_1,...,x_N)^T \in \mathbb{C}^N\) \(y = (y_1,...,y_N)^T \in \mathbb{C}^N\)

- Associated with L is the Laplacian quadratic form

\[
Q_L(x) = \frac{1}{2N} \langle x, Lx \rangle
\]
### Laplacian phase potentials and Laplacian spacing potentials

<table>
<thead>
<tr>
<th></th>
<th>Potentials</th>
<th>Gradient of potentials</th>
<th>Control</th>
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<tbody>
<tr>
<td><strong>Laplacian phase potentials</strong></td>
<td>( W_1(\theta) = Q_L(e^{i\theta}) = \frac{1}{2N} \langle e^{i\theta}, L e^{i\theta} \rangle )</td>
<td>( \dot{W}<em>1(\theta) = \frac{K_1}{N^2} \sum</em>{k=1}^{N} \langle ie^{i\theta_k}, L_k e^{i\theta} \rangle^2 )</td>
<td>( u_{k}^{\text{ori}} = \frac{K_1}{N} \langle ie^{i\theta}, L e^{i\theta} \rangle, k = 1, ..., N. )</td>
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<tr>
<td><strong>Laplacian spacing potentials</strong></td>
<td>( S(r, \theta) = Q_L(c) = \frac{1}{2N} \langle c, Lc \rangle )</td>
<td>( \dot{S}(r, \theta) = -\frac{K_0}{N^2} \sum_{k=1}^{N} \langle e^{i\theta_k}, L_k c \rangle^2 \leq 0 ) for ( K_0 &gt; 0 )</td>
<td>( u_{k}^{\text{pac}} = \omega_0 \frac{K_0}{N} \langle e^{i\theta_k}, L_k c \rangle, k = 1, ..., N. )</td>
</tr>
<tr>
<td><strong>Phase potentials</strong></td>
<td>( U_1(\theta) = \frac{N}{2}</td>
<td>P_\theta</td>
<td>^2 )</td>
</tr>
</tbody>
</table>

- The Laplacian phase control
  - For \( K_1 < 0 \): The phases are synchronized; for \( K_1 > 0 \): the phase are balanced
- The Laplacian spacing control
  - For \( K_0 > 0 \), and \( \omega_0 \neq 0 \): Circular formation
Synchronized and balanced circular formations

- Consider the composite potential form

\[ V(r, \theta) = K_0 S(r, \theta) + K_1 \omega_0 W_1(\theta) \]

where \( K_0 > 0 \), and \( \omega_0 \neq 0 \)

- As we know, the center of the kth circle is \( c_k = r_k + \omega_0^{-1} e^{i\theta} \)

- Then

\[ u_k = \omega_0 + u_k^{\text{ori}} + u_k^{\text{spac}} \]

\[ = \omega_0 + \frac{K_1}{N} < i e^{i\theta_k}, L_k e^{i\theta} > + \omega_0 \frac{K_0}{N} < e^{i\theta_k}, L_k c > \]

\[ = \omega_0 + \frac{K_1 - K_0}{N} < i e^{i\theta_k}, L_k e^{i\theta} > + \omega_0 \frac{K_0}{N} < e^{i\theta_k}, L_k r > \]
Synchronized and balanced circular formations

- Use
  \[ u_k = \omega_0 + \frac{K_1}{N} \langle ie^{i\theta_k}, L_k e^{i\theta} \rangle + \omega_0 \frac{K_0}{N} \langle e^{i\theta_k}, L_k c \rangle \]
- we will get the following result.

- Each figure has \( N=12 \), \( \omega_0 = 0.1 \) and \( K_0 = N\omega_0 \)
- (a) \( k1<0 \); the synchronized circular formation
- (b) \( k1>0 \); the splay circular formation
Symmetric circular formations

- Let the positive integer $M$ be a divisor of $N$. An $(M,N)$-pattern is a symmetric arrangement of $N$ phases consisting of $M$ clusters uniformly spaced around the unit circle.
- Notice: $(1,N)$-pattern the synchronized state
- $(N,N)$-pattern the splay state
- Under all-to-all interaction, we define

$$p_{m\theta} = \frac{1}{mN} \sum_{k=1}^{N} e^{im\theta_k} \quad U_m(\theta) = \frac{N}{2} |p_{m\theta}|^2$$

$$V^{M,N}(r) = K_0 S(r, \theta) + \sum_{m=1}^{D} K_m U_m(\theta)$$
Symmetric circular formations with all-to-all interaction and $N = 12$. The patterns correspond to $M = 1, 2, 3, 4, 6,$ and $12$ evenly spaced clusters with $\omega_0 = K_0 = 0.1$, $K_m > 0$ for $m = 1, \ldots, M - 1$, and $K_M < 0$. The top left is the synchronized circular formation, while the bottom right is the splay circular formation.
Symmetry breaking

- Break the rotational symmetry by adding a reference phase \( \theta_0 \) and we have \( \dot{\theta}_0 = \omega_0 \).

- The potential \( W_1(\theta) + 1 - \cos(\theta_0 - \theta_N) \) which is minimized by \( \theta_k = \theta_0 \) for \( k = 1, \ldots, N \).

- \( \omega_0 \neq 0 \) Corresponds to a circular formation in which every particle’s phase is synchronized with the reference phase.

- \( \omega_0 = 0 \) Corresponds to a parallel Motion in the direction of the reference phase.

Collective trajectory tracking in a parallel formation with \( N = 12 \). The control (14) tracks a piecewise-linear reference trajectory with \( N = 12, \omega_0 = 0, \) and \( K_1 < 0 \). The sequence of reference phases is \( \pi/8, -3\pi/8, \pi/8, \) and \(-3\pi/8\).
Future research

- The relation between optimal data collection and energy optimality
- 2-D to 3-D
- Shapes other than circle: ellipse, triangle, square and other formations
- Analysis of the convergence time
- Evaluate the sampling performance of a mobile sensor networks
References


Thanks