On non-local stability properties of extremum seeking control
Author: Y. Tan, D. Nesic, I. Mareels

Reporter: Chun Yin

School of Mathematics Science,
University of Electronic Science and Technology of China,
Chengdu, Sichuan, 611731, P.R. China.
Utah State University, Logan Utah, USA

Nov.18 2011
1. Introduction

2. Extremum seeking control: Convergence Analysis
   - First order extremum seeking scheme
   - Higher order extremum seeking scheme

3. Simulation example
   - The first order scheme
   - Comparison of the first and higher order schemes

4. Discussion

Reporter: Chun Yin
ESC
Outline

1. Introduction

2. Extremum seeking control: Convergence Analysis
   - First order extremum seeking scheme
   - Higher order extremum seeking scheme

3. Simulation example
   - The first order scheme
   - Comparison of the first and higher order schemes

4. Discussion
Outline

1. Introduction

2. Extremum seeking control: Convergence Analysis
   - First order extremum seeking scheme
   - Higher order extremum seeking scheme

3. Simulation example
   - The first order scheme
   - Comparison of the first and higher order schemes

4. Discussion

Reporter: Chun Yin
ESC
Outline

1. Introduction
2. Extremum seeking control: Convergence Analysis
   - First order extremum seeking scheme
   - Higher order extremum seeking scheme
3. Simulation example
   - The first order scheme
   - Comparison of the first and higher order schemes
4. Discussion
Consider the SISO nonlinear model

\[ \dot{x} = f(x, u), \quad y = h(x). \]

Suppose that there exists a unique \( x^* \) such that \( y^* = h(x^*) \) is the extremum of the map \( h(\cdot) \).

Due to uncertainty, it is often reasonable to assume that neither \( x^* \) nor \( h(\cdot) \) are precisely known to the control designer.

The main objective in extremum seeking control is to force the solutions of the closed-loop system to eventually converge to \( x^* \) and to do so without any precise knowledge about \( x^* \) and \( h(\cdot) \).
Consider the SISO nonlinear model

\[ \dot{x} = f(x, u), \quad y = h(x). \]

suppose that there exists a unique \( x^* \). such that 
\[ y^* = h(x^*) \] is the extremum of the map \( h(\cdot) \).

Due to uncertainty, it is often reasonable to assume that neither \( x^* \). nor \( h(\cdot) \) are precisely known to the control designer.

The main objective in extremum seeking control is to force the solutions of the closed-loop system to eventually converge to \( x^* \). and to do so without any precise knowledge about \( x^* \) and \( h(\cdot) \).
Consider the SISO nonlinear model

\[ \dot{x} = f(x, u), \quad y = h(x). \]

Suppose that there exists a unique \( x^* \) such that \( y^* = h(x^*) \) is the extremum of the map \( h(\cdot) \).

Due to uncertainty, it is often reasonable to assume that neither \( x^* \) nor \( h(\cdot) \) are precisely known to the control designer. The main objective in extremum seeking control is to force the solutions of the closed-loop system to eventually converge to \( x^* \) and to do so without any precise knowledge about \( x^* \) and \( h(\cdot) \).
The aims of this paper are to:

- prove non-local stability properties of several extremum seeking controllers.
- reduce the size of the parameters to slow down the convergence rate of the extremum seeking controllers and enlarge the domain of the attraction.
The aims of this paper are to:

- prove non-local stability properties of several extremum seeking controllers.
- reduce the size of the parameters to slow down the convergence rate of the extremum seeking controllers and enlarge the domain of the attraction.
Consider a parameterized family of systems:

$$\dot{x} = f(t, x, \epsilon) \quad (2.1)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R} \geq 0$, $\epsilon \in \mathbb{R}^l > 0$ are respectively the state of the system, the time variable and the parameter vector.

The stability of the system can depend in an intricate way on the parameters.

The next definition applies to arbitrary nonlinear multi-parameter system (2.1).
Consider a parameterized family of systems:

$$\dot{x} = f(t, x, \epsilon)$$  \hspace{1cm} (2.1)

where $x \in \mathbb{R}^n$, $t \in \mathbb{R} \geq 0$, $\epsilon \in \mathbb{R}^l > 0$ are respectively the state of the system, the time variable and the parameter vector.

The stability of the system can depend in an intricate way on the parameters.

The next definition applies to arbitrary nonlinear multi-parameter system (2.1).
Definition 1.

System (2.1) with parameter $\epsilon$ is said to be semi-globally practically asymptotically (SPA) stable, uniformly in $(\epsilon_1, \cdots, \epsilon_j)$, if there exists $\beta \in \mathbb{K}_L$ such that the following holds. For each pair of strictly positive real numbers $(\Delta, \nu)$, there exist real numbers $\epsilon_k^* = \epsilon_k^*(\Delta, \nu) > 0$, $k = 1, \cdots, j$ and for each fixed $\epsilon_k \in (0, \epsilon_k^*)$, $k = 1, \cdots, j$ there exist $\epsilon_i = \epsilon_i(\epsilon_1, \cdots, \epsilon_{i-1}, \Delta, \nu)$, with $i = j + 1, \cdots, \ell$, such that the solutions of (2.1) with the so constructed parameters $\epsilon = (\epsilon_1, \cdots, \epsilon_\ell)$ satisfy:

$$|x(t)| \leq \beta(|x_0|, (\epsilon_1 \cdot \epsilon_2 \cdots \cdot \epsilon_\ell)(t - t_0)) + \nu$$

for all $t \geq t_0 \geq 0$, $x(t_0) = x_0$ with $|x_0| \leq \Delta$. If we have that $j = \ell$, then we say that the system is SPA stable, uniformly in $\epsilon$.
Consider a family of control laws of the following form:

\[ u = \alpha(x, \theta), \quad (2.2) \]

where \( \theta \in \mathbb{R} \) is a scalar parameter. The following assumptions are used to obtain the result.
A1. There exists a function \( l : R \rightarrow R^n \) such that 
\[ f(x, \alpha(x, \theta)) = 0 \] if and only if \( x = l(\theta) \).

A2. For each \( \theta \in R \), the equilibrium \( x = l(\theta) \) of system (2.1) with the control laws (2.2) is globally asymptotically stable, uniformly in \( \theta \).

A3. Denoting \( Q(\cdot) = h \circ l(\cdot) \), there exists a unique \( \theta^* \) maximizing \( Q(\cdot) \) and, the following hold :

\[
Q'(\theta) = 0, \quad Q''(\theta) < 0, \quad (2.3) \\
Q'(\theta + \xi)\xi < 0, \quad \forall \xi \neq 0. \quad (2.4)
\]
A1. There exists a function $l : R \to R^n$ such that $f(x, \alpha(x, \theta)) = 0$ if and only if $x = l(\theta)$.

A2. For each $\theta \in R$, the equilibrium $x = l(\theta)$ of system (2.1) with the control laws (2.2) is globally asymptotically stable, uniformly in $\theta$.

A3. Denoting $Q(\cdot) = h \circ l(\cdot)$, there exists a unique $\theta^*$ maximizing $Q(\cdot)$ and, the following hold:

$$Q'(\theta) = 0, \quad Q''(\theta) < 0, \quad (2.3)$$

$$Q'(\theta + \xi) \xi < 0, \quad \forall \xi \neq 0. \quad (2.4)$$
A1. There exists a function $l : \mathbb{R} \rightarrow \mathbb{R}^n$ such that $f(x, \alpha(x, \theta)) = 0$ if and only if $x = l(\theta)$.

A2. For each $\theta \in \mathbb{R}$, the equilibrium $x = l(\theta)$ of system (2.1) with the control laws (2.2) is globally asymptotically stable, uniformly in $\theta$.

A3. Denoting $Q(\cdot) = h \circ l(\cdot)$, there exists a unique $\theta^*$ maximizing $Q(\cdot)$ and, the following hold:

\[ Q'(\theta) = 0, \quad Q''(\theta) < 0, \quad (2.3) \]
\[ Q'(\theta + \xi)\xi < 0, \quad \forall \xi \neq 0. \quad (2.4) \]
Consider the first order extremum seeking scheme, as shown in the figure 6, with the following dynamics:

\[
\dot{x} = f(x, \alpha(x, \theta)), \\
y = h(x), \\
\theta
\]

Figure: The simplest peak seeking feedback scheme.

Reporter: Chun Yin
\[ \dot{x} = f(x, \alpha(x, \hat{\theta} + a \sin(\omega t))), \]
\[ \dot{\hat{\theta}} = kh(x)b \sin(\omega t), \quad (2.5) \]
where \((k, a, b, \omega)\) are tuning parameters.

Introduce the change of the coordinates, \(\tilde{x} = x - x^*\), \(\tilde{\theta} = \hat{\theta} - \theta^*\), the system can be rewritten as

\[ \dot{\tilde{x}} = f(\tilde{x} + x^*, \alpha(\tilde{x} + x^*, \tilde{\theta} + \theta^* + a \sin(\omega t))), \]
\[ \dot{\tilde{\theta}} = kh(\tilde{x} + x^*)b \sin(\omega t), \quad (2.6) \]
let \(k = \omega \delta K, \sigma = \omega t\). Define \(\epsilon = [a^2 \quad \delta \quad \omega]\).
\[ \dot{x} = f(x, \alpha(x, \hat{\theta} + a \sin(\omega t))), \]
\[ \dot{\hat{\theta}} = k h(x) b \sin(\omega t), \] (2.5)

where \((k, a, b, \omega)\) are tuning parameters.

Introduce the change of the coordinates, \(\tilde{x} = x - x^*, \tilde{\theta} = \hat{\theta} - \theta^*\), the system can be rewritten as

\[ \dot{\tilde{x}} = f(\tilde{x} + x^*, \alpha(\tilde{x} + x^*, \tilde{\theta} + \theta^* + a \sin(\omega t))), \]
\[ \dot{\tilde{\theta}} = k h(\tilde{x} + x^*) b \sin(\omega t), \] (2.6)

let \(k = \omega \delta K, \sigma = \omega t\). Define \(\epsilon = [a^2 \quad \delta \quad \omega]\).
The following result can be obtained.

**Theorem 1**

Suppose that Assumptions 1–3 hold. Then, system (2.6) (when $b = a$) with parameter $\epsilon$ is SPA stable, uniformly in $(a^2, \delta)$. 
Outline

1. Introduction

2. Extremum seeking control: Convergence Analysis
   - First order extremum seeking scheme
   - Higher order extremum seeking scheme

3. Simulation example
   - The first order scheme
   - Comparison of the first and higher order schemes

4. Discussion

Reporter: Chun Yin
Consider the extremum seeking controller given in the following figure.

\[ \dot{x} = f(x, \alpha(x, \theta)), \]
\[ y = h(x), \]
\[ \theta 
\]
\[ W_L(s) \]
\[ b \sin(\omega t) \]
\[ W_H(s) \]
\[ a \sin(\omega t) \]

**Figure:** A peak seeking feedback scheme with a low-pass filter and high-pass filter.
When $\mathcal{W}_L(s) = (\omega_l/s + \omega_l)$ and $\mathcal{W}_H(s) = 1$, the following equations describe the closed-loop system

\[
\begin{align*}
\dot{x} &= f(x, \alpha(x, \hat{\theta} + a\sin(\omega t))), \\
\dot{\hat{\theta}} &= k\xi, \\
\dot{\xi} &= -\omega_l\xi + \omega_l h(x)b\sin(\omega t),
\end{align*}
\] (2.7) 

let $b = a$. 

Introduce the change of the coordinates, \( \tilde{x} = x - x^* \), \( \tilde{\theta} = \hat{\theta} - \theta^* \), \( \tilde{\xi} = \xi \). The system in new coordinates takes the following form:

\[
\begin{align*}
\dot{\hat{x}} &= f(\tilde{x} + x^*, \alpha(\tilde{x} + x^*, \tilde{\theta} + \theta^* + a \sin(\omega t))), \\
\dot{\tilde{\theta}} &= k \tilde{\xi}, \\
\dot{\tilde{\xi}} &= -\omega l [\tilde{\xi} - h(\tilde{x} + x^*) b \sin(\omega t)],
\end{align*}
\]

(2.8)

define \( \omega l = \omega \delta \omega_L \), \( k = \omega \delta K \), \( \epsilon = [a^2 \quad \delta \quad \omega] \).
The following result can be derived.

**Theorem**

Suppose that A 1–3 hold. Then, the closed-loop system with parameter $\epsilon$ is SPA stable, uniformly in $a^2$.

When $W_L(s) = (\omega_1/s + \omega_1)$ and $W_H(s) = (s/s + \omega_h)$, the following equations describe the closed-loop system

\[
\begin{align*}
\dot{x} &= f(x, \alpha(x, \hat{\theta} + a \sin(\omega t))), \\
\dot{\hat{\theta}} &= k\xi, \\
\dot{\xi} &= -\omega_1 \xi + \omega_1(y - \eta)b \sin(\omega t), \\
\dot{\eta} &= -\omega_h \eta + \omega_h y,
\end{align*}
\]  

(2.9)
The following result can be derived.

**Theorem**

Suppose that A 1–3 hold. Then, the closed-loop system with parameter $\epsilon$ is SPA stable, uniformly in $a^2$.

When $W_L(s) = (\omega_l/s + \omega_l)$ and $W_H(s) = (s/s + \omega_h)$, the following equations describe the closed-loop system

\[
\begin{align*}
\dot{x} &= f(x, \alpha(x, \hat{\theta} + a \sin(\omega t))), \\
\dot{\hat{\theta}} &= k\xi, \\
\dot{\xi} &= -\omega_l\xi + \omega_l(y - \eta)b \sin(\omega t), \\
\dot{\eta} &= -\omega_h\eta + \omega_h y,
\end{align*}
\]  
(2.9)
The result can be obtained.

**Theorem**

*Suppose that A 1–3 hold. Then, the closed-loop system with parameter $\epsilon$ is SPA stable, uniformly in $a^2$.***
Consider the following system

\[
\begin{align*}
\dot{x} &= -x + u^2 + 4u, \\
y &= -(x + 4)^2
\end{align*}
\] (3.1)

The initial condition is chosen as \(x(0) = 2\). It is obvious that when \(x = -4\), \(y\) reaches its global maximum \(y^* = 0\). Let control input \(u = \theta\), we have \(\theta^* = -2\), \(x^* = -4\) and \(y^* = 0\). The initial value of \(x(0)\) is far away from the desired one \(x^* = -4\).
Outline

1. Introduction

2. Extremum seeking control: Convergence Analysis
   - First order extremum seeking scheme
   - Higher order extremum seeking scheme

3. Simulation example
   - The first order scheme
   - Comparison of the first and higher order schemes

4. Discussion

Report: Chun Yin
ESC
Increase $a$ will get a fast convergent speed

Let $\hat{\theta}(0) = 0$. By choosing $a = 0.15$, $\delta = 0.5(K = 4)$ and $\omega = 0.5$, the performance of the first order scheme is shown in following Figure where $|z| = |(\tilde{x}, \tilde{\theta})^T|$. It can be seen that, the state $z$ converges to the neighborhood of the origin. The output also converges to the vicinity of the extremum value. We increase $a$ such that $a = 0.35$ while keeping $\delta = 0.5(K = 4)$ and $\omega = 0.5$. From the result of Theorem 1, increasing $a$ will get a fast convergent speed, while the domain of the attraction would be smaller. It can be seen clearly from the next Figure that, though both $y(t)$ and $|z|$ converge very fast, it converges to a much larger neighborhood of the optimal values.
Figure: The performance of the simplest extremum seeking scheme.
Increase $\delta$ will get a fast convergent speed

Now, we fix $a = 0.3$ and $\omega = 0.25$, first let $\delta = 0.25$, as seen from the next Figure, the state $z$ converges to the neighborhood of the origin. The output also converges to the vicinity of the extremum value. Similarly, we increase $\delta$ to be $0.55$, the performance of the extremum seeking scheme is shown in Figure. The convergence speed of latter one is much faster.
Figure: The performance of the simplest extremum seeking scheme.
Increase $\omega$ will get a fast convergent seed

By choosing $a = 0.3$, $\delta = 0.5$ and $\omega = 0.1$, the performance of the first order scheme is shown in following Figure. It can be seen that, the state $z$ converges to the neighborhood of origin. The output also slowly converges to the vicinity of the extremum value. If we increase $\omega$ such that $\omega = 0.5$ while keeping $a = 0.5$ and $\delta = 0.5$, we can see from Figure that, though both $y(t)$ and $|z|$ converge very fast, they converge to a larger neighborhood of the desired one compared with the performance when a smaller $\delta (\delta = 0.1)$ is employed.
Figure: The performance of the simplest extremum seeking scheme.
Outline

1. Introduction
2. Extremum seeking control: Convergence Analysis
   - First order extremum seeking scheme
   - Higher order extremum seeking scheme
3. Simulation example
   - The first order scheme
   - Comparison of the first and higher order schemes
4. Discussion

Reporter: Chun Yin
In the simulation, two extremum seeking schemes (first and higher order schemes) are compared with same dynamics. We fix the parameters $a = 0.3$, $k = 0.2(\delta = 0.1, K = 4)$ and $\omega = 0.5$ for both schemes. In the extremum seeking scheme with a high-pass filter, let $\omega_h = 0.2$ while keeping other parameters the same as the first order scheme, the performances of two extremum seeking schemes are shown in next Figure, where $|z| = |(x - x^*, \theta - \theta^*, \xi)|$. It can be seen clearly that, the steady-state of the two schemes are comparable.
Figure: The performance of the simplest extremum seeking scheme.
Consider using a extremum-seeking sliding mode controller.

Consider using a extremum-seeking fractional-order controller.
Consider using a extremum-seeking sliding mode controller.

Consider using a extremum-seeking fractional-order controller.