Discrete Fractional Hartley and Fourier Transforms

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Abstract—This paper is concerned with the definitions of the discrete fractional Hartley transform (DFRHT) and the discrete fractional Fourier transform (DFRFT). First, the eigenvalues and eigenvectors of the discrete Fourier and Hartley transform matrices are investigated. Then, the results of the eigendecompositions of the transform matrices are used to define DFRHT and DFRFT. Also, an important relationship between DFRHT and DFRFT is described, and numerical examples are illustrated to demonstrate that the proposed DFRFT is a better approximation to the continuous fractional Fourier transform than the conventional defined DFRFT. Finally, a filtering technique in the fractional Fourier transform domain is applied to remove chirp interference.

Index Terms—Discrete Fourier transform, discrete fractional Fourier transform, discrete fractional Hartley transform, discrete Hartley transform.

I. INTRODUCTION

In recent years, the concept of fractional operator and measure have been investigated extensively in many engineering applications and science. Four typical examples are described as follows. The first is that the fractional derivative and integral are defined by many mathematicians and applied to solve some physical problems [1]. The second is that the fractional Fourier transform has been studied in the optical community and signal processing area [2], [3]. The third is that the fractional dimension is used to measure some real-world data such as coastline, clouds, dust in the air, and networks of neurons in the body. The fractional dimension has been applied widely to pattern recognition and classification [4]. The last is that the fractional lower order moment has been used to analyze non-Gaussian signals, which is more realistic than the Gaussian model in signal processing applications [5].

On the other hand, various unitary transforms have been widely used in image compression and adaptive filtering [6], [7]. Some typical ones are the discrete cosine transform (DCT), the discrete Hartley transform (DHT), and the discrete Fourier transform (DFT), among others. So far, the fractional version of these transforms has not been investigated yet, except for the Fourier transform. A definition of the discrete fractional Fourier transform (DFRFT) can be found in [8]. One of the natural criteria to evaluate the definition of the DFRFT is to compare the transformed results of the DFRFT with those of the continuous fractional Fourier transform (FRFT) for the same transform signal quantitatively. The more similar the transformed results, the better the fractional transform is defined.

The purpose of this paper is to define a discrete fractional Hartley transform (DFRHT) and a discrete fractional Fourier transform (DFRFT). The paper is organized as follows. In Section II, preliminaries about the fractional Fourier transform are given, including the definitions of the continuous and discrete transforms. In Section III, the eigenvalues and eigenvectors of the DHT matrix are first studied. Then, an eigendecomposition of the DHT matrix is presented. In Section IV, the DFRHT and DFRFT are defined by imposing some constraints to resolve the ambiguities existing in transform matrices. Moreover, a relationship between DFRFT and DFRHT is described, and numerical examples are presented. In the last section, a filtering technique in the fractional Fourier transform domain is applied to remove chirp interference.

II. PRELIMINARIES

In this section, the definition of the continuous fractional Fourier transform is first reviewed. Then, the eigenvalues and eigenvectors of the discrete Fourier transform matrix are briefly described. Finally, the discrete fractional Fourier transform defined by Santhanam and McClellan is stated.

A. Continuous Fractional Fourier Transform

The continuous fractional Fourier transform (FRFT) is defined as [3]

$$F^\alpha[x(t)] = \int_{-\infty}^{\infty} x(t)K_\alpha(t, \omega)dt$$  \hspace{1cm} (1)

where the transform kernel is given by

$$K_\alpha(t, \omega) = \sqrt{\frac{1-j \cos \alpha}{2\pi}} e^{j[t^2+\omega^2]/2} e^{j \omega t \cos \alpha},$$  \hspace{1cm} (2)

if $\alpha$ is not a multiple of $\pi$

$$\delta(t-\omega),$$  \hspace{1cm} if $\alpha$ is a multiple of $2\pi$

$$\delta(t+\omega),$$  \hspace{1cm} if $\alpha + \pi$ is a multiple of $2\pi$.

After some manipulation, it is easy to show that

$$F^{\alpha+\beta}[x(t)] = F^\beta\{F^\alpha[x(t)]\}.$$  \hspace{1cm} (3)

This implies that the angle additivity property is satisfied, i.e., application of the transform with angular parameter $\alpha$ followed
Fig. 1. Continuous FRFT of rectangle window function for various angles [3]: (a) $\alpha = 0.01$, (b) $\alpha = 0.05$, (c) $\alpha = 0.2$, (d) $\alpha = 0.4$, (e) $\alpha = \pi/4$, and (f) $\alpha = \pi/2$.

by an application of the transform with angular parameter $\beta$ is equivalent to the application of the transform with angular parameter $\alpha + \beta$. Moreover, a complete set of eigenfunctions of the fractional Fourier transform are the Hermite Gaussian functions [2]:

$$
\Gamma^\alpha[\psi_n(x)] = e^{\alpha \beta} \psi_n(x)
$$
$$
\psi_n(x) = \frac{2^{\alpha/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi} x) e^{-\pi x^2}
$$

where $H_n(x)$ is the $n$th-order Hermite polynomial. Fig. 1 shows the FRFT of the rectangular window function:

$$
x(t) = \begin{cases} 
1, & \text{for } |t| \leq 2 \\
0, & \text{otherwise}
\end{cases}
$$

for various angles. The real parts of the FRFT or DFRFT in this paper are plotted by solid lines, and the imaginary parts of the FRFT or DFRFT are plotted by dashed or dotted lines.

B. Eigenvalues and Eigenvectors of DFT Matrix

Now, we review the properties of the eigenvalues and eigenvectors of the DFT matrix $F$ whose elements are defined...
From the results in [9], [10], the properties of the eigenvalues and eigenvectors of the DFT matrix can be summarized as the following two properties.

**Property 1:** The eigenvalues of $F$ are $\{1, -1, j, -j\}$ and its multiplicities are listed in the table, shown at the bottom of the page.

**Proof:** See [9].

**Property 2:** Let $\omega = 2\pi/N$, and matrix $S$ be as shown in (7), at the bottom of the page, then it can be shown that $FS = SF$.

**Proof:** See [10].

Because matrix $S$, with distinct eigenvalues, commutes with $F$, the eigenvectors of $S$ will be the desired set of eigenvectors of $F$. Note that $S$ is a real and symmetric matrix, so its eigenvectors will be real and orthogonal.

### C. Discrete Fractional Fourier Transform

Let the data vector be $x$. Santhanam and McClellan defined the discrete fractional Fourier transform as [8]

$$z_{\alpha} = F^{2\alpha/\pi} x.$$  \hfill (8)

The $(2\alpha/\pi)$th power of the DFT matrix $F$ is found by the equation

$$F^{2\alpha/\pi} = \sum_{i=0}^{3} a_i(\alpha) F^i$$  \hfill (9)

where the coefficients $a_i(\alpha)$ are given by

$$a_0(\alpha) = \frac{1}{2} \left(1 + e^{j\alpha}\right) \cos \alpha$$  
$$a_1(\alpha) = \frac{1}{2} \left(1 - j e^{j\alpha}\right) \sin \alpha$$  
$$a_2(\alpha) = \frac{1}{2} \left(e^{j\alpha} - 1\right) \cos \alpha$$  
$$a_3(\alpha) = \frac{1}{2} \left(-1 - j e^{j\alpha}\right) \sin \alpha.$$  

Although this definition of DFRFT obeys the angle additivity property, it is not the discrete version of the continuous transform defined in (2). A numerical example is illustrated as follows. Fig. 2 shows the results of this DFRFT produced by a discrete rectangular window defined as

$$x(n) = \begin{cases} 1, & \text{for } |n| \leq 6 \\ 0, & \text{for } 7 \leq |n| \leq 36. \end{cases}$$  \hfill (10)

In this example, we choose $N = 73$. The results shown in Fig. 2 are far from the results in Fig. 1. Thus, a more suitable definition of DFRFT must be developed. This is one of the purposes of this paper.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Mul. of 1</th>
<th>Mul. of $-1$</th>
<th>Mul. of $-j$</th>
<th>Mul. of $j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4m$</td>
<td>$m+1$</td>
<td>$m$</td>
<td>$m$</td>
<td>$m+1$</td>
</tr>
<tr>
<td>$4m+1$</td>
<td>$m+1$</td>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td>$4m+2$</td>
<td>$m+1$</td>
<td>$m+1$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td>$4m+3$</td>
<td>$m+1$</td>
<td>$m+1$</td>
<td>$m+1$</td>
<td>$m$</td>
</tr>
</tbody>
</table>

$$S = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 2 \cos(\omega) & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 \cos(2\omega) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \cos[(N-2)\omega] & 1 \\ 1 & 0 & 0 & \cdots & 2 \cos[(N-1)\omega] & 2 \cos[(N-1)\omega] \end{bmatrix}$$  \hfill (7)
III. EIGENVALUES AND EIGENVECTORS OF THE DHT MATRIX

In this section, an eigendecomposition of an \( N \times N \) DHT matrix \( H \) whose elements are given by

\[
H_{nk} = \frac{1}{\sqrt{N}} \left[ \cos \left( \frac{2\pi kn}{N} \right) + \sin \left( \frac{2\pi kn}{N} \right) \right],
\]

will be studied. For convenience of further discussion, we define two matrices \( F_r \) and \( F_i \) as follows:

\[
F_{r_{nk}} = \frac{1}{\sqrt{N}} \cos \left( \frac{2\pi kn}{N} \right), \quad 0 \leq n, k \leq N - 1
\]

\[
F_{i_{nk}} = \frac{1}{\sqrt{N}} \sin \left( \frac{2\pi kn}{N} \right), \quad 0 \leq n, k \leq N - 1.
\]

Then, the matrix \( H \) and \( F \) can be rewritten as

\[
H = F_r + F_i \tag{12}
\]

\[
F = F_r - jF_i. \tag{13}
\]

Now, the eigenvectors of the matrix \( H \) are summarized below:

**Property 3:** It can be shown that \( HS = SH \). That is, the eigenvectors \( \{v_1, v_2, \ldots, v_N\} \) of \( S \) are also the eigenvectors of \( H \).

**Proof:** From Property 2 and (13), we have

\[
(F_r - jF_i)S = S(F_r - jF_i). \tag{14}
\]

Since \( S \) is a real matrix, we obtain

\[
F_rS = SF_r, \quad F_iS = SF_i.
\]

This equation implies that

\[
HS = (F_r + F_i)S = SH. \tag{15}
\]

The proof is completed.

From this proof, it is clear that the matrices \( F_r \) and \( F_i \) also commute with \( S \). Thus, the eigenvectors of \( S \) are also the eigenvectors of \( F_r \) and \( F_i \). After the discussion of eigenvectors of \( H \), the eigenvalues of \( H \) and their multiplicities are summarized as the following fact.

**Property 4:** The eigenvalues of \( H \) are \( \{1, -1\} \) and their multiplicities are listed below:

<table>
<thead>
<tr>
<th>( N )</th>
<th>Mul. of 1</th>
<th>Mul. of -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>4m</td>
<td>2m + 1</td>
<td>2m - 1</td>
</tr>
<tr>
<td>4m+1</td>
<td>2m + 1</td>
<td>2m</td>
</tr>
<tr>
<td>4m+2</td>
<td>2m + 1</td>
<td>2m + 1</td>
</tr>
<tr>
<td>4m+3</td>
<td>2m + 2</td>
<td>2m + 1</td>
</tr>
</tbody>
</table>

**Proof:** From Property 1 and the relations \( H = F_r + F_i \), \( F = F_r - jF_i \), this property can be proved trivially.

From this proof, we have that the eigenvectors of \( H \) with eigenvalue 1 correspond to the eigenvectors of \( F \) with eigenvalues 1 and \(-j\). Moreover, the eigenvectors of \( H \) with eigenvalue \(-1\) correspond to the eigenvectors of \( F \) with eigenvalues \(-1\) and \( j \). The byproduct of this proof is that the matrix \( F_r \) has eigenvalues \( \{1, -1, 0\} \), and their multiplicities are given as shown in (15a)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Mul. of 1</th>
<th>Mul. of -1</th>
<th>Mul. of 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>4m</td>
<td>m + 1</td>
<td>m</td>
<td>2m - 1</td>
</tr>
<tr>
<td>4m+1</td>
<td>m + 1</td>
<td>m</td>
<td>2m</td>
</tr>
<tr>
<td>4m+2</td>
<td>m + 1</td>
<td>m + 1</td>
<td>2m</td>
</tr>
<tr>
<td>4m+3</td>
<td>m + 1</td>
<td>m + 1</td>
<td>2m + 1</td>
</tr>
</tbody>
</table>

and that the matrix \( F_i \) has eigenvalues \( \{1, -1, 0\} \) and their multiplicities are given as shown in (15b)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Mul. of 1</th>
<th>Mul. of -1</th>
<th>Mul. of 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>4m</td>
<td>m</td>
<td>m - 1</td>
<td>2m + 1</td>
</tr>
<tr>
<td>4m+1</td>
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<td>m</td>
<td>2m + 1</td>
</tr>
<tr>
<td>4m+2</td>
<td>m</td>
<td>m</td>
<td>2m + 2</td>
</tr>
<tr>
<td>4m+3</td>
<td>m</td>
<td>m</td>
<td>2m + 2</td>
</tr>
</tbody>
</table>

After the discussion of eigenvalues of \( H \), we investigate the eigendecomposition of the matrices \( F_r \), \( F_i \), and \( H \). Before this, we are required to show the following property.

**Property 5:** The following can be shown.

a) \( F_r^2 = (I + P)/2 \), \( F_i^2 = (I - P)/2 \), and \( F_rF_i = F_iF_r = 0 \). The \( I \) is an \( N \times N \) identity matrix, and \( P \) is a circular flip matrix defined by

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & J \end{bmatrix}
\]

where \( J \) is an \((N - 1) \times (N - 1)\) matrix with ones on the antidiagonal.

b) If \( F_ru = \lambda u \) and \( \lambda \neq 0 \), then \( F_iu = 0 \).

c) If \( F_iu = \lambda u \) and \( \lambda \neq 0 \), then \( F_ru = 0 \).

**Proof:**

a) Since the inverse Fourier transform matrix is \( F_r + jF_i \), we have

\[
(F_r + jF_i)(F_r - jF_i) = I.
\]

From [9, Lemma 1] we obtain

\[
(F_r + jF_i)(F_r - jF_i) = P. \tag{18}
\]

Since \( F_r \), \( F_i \), \( I \), and \( P \) are all real matrices, the following expressions are valid by decomposing the left sides of (17) and (18) into real parts and imaginary parts:

\[
F_r^2 + F_i^2 = I
\]

\[
F_r^2 - F_i^2 = P
\]

\[
F_rF_r - F_iF_i = 0
\]

\[
F_rF_i + F_iF_r = 0.
\]

Solving these simultaneous equations, we have

\[
F_r^2 = \frac{I + P}{2}, \quad F_i^2 = \frac{I - P}{2}, \quad F_rF_i = F_iF_r = 0. \tag{19}
\]

b) Since \( F_ru = \lambda u \) (\( \lambda \neq 0 \)) and \( F_iF_r = 0 \), we have

\[
F_iu = \frac{1}{\lambda} F_rF_iu = 0.
\]

c) The proof is similar to b).
This property tells us that the matrices $F_r$ and $F_i$ are mutually orthogonal. Thus, the eigenvector of $F_r$ with nonzero eigenvalue is the eigenvector of $F_i$ with zero eigenvalue. Also, the eigenvector of $F_i$ with nonzero eigenvalue is the eigenvector of $F_r$ with zero eigenvalue. Based on the above facts, it is clear that the eigendecompositions of $F_r$, $F_i$, and $H$ can be written as follows:

$$F_r = [U_1 \ U_2 \ U_3 \ U_4]
\begin{bmatrix}
I_1 & 0 & 0 & 0 \\
0 & -I_2 & 0 & 0 \\
0 & 0 & I_3 & 0 \\
0 & 0 & 0 & -I_4
\end{bmatrix}
\cdot [U_1 \ U_2 \ U_3 \ U_4]^T \quad (21)$$

$$F_i = [U_1 \ U_2 \ U_3 \ U_4]
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & I_3 & 0 \\
0 & 0 & 0 & -I_4
\end{bmatrix}
\cdot [U_1 \ U_2 \ U_3 \ U_4]^T \quad (22)$$

$$H = [U_1 \ U_2 \ U_3 \ U_4]
\begin{bmatrix}
I_1 & 0 & 0 & 0 \\
0 & -I_2 & 0 & 0 \\
0 & 0 & I_3 & 0 \\
0 & 0 & 0 & -I_4
\end{bmatrix}
\cdot [U_1 \ U_2 \ U_3 \ U_4]^T \quad (23)$$

where $I_i$ is the identity matrix with size $N_i \times N_i$. The values of $N_i$ are listed below.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>$N_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4m$</td>
<td>$m + 1$</td>
<td>$m$</td>
<td>$m$</td>
<td>$m - 1$</td>
</tr>
<tr>
<td>$4m + 1$</td>
<td>$m + 1$</td>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td>$4m + 2$</td>
<td>$m + 1$</td>
<td>$m + 1$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td>$4m + 3$</td>
<td>$m + 1$</td>
<td>$m + 1$</td>
<td>$m + 1$</td>
<td>$m$</td>
</tr>
</tbody>
</table>

The matrices $U_i$ are given by the following.

1) $U_1$ is constructed by the eigenvectors $v$ of matrix $S$ which satisfy $F_r v = v$.
2) $U_2$ is constructed by the eigenvectors $v$ of matrix $S$ which satisfy $F_i v = v$.
3) $U_3$ is constructed by the eigenvectors $v$ of matrix $S$ which satisfy $F_r v = -v$.
4) $U_4$ is constructed by the eigenvectors $v$ of matrix $S$ which satisfy $F_i v = -v$.

In the next section, we will use the eigendecompositions of $H$ and $F$ to define discrete fractional Hartley and Fourier transforms, and use the eigendecompositions of $F_r$ and $F_i$ to study the relationship between DFRHT and DFRFT.

### IV. Definitions of DFRHT and DFRFT

#### A. Discrete Fractional Hartley Transform

Let data vector be $x$; then its fractional Hartley transform $y_{\tau}$ is defined by

$$y_{\tau} = H^\tau x. \quad (24)$$

When the power $\tau$ is chosen as 1, the DFRHT becomes the conventional Hartley transform. When $\tau = 0$, it is clear that $y_0 = x$. Since expression $H^{\tau + p} x = H^\tau H^p x$ is valid, the angle additivity property is satisfied. Now, the problem is how to compute the matrix $H^\tau$. The matrix $H^\tau$ can be obtained by taking the $\tau$th power of the eigenvalues of the matrix $H$, that is,

$$H^\tau = [U_1 \ U_2 \ U_3 \ U_4]
\begin{bmatrix}
I_1 & 0 & 0 & 0 \\
0 & (-I_2)^\tau & 0 & 0 \\
0 & 0 & I_3 & 0 \\
0 & 0 & 0 & (-I_4)^\tau
\end{bmatrix}
\cdot [U_1 \ U_2 \ U_3 \ U_4]^T. \quad (25)$$

It is clear that there are two ambiguities in this decomposition to make the computation of $H^\tau$ not unique. They are the following.

A1) Since the following two expressions are valid:

$$1^\tau = e^{-j2k\pi \tau}, \quad (-1)^\tau = e^{-j(2k+1)\pi \tau}, \quad \text{for all integer } k$$

the matrices $I_i^\tau (i = 1, 3)$ and $(-I_i)^\tau (i = 2, 4)$ are not unique. Additional constraint must be imposed such that this ambiguity can be removed.

A2) Since $U_1$ is constructed by the eigenvectors $v$ of the matrix $S$ which satisfies $F_r v = v$, any two column vectors of $U_1$ can be interchanged. Thus, there exist $N_1$! valid matrices $U_1$. Similarly, the matrices $U_3(i = 1, 3)$ and $U_5(i = 2, 4)$ suffer the same trouble. In order to make $U_i$ be unique, it is necessary to propose a rule to arrange the order of the column vectors in $U_i$.

In the following, we will provide a simple assignment rule to remove the ambiguities A1) and A2). For removing ambiguity A1), we choose $I_i^\tau$ and $(-I_i)^\tau$ as follows:

$$I_i^\tau = \begin{bmatrix}
e^{-j0\pi \tau} & 0 & 0 & \cdots & 0 & 0 \\
0 & e^{-j2\pi \tau} & 0 & \cdots & 0 & 0 \\
0 & 0 & e^{-j4\pi \tau} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & e^{-j(2(N_i-1)\pi \tau)} & 0
\end{bmatrix}, \quad i = 1, 3$$

$$(-I_i)^\tau = \begin{bmatrix}
e^{-j\pi \tau} & 0 & 0 & \cdots & 0 & 0 \\
0 & e^{-j3\pi \tau} & 0 & \cdots & 0 & 0 \\
0 & 0 & e^{-j5\pi \tau} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & e^{-j(2(N_i-1)\pi \tau)} & 0
\end{bmatrix}, \quad i = 2, 4. \quad (26)$$

Moreover, we remove ambiguity A2) by arranging the column vectors of the matrix $U_i$ in terms of the following description. Let $u_{im}$ and $u_{in}$ be two column vectors of the matrix $U_i$, then there exist $\lambda_m$ and $\lambda_n$ such that

$$S u_{im} = \lambda_m u_{im}, \quad S u_{in} = \lambda_n u_{in}.$$  

The constraint imposed on the column vectors $u_{im}$ and $u_{in}$ is that

$$\lambda_m > \lambda_n, \quad \text{if } m < n. \quad (28)$$
Because the eigenvalues of $S$ are distinct, the matrix $U_i$ can be uniquely specified by the above constraint.

Although the rule to remove ambiguities A1) and A2) has many choices, the proposed rule is very simple. In the sequel, we will use numerical examples to describe the advantage of this choice. Finally, we summarize the computation procedure of fractional Hartley transform as follows.

Procedure 1: Given data vector $x$, matrix $S$, and power $\tau$, use the following steps to compute $y_{\tau}$:

1) Compute the eigenvalues and eigenvectors of matrix $S$.
2) Use (28) to construct the matrices $U_i$ ($i = 1, \cdots, 4$).
3) Use (26), (27) to compute $I_i$ ($i = 1, 3$) and $(-I_i)^{\tau}$ ($i = 2, 4$).
4) Use (25) to calculate matrix $H^\tau$.
5) Use (24) to compute $y_{\tau} = H^\tau x$.

When data vector $x$ is real, its fractional Hartley transform $y_{\tau}$ is complex, except when $\tau$ is an integer. Moreover, it is easy to show that

$$H^{2m} x = x$$
$$H^{2m+1} x = H x.$$  

This expression is intuitively valid because the eigenvalues of the matrix $H$ are 1 or $-1$.

B. Discrete Fractional Fourier Transform

Let data vector be $x$; then its fractional Fourier transform is defined by

$$z_{\tau} = F^{\tau} x.$$  (29)

Compare (29) with (9); we obtain $\tau = 2\nu/\pi$. Instead of using (9) to compute the matrix $F^{\tau}$, we develop a new procedure to calculate it. Since

\[ F^\tau = [F_{\nu} + (-j) F_i]^\tau \]  

substitute (21), (22) into (30); we obtain

\[ F^\tau = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix} \begin{bmatrix} (I_1)^\tau & 0 & 0 & 0 \\ 0 & (I_2)^\tau & 0 & 0 \\ 0 & 0 & (I_3)^\tau & 0 \\ 0 & 0 & 0 & (I_4)^\tau \end{bmatrix} \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix}^\dagger \]

\[ = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix} \begin{bmatrix} (I_1)^\tau & 0 & 0 & 0 \\ 0 & (I_2)^\tau & 0 & 0 \\ 0 & 0 & (I_3)^\tau & 0 \\ 0 & 0 & 0 & (I_4)^\tau \end{bmatrix} \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix}^\dagger. \]  (31)

It is clear that there are three ambiguities in this expression. The ambiguities of matrices $I_j$, $(-I_j)^{\tau}$, and $U_i$ are all the same as those in the DFRHT case. Thus, these ambiguities can be removed by using the same constraints. Since the following expression is valid:

$$(-j)^{\tau} = e^{-j(2\nu+1)/2\pi}, \quad \text{for all integer } k$$  (32)

the value of $(-j)^{\tau}$ is not unique. We choose the value

$$(-j)^{\tau} = e^{-j(\pi/2)\tau}.$$  (33)

After these choices, we have the following facts.

Property 6: The following can be shown.

a) When $N = 4m + 1$ or $N = 4m + 3$, the eigenvalues of $F^\tau$ are given by $e^{-j\pi m/2}$, $0 \leq n \leq N - 1$. (34)

b) When $N = 4m$ or $N = 4m + 2$, the eigenvalues of $F^\tau$ are given by $e^{-j\pi (n/2)}$ and $0 \leq n \leq N - 2$. (35)

Proof: Substitute (26), (27), and (33) into (31); the fact can be proven easily.

From Property 6 and (4), it is clear that the eigenvalues of the proposed DFRFT matrix $F^\tau$ are consistent with those of the continuous fractional Fourier transform defined in (2) when we choose $\tau = 2\nu/\pi$. However, the eigenvalues of the DFRFT matrix $F^{2\nu/\pi}$ defined in (9) do not have this consistent property. Thus, the transformed results of our DFRFT are more similar to those of the continuous FRFT than the results of the DFRFT defined in (9) for the same data vector $x$. In the sequel, two numerical examples and an analytical method are used to illustrate this fact. Finally, we summarize the computation procedure of the fractional Fourier transform as follows.

Procedure 2: Given data vector $x$, matrix $S$, and power $\tau$, use the following steps to compute $z_{\tau}$:

1) compute the eigenvalues and eigenvectors of matrix $S$;
2) use (28) to construct the matrices $U_i$ ($i = 1, \cdots, 4$);
3) use (26), (27) to compute $I_i$ ($i = 1, 3$) and $(-I_i)^{\tau}$ ($i = 2, 4$);
4) use (33) to calculate $(-j)^{\tau}$;
5) use (31) to calculate matrix $F^\tau$;
6) use (29) to compute $z_{\tau} = F^\tau x$.

When data vector $x$ is real, its fractional Fourier transform $z_{\tau}$ is complex, except that $\tau$ is an integer.

C. Relationship Between DFRHT and DFRFT

In the following, the relationship between the DFRHT and DFRFT will be investigated. Before this, it is useful to review the relationship between the DFT and DHT. The main result is summarized as follows.

Property 7: Let the DHT and DFT of $x$ be denoted by $x_F = Fx$, $x_H = Hx$. (36)

Then it can be shown that

$$\text{Real}(x_F) = \left( \frac{1+p}{2} \right) x_H$$
$$\text{Imag}(x_F) = -\left( \frac{1-p}{2} \right) x_H.$$
where \( \text{Real}(\cdot) \) and \( \text{Imag}(\cdot) \) denote the real part and imaginary parts of the vector.

**Proof:** See [11].

Property 7 tells us that the even part of DHT is the real part of DFT, and the odd part of DHT corresponds to the imaginary part of the DFT. Thus, the DFT can be computed from the results of DHT by using Property 7. Now, we will address how to calculate DFRFT when DFRHT has been computed. The following property will help us solve this problem.

Property 8: Let DFRHT \( y_r = H^* x \), and define its even part and odd parts as

\[
y_r^e = \frac{I + P}{2} y_r, \quad y_r^o = \frac{I - P}{2} y_r.
\]

Then it can be shown that

\[
y_r^e = F_r^o x, \quad y_r^o = F_r^e x.
\]

**Proof:** From Property 5, we have

\[
y_r^e = \frac{I + P}{2} y_r = F_r^o H^* x.
\]

Moreover, from (21), (22), and (25), we get

\[
H^* = F_r^o + F_r^e.
\]

Substituting (38) into (37) we obtain

\[
y_r^e = F_r^o x + F_r^o F_r^e x.
\]

Using the eigendecompositions of \( F_r \) and \( F_r^e \) described in (21), (22), it can be shown that

\[
F_r^{2 \tau} = F_r^e, \quad F_r^{2 \tau} F_r^e = 0.
\]

Thus, we obtain

\[
y_r^e = F_r^e x.
\]

The proof is completed. As for the proof of \( y_r^o = F_r^e x \), it can be shown similarly.

This property tells that the even part of DFRHT is equal to \( F_r^o x \), and the odd part of DFRHT is equal to \( F_r^e x \). From (31), (33), it is easy to show that

\[
F_r^o = F_r^e + e^{-j(\pi/2)\tau} F_r^e.
\]

Multiply both sides by data vector \( x \), we obtain

\[
z_r = y_r^e + e^{-j(\pi/2)\tau} y_r^o = \frac{I + P}{2} y_r + e^{-j(\pi/2)\tau} \frac{I - P}{2} y_r.
\]

This expression is the relationship between DFRHT and DFRFT. It can be used to compute DFRFT from the results of DFRHT easily. When \( \tau = 1 \), (41) reduces to the conventional relationship between DFT and DHT described in Property 7. Thus, one particular feature of our definitions of DFRHT and DFRFT is that the relation in Property 7 is still preserved. Moreover, if the real part and imaginary part of the DFRHT \( y_r \) are both even symmetric, then it can be shown that

\[
\frac{I + P}{2} y_r = y_r, \quad \frac{I - P}{2} y_r = 0.
\]

This means that \( z_r = y_r \), i.e., DFRFT is equal to DFRHT.

### D. Numerical Examples

**Example 1. Rectangular Window Function:** In this example, we deal with the transformation of the rectangular window defined in (10). By using the definitions of DFRHT and DFRFT developed in this section, the transform results are shown in Fig. 3 for various angular parameters \( \tau = 2\alpha/\pi \). Compared to the results in Fig. 3(a) with Fig. 1, we observe that the transform results in Fig. 3(a) are more similar to those in Fig. 1 than those in Fig. 2. Thus, our definition of DFRFT is a better approximation of the continuous fractional Fourier transform than the DFRFT defined in [8]. However, only 73 samples are used in this experiment; the curves in Fig. 3(a) are much smoother than those in Fig. 1. Finally, it is worth mentioning that the results in Fig. 3(a) are also obtained from the results in Fig. 3(b) by using (41), which is the relationship between the DFRHT and DFRFT. In fact, the DFRFT is equal to the DFRHT in this example because DFRHT \( y_r \) is even symmetric for all \( \tau \). This is owing to the fact that the rectangle function is even symmetric.

**Example 2. Two-Impulse Function:** In this example, we further consider the fractional transform of the following continuous signal:

\[
x(t) = 3\delta(t - d) + \delta(t + d)
\]

where \( \delta(t) \) is the impulse function. Using the results in [3], the continuous FRFT of this special signal has the closed formula given by

\[
F^\alpha[x(t)] = \sqrt{\frac{1 - j \cot(\alpha)}{2\pi}} e^{j(d^2 + \alpha^2) / 2\cot(\alpha)} \cdot [3\cos(k_0 \cos(\alpha) + e^{j\omega_0 \cos(\alpha)}].
\]

Fig. 4(a) shows the continuous FRFT of the signal \( x(t) \) for various angle \( \alpha \) and \( d = 0.4 \). For comparison, we examine the DFRFT of the digital signal

\[
x(n) = 3\delta(n - 1) + \delta(n + 1)
\]

\[
= \begin{cases} 
3, & n = 1 \\
1, & n = -1 \\
0, & |n| \leq 20
\end{cases}
\]

where \( \delta(n) \) is the unit sample function and the length of \( x(n) \) is 41. Fig. 4(b) and (c) shows the transform results of the proposed DFRFT and DFRHT. It is clear that the results of our DFRFT are very similar to those of the continuous FRFT shown in Fig. 4(a). Moreover, the results of DFRHT are not the same as those of DFRFT because the signal \( x(n) \) is not even symmetric. Finally, it is worth mentioning that the results in Fig. 4(b) also can be obtained from the results in Fig. 4(c) by using (41), which is the relationship between the DFRHT and DFRFT.

### E. Discussion

In the following, an analytical approach is used to show that our DFRFT is a better discrete version of the continuous FRFT than the conventional DFRFT defined in [8]. First, we show that the eigenvector of the matrix \( S \) defined in (7) is a discrete counterpart of the Hermite function which is
an eigenfunction of fractional Fourier transform. Let vector \( \mathbf{v}_i = [v_{i0}, v_{i1}, \ldots, v_{i(N-1)}]^T \) be an eigenvector of matrix \( S \) with eigenvalue \( \gamma_i \); then it will satisfy the following difference equation [10]:

\[
\delta^2 v_{ik} + \left[ 2 \cos \left( k \frac{2\pi}{N} \right) - (\gamma_i - 2) \right] v_{ik} = 0 \quad (45)
\]

where \( \delta^2 v_{ik} = v_{i(k+1)} - 2v_{ik} + v_{i(k-1)} \) is the central second difference operator. Because (45) can be treated as a discrete version of the second-order differential equation

\[
\frac{d^2 v(t)}{dt^2} + [2 \cos(2\pi t) - (\gamma - 2)]v(t) = 0 \quad (46)
\]

whose periodic solutions are the Mathieu functions [10], the eigenvectors of \( S \) may be thought of as discrete Mathieu functions. Since the Mathieu functions can converge to the Hermite functions [17], the eigenvectors of matrix \( S \) also can be treated as the discrete Hermite functions. Thus, the eigenvectors of our fractional Fourier transform matrix \( F^\tau \) are approximate discrete Hermite functions which are the eigenfunctions of the continuous FRFT because matrices \( S \) and \( F^\tau \) have the same eigenvectors. This fact tells us that the eigenfunctions of our DFRFT and continuous FRFT are consistent. Moreover, Property 6 states that the eigenvalues of the proposed DFRFT matrix \( F^\tau \) are consistent with those of the continuous FRFT. Due to these two agreements, the transform results of our DFRFT are similar to those of the continuous FRFT. As for the definition in [8], their eigenvectors and eigenvalues of the transform matrix are not consistent with continuous ones, so its transform results are not similar to the continuous ones.

V. CHIRP FILTERING IN THE DFRFT DOMAIN

The fractional Fourier transform (FRFT) has been successfully used in many applications such as signal detection, pattern recognition, time-variant filtering, multiplexing, data compression, and the study of time–frequency distributions [12]–[16]. In this section, we concentrate on the applications of the chirp interference removal. The details of the continuous chirp case have been investigated in [16]. Here, we only extend the technique developed in [16] to the discrete chirp case. Since the FRFT of the chirp signal is the line delta function in the appropriate fractional Fourier domain, we can remove this impulse of the chirp component in the FRFT domain by multiplying a narrow bandstop mask. The narrower the bandstop mask is, the less distortion the nonchirp part has.

Given the angular parameter \( \tau \) and the signal \( x(n) \) composed of a desired signal and a chirp interference, the procedure of filtering out this chirp component in DFRFT domain is summarized as follows.

- **Step 1)** Compute the DFRFT \( x_1(n) \) of the signal \( x(n) \) with angular parameter \( \tau \).

- **Step 2)** Multiply the transform result \( x_1(n) \) by the bandstop mask \( m(n) \). The masking result is denoted by \( x_2(n) = x_1(n)m(n) \).
Fig. 4. (a) Continuous FRFT of two impulse function for various angles: (top left) $\alpha = 0.25^{\circ}\pi$, (top right) $\alpha = 0.3^{\circ}\pi$, (middle left) $\alpha = 0.35^{\circ}\pi$, (middle right) $\alpha = 0.4^{\circ}\pi$, (bottom left) $\alpha = 0.45^{\circ}\pi$, and (bottom right) $\alpha = 0.5^{\circ}\pi$. (b) DFRFT of two impulse function for various angles: (top left) $\alpha = 0.25^{\circ}\pi$, (top right) $\alpha = 0.3^{\circ}\pi$, (middle left) $\alpha = 0.35^{\circ}\pi$, (middle right) $\alpha = 0.4^{\circ}\pi$, (bottom left) $\alpha = 0.45^{\circ}\pi$, and (bottom right) $\alpha = 0.5^{\circ}\pi$. (c) DFRHT of two impulse function for various angles: (top left) $\alpha = 0.25^{\circ}\pi$, (top right) $\alpha = 0.3^{\circ}\pi$, (middle left) $\alpha = 0.35^{\circ}\pi$, (middle right) $\alpha = 0.4^{\circ}\pi$, (bottom left) $\alpha = 0.45^{\circ}\pi$, and (bottom right) $\alpha = 0.5^{\circ}\pi$. 
eliminated by the proposed removal method. The Gaussian signal-to-chirp-noise ratio is improved from $-3.8$ to $6.3$ dB.

VI. CONCLUSIONS

In this paper, the definitions of the discrete fractional Hartley transform (DFRHT) and the discrete fractional Fourier transform (DFRFT) have been presented. First, the eigenvalues and eigenvectors of the discrete Fourier and Hartley transform matrices are investigated. Then, the results of the eigendecompositions of the transform matrices are used to define the DFRHT and DFRFT. Also, an important relationship between DFRHT and DFRFT is described, and numerical examples are illustrated to demonstrate that the proposed DFRFT is a better approximation to the continuous fractional Fourier transform than the conventional defined DFRFT. Finally, a filtering technique in the fractional Fourier transform domain is applied to eliminate the chirp interference. However, only fractional Hartley and Fourier transforms are defined. Thus, it is interesting to develop other types of fractional transforms such as the Hardmard transform and DCT. This topic will be investigated in the future.

REFERENCES

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