How to design a fuzzy adaptive controller based on observers for uncertain affine nonlinear systems

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Abstract

This paper focuses on the construction of a fuzzy adaptive output feedback control based on any observer (high-gain (HG) observer, sliding mode (like) observer, etc.) for a class of single-input–single-output (SISO) uncertain or ill-defined affine nonlinear systems. Indeed, the corrective term of the proposed observer involves a well-defined design function which is shown to be satisfied by the commonly used HG based observers, namely for the usual HG observers and the sliding mode observers together with their implementable versions. The design of the underlying update law as well as the robust control term is based on an appropriate filtering of the output tracking error. This particularly allows to overcome the output observation error filtering or the necessity of the famous strictly positive real (SPR) condition.

Keywords: Nonlinear systems; Fuzzy adaptive control; Sliding mode observer; High-gain observer

1. Introduction

In recent years, the nonlinear control based on the universal function approximators (fuzzy systems, neural networks) has received much attention [16,14,57,36,51,42,10,6,4,3,37,52]. Fuzzy control, in particular, has an impact in the control community because the fuzzy controllers provide a systematic and efficient framework to incorporate linguistic fuzzy information from human expert. However, in complicated situations, where the plant parameters are subject to perturbations or when the dynamics of the systems are too complex for a mathematical model to describe, adaptive schemes have to be used online to gather data and adjust the control parameters automatically.

Based on the universal approximation theorem [59,52], several adaptive fuzzy control schemes have been developed for a class of nonlinear and uncertain systems, [51,42,41,10,6,4,3,52]. In such schemes, the stability of the closed-loop system is established according to Lyapunov’s theory. To cope with approximation errors (i.e. modelling errors or construction errors) and external disturbances, these adaptive fuzzy controllers are augmented by a robust compensator (i.e. a robustifying term or a robust control term) that can be a supervisory control [51,52], sliding

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mode control [42,41,10], and/or $H^\infty$ control [6,4,3]. A key assumption in these methods is that all the states of the plant are available for measurement, i.e. the fuzzy systems used to approximate the system nonlinearities (in the indirect versions) or the ideal control (in the direct versions) are functions of the system states which are supposed available for measurement.

In many control problems, state variables may be partially unavailable, under such cases, observer-based control schemes (or output feedback control schemes) should be applied. Observer design has been a very active field during the last decade, and has turned out to be much more challenging than the control problem. How to design a robust observer for an uncertain and perturbed nonlinear system is a very demanding problem. Based on state/or error observer, the direct or indirect adaptive fuzzy control schemes were developed in [27–29,13,54–56,50,48,45,17,12]. These schemes require strictly positive real (SPR) conditions on the estimation error dynamics (i.e., the observation error dynamics) so that they can use Meyer–Kalman–Yakubovich (MKY) lemma for stability analysis. The original observation error dynamics, which are not SPR in general, are augmented by a low-pass filter designed to satisfy the SPR condition of a transfer function associated with the Lyapunov stability analysis. However, these schemes result in the filtering of the fuzzy basis functions (FBF), which makes the dynamic order of the system controller/observer very large. Moreover, we think that there exist some drawbacks in these fuzzy adaptive control algorithms. The major problem that these errors are propagated from one paper to another. In order to overcome these drawbacks in the future works, the most significant errors in these papers are given in Section 5.

Other schemes of adaptive fuzzy controllers based on observer were developed in [33,35,46,47,43,25]. In these schemes no SPR condition is need for the stability proof. In [35,25], to avoid the use of the famous SPR condition and the filtering of the FBF, the output observation error is filtered and the state variables of the filter are used to design the underlying update law as well as the robust control term. However, in these control schemes, there is a kind of redundancy: on the one hand an observer is designed to evaluate the system states which will be employed in FBF construction, on the other hand a chain of integrators to estimate the filter states which will be used in the design of the robust control term and the parameters update. This chain of integrators is quite similar to an observer. In [47,46], the authors developed indirect and direct adaptive fuzzy controllers based on HG observer for a class of SISO and MIMO nonlinear systems, respectively. Although this kind of adaptive fuzzy control schemes can ensure the stability of the closed-loop system and achieve a prescribed tracking performance, this observer sometimes exhibits a peaking phenomenon in the transient behavior due to the high-gain (HG). To overcome this kind of problems, the solutions are proposed in [46,39,11].

Unlike the above contributions, in this paper, a fuzzy adaptive controller based on any observer (HG observer, sliding mode (like) observer, etc.) is designed. There are three main contributions that are worth to be emphasized. Firstly, there are still no reports of the fuzzy (or neural networks) adaptive control with sliding mode (like) observers independently of the nature of the system to be controlled, namely the MIMO or SISO unknown nonlinear dynamical systems. In the literature, the design methods of the observer-based control system are not general and the states are commonly estimated by using a relatively simple observer: a linear error observer [27–30,33,55,56,13,50,15,48,58], an HG observer [46,39,11] and an adaptive observer [54,35,47,45,17,12]). Secondly, the stability analysis as well as the controller implementation are carried out without resorting to the famous SPR condition (as in [27–29,13,54–56,50,48,45,17,12]) or the output observation error filtering (as in [35,25]). Of particular interest, in this present work, a suitable filtering of the output tracking error is made to design the update laws and the robust control term. Finally, a set of comments on the previously algorithms concerning "observer-based fuzzy adaptive control" is addressed in Section 5.

2. Notation and problem statement

Let $R$ denote the real numbers, $R^n$ the real $n$-vectors and $R^{n \times m}$ the real $m \times n$ matrices. We define the norm of a vector $\mathbf{x} \in R^n$ as $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and the norm of a matrix $A \in R^{n \times m}$ as $\|A\|_2 = \sqrt{\lambda_{\text{max}}(A^T A)} = \sigma_{\text{max}}(A)$, $\lambda_{\text{max}}(.)$ and $\lambda_{\text{min}}(.)$ are largest and smallest eigenvalues of a matrix and $\sigma_{\text{max}}(A)$ is the maximum singular value. The absolute value is denoted by $|.|$.

In this paper, we consider the $n$th order nonlinear dynamical system of the form [52]

\[ x^{(n)} = f(x, \dot{x}, \ldots, x^{(n-1)}) + g(x, \dot{x}, \ldots, x^{(n-1)})u + d(t), \]

\[ y = x \]  

(1)
or equivalently of the form
\[
\dot{x} = A\hat{x} + B[ f(\hat{x}) + g(\hat{x})u + d(t)],
\]
\[
y = C\hat{x},
\]
where
\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}, \quad C^T = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

\(u \in \mathbb{R}\) is the input and \(y \in \mathbb{R}\) is the output, \(f\) and \(g\) are unknown but continuous functions, \(d\) is the external bounded disturbance, and \(\hat{x} = [x, \dot{x}, \ldots, x^{(n-1)}]^T = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n\) is the state vector where not all \(x_i\) are assumed to be available for measurement. Only the system output \(y\) is assumed to be measurable.

Let \(y_m\) be a bounded reference signal, \(e = y_m - y\) the output tracking error, and \(\hat{x}\) the estimate of \(x\). Denote \(\dot{y}_m = \dot{y}_m, \ldots, y_m^{(n-1)}\), \(\hat{y}_m = [e, \dot{e}, \ldots, e^{(n-1)}]^T\), \(\hat{\hat{y}}_m = [\hat{e}, \dot{\hat{e}}, \ldots, \hat{e}^{(n-1)}]^T\), \(\hat{\hat{y}} = \hat{x} - \hat{\hat{y}}\).

So that (2) be controllable and feedback linearizable, it is required that \(\hat{g}(\hat{x}) \neq 0\) for \(\hat{x}\) in a certain controllability region \(\Omega_{\hat{x}} \subset \mathbb{R}^n\). Without loss of generality, the following assumption is made:

Assumption 1 (Wang et al. [56,55,58], Slotine and Li [40], Wang [53]). Assume that \(\hat{g}(\hat{x})\) and \(d(t)\) satisfy: \(0 < \hat{g}(\hat{x}) \leq g_H\), and \(|d(t)| \leq D\), respectively, for all \(\hat{x} \in \Omega_{\hat{x}} \subset \mathbb{R}^n\), where \(D\) and \(g_H\) are positive constants.

It is worth noting that there are several nonlinear systems can be given the form (1) and satisfy the assumption “\(0 < \hat{g}(\hat{x}) \leq g_H\)”, e.g. Duffing chaotic system, Chua’s circuit, aircraft wing rock, induction servo-motor drive, inverted pendulum, and many others.

The control problem consists in determining the control input \(u\) to steer the state variables of the system \(\hat{x}\) close to the reference signal \(y_m\), while ensuring all involved signals in the closed-loop remain bounded.

If \(d(t) = 0\) and using the feedback linearization, we know that there exists some ideal controller
\[
u^* = \frac{1}{\hat{g}(\hat{x})}[-f(\hat{x}) + y_m^{(n)} + K_c^T e],
\]
where \(K_c\) is the feedback gain vector, selected such that the characteristic polynomial of \(A - BK_c^T\) is Hurwitz because \((A, B)\) is controllable. However, the control law (3) is inapplicable, since the nonlinear functions \((f(\hat{x})\) and \(g(\hat{x})\) are not known, and the system states \(\hat{x}\) are not available. Thereafter, to overcome such problems, a fuzzy system will be used to approximate directly the ideal control \(u^*\) and an observer will be designed to estimate the system states.

3. Description of the used fuzzy logic system

The basic configuration of a fuzzy logic system consists of a fuzzifier, some fuzzy IF–THEN rules, a fuzzy inference engine and a defuzzifier, as shown in Fig. 1. The fuzzy inference engine uses the fuzzy IF–THEN rules to perform a mapping from an input vector \(\hat{x}^T = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n\) to an output \(\hat{f} \in \mathbb{R}\).

The \(i\)th fuzzy rule is written as
\[
R^{(i)}: \text{if } x_1 \text{ is } A^{(i)}_1 \text{ and } \ldots \text{ and } x_n \text{ is } A^{(i)}_n \text{ then } \hat{f} \text{ is } f^i,
\]
where \(A^{(i)}_1, A^{(i)}_2, \ldots, A^{(i)}_n\) are fuzzy variables and \(f^i\) is the fuzzy singleton for the output in the \(i\)th rule. By using the singleton fuzzifier, product inference, and center-average defuzzifier, the output of the fuzzy system can be expressed
as follows:

\[ \hat{f}(x) = \frac{\sum_{i=1}^{r} f^i(\prod_{j=1}^{n} \mu_{A_j^i}(x_j))}{\sum_{i=1}^{r} (\prod_{j=1}^{n} \mu_{A_j^i}(x_j))} = \theta^T \psi(x), \]  

(5)

where \( \mu_{A_j^i}(x_j) \) is the membership function of the fuzzy set \( A_j^i \), \( r \) is the number of fuzzy rules, \( \theta^T = [f^1, f^2, \ldots, f^r] \) is the adjustable parameter vector (composed of consequent parameters), and \( \psi^T = [\psi^1 \psi^2 \ldots \psi^r] \), where

\[ \psi^i(x) = \frac{(\prod_{j=1}^{n} \mu_{A_j^i}(x_j))}{\sum_{i=1}^{r} (\prod_{j=1}^{n} \mu_{A_j^i}(x_j))} \]  

(6)

is the FBF (i.e. the normalized firing strength). In this paper, it is assumed that there exists always at least one active rule, i.e. \( \sum_{i=1}^{r} (\prod_{j=1}^{n} \mu_{A_j^i}(x_j)) > 0 \).

It is worth noting that the fuzzy system (5) is the most frequently used one in control applications. Following the universal approximation results [59,52], the fuzzy system (5) is able to approximate any nonlinear smooth function \( f \) on a compact operating space to any degree of accuracy. In this paper, like the majority of the available results, it is assumed that the structure of the fuzzy system and the membership function parameters are properly specified in advance by the designer. This means that the designer decision is needed to determine the structure of the fuzzy system (that is, pertinent inputs, number of membership functions for each input, membership function parameters, and numbers of rules), and the consequent parameters, i.e. \( \theta \), must be calculated by learning algorithms.

4. Fuzzy adaptive controller and observer design

In this section, a direct fuzzy adaptive controller based on any observer (HG observer, sliding mode observer, etc.) is proposed.

4.1. Fuzzy adaptive controller design

After adding and subtracting the term \( g(x)u^* \), system (2) becomes

\[ \dot{x} = Ax + B[f(x) + g(x)u^* + g(x)(u - u^*) + d(t)], \]

\[ y = Cx. \]

(7)

Let us define the tracking error vector as \( e = y_{m} - x \). And substituting (3) into (7), after some manipulations, it is obtained that

\[ \dot{e} = [A - BK_e^T]e + B[-g(x)(u - u^*) - d(t)], \]

\[ e = Ce. \]

(8)
The control input for system (2) can be determined as

$$u = u_a - u_r,$$

where $u_a$ is the fuzzy adaptive term control which is designed to approximate the ideal control law $u^*$ (3), and the bounded robust term $u_r$ is employed to compensate the fuzzy approximation error and the external disturbances. Substituting (9) into (8) yields

$$\dot{e} = [A - BK_e^T]e + B[-g(x)(u_a - u^*) + g(x)u_r - d(t)],$$

$$e = Ce.$$

(10)

The ideal control $u^*$ can be approximated according to the universal approximation theorem [59,52] such that

$$u^* = u^*(x, \theta^*) + \delta(x)$$

$$= \theta^T\psi(x) + \delta(x),$$

(11)

where $\delta(x)$ is the fuzzy approximation error and $\theta^*$ is the ideal parameter vector which is defined as follows:

$$\theta^* = \arg \min_{\theta \in \Omega_\theta} \left[ \sup_{x \in \Omega_x} |\theta^T\psi(x) - u^*| \right],$$

(12)

where $\Omega_\theta$ denotes the set of suitable bounds on $\theta$. Note that the ideal parameter vector $\theta^*$ is an artificial constant quantity introduced only for analysis purpose and its value is not needed when implementing the controller. However, we need the following assumption for the ideal parameter vector [52].

**Assumption 2.** The ideal parameter vector satisfies

$$\|\theta^*\| \leq M_\theta,$$

(13)

where $M_\theta$ is unknown positive constant.

According to the universal approximation theorem [59,52] for the fuzzy logic systems, there exists a positive constant $c_0$ such that the following inequality holds for all $x \in \Omega_x$:

$$|\delta(x)| \leq c_0.$$

(14)

From (5), the fuzzy adaptive control term $u_a$ can be described as

$$u_a = \theta^T\psi(\hat{x}).$$

(15)

Using (15), (10) becomes

$$\dot{e} = (A - BK_e^T)e + B[-g(x)\tilde{\theta}^T\psi(\hat{x}) + g(x)u_r + w],$$

$$e = Ce,$$

(16)

where $\tilde{\theta} = \theta - \theta^*$ is the parameter error vector and

$$w = g(x)([u^* - \theta^T\psi(x)] + \theta^T[\psi(x) - \psi(\hat{x})]) - d(t)$$

(17)

which represents the lumped disturbance term generated by the state estimation error, the fuzzy approximation error and external disturbances.

Now, the output tracking error dynamics of (16) can be given by

$$e = H(s)[-g(x)\tilde{\theta}^T\psi(\hat{x}) + g(x)u_r + w],$$

(18)

where $s$ is the Laplace variable, and $H(s) = C(sI - (A - BK_e^T))^{-1}B$ is the transfer function of (16).
If we select \( K_T = [C_n x^n, \ldots, C_n^2 x^2, C_n^1 x] \) with \( C_n^i = n! / ((n - i)! i!) \), it can easily be shown that

\[
H(s) = \frac{1}{(s + z)^n},
\]

(19)

where \( z \) is a positive design parameter.

In general, the problem of the observer-based fuzzy adaptive control consists in addressing the following question: How to design the update laws for the consequent parameters and the robust control term using only the available signals, i.e. measured and/or estimated signals? In [27,28,54–56,13,50,12], the authors use an appropriate filtering on the observation error dynamics to enforce the SPR condition, inducing thereby the filtering of the regressor vector. Then, the output observation error which is available for measurement is used to design the update law and the robust control term. In [35,25], instead of seeking to get the SPR condition, the output observation error is filtered and the state variables of the involved filter are used to update the fuzzy parameters. Unlike the above contributions, we propose to filter the output tracking error to design the update law and the robust control term.

Motivated by the works in [38,63], we consider a new variable \( e_f \) defined as follows:

\[
\dot{e}_f + K e_f = z_0 (\dot{e} + xe).
\]

(20)

From (20), we have

\[
e_f = \left[ \frac{z_0 (s + z)}{(s + K)} \right] e,
\]

(21)

where \( K \) and \( z_0 \) are positive design parameters.

Then, using (21), Eq. (18) can be expressed as follows:

\[
e_f = \frac{L(s)}{s + K} [-g(x) \tilde{\psi}(\hat{x}) + g(x) u_r + w]
\]

(22)

with \( L(s) \) is a stable filter given by: \( L(s) = z_0 / (s + z)^{n-1} \).

From (22), the differential equation, which allows to generate \( e_f \), can be obtained as

\[
\dot{e}_f + K e_f = L(s)[-g(x) \tilde{\psi}(\hat{x}) + g(x) u_r + w]
\]

(23)

or, (23) can be rewritten as follows:

\[
\dot{e}_f + K e_f = -\tilde{\psi}(\hat{x}) + u_r + w_1,
\]

(24)

where

\[
w_1 = \tilde{\psi}(\hat{x}) - u_r + L(s)[-g(x) \tilde{\psi}(\hat{x}) + g(x) u_r + w].
\]

(25)

Note that though the right-hand side of (24) contains unknown parameters and signals, we can obtain the signal \( e_f \) via filtering of \( e \) which is available (i.e. using Eq. (20)). Then, the dynamical equation (24) is used for analysis purpose only. From (20), we can generate the signal \( e_f \) as follows:

\[
e_f = -\int (K e_f - z_0 e) \, dt + z_0 e \quad \text{with} \ e_f(0) = 0.
\]

(26)

Since the signal \( e_f \) is available, the parameters update and the robust control term \( u_r \) can be designed as follows:

\[
\dot{\theta} = \gamma [e_f \tilde{\psi}(\hat{x}) - \sigma \theta],
\]

(27)

\[
u_r = -\rho \text{sign}(e_f)
\]

(28)

with \( \gamma \) and \( \sigma \) are positive design parameters, \( \rho > 0 \) will be defined later.
4.2. Observer design

In order to solve the problem of the unavailable states, a more general observer in this subsection is proposed. Before giving our general observer, we introduce the following interesting notations:

1. Let \( D = \text{diag} \left[ 1, \frac{1}{\lambda}, \ldots, \frac{1}{\lambda^{n-1}} \right] \), \( \lambda > 1 \) is a real number.

2. Let \( S \) be the unique solution of the following algebraic Lyapunov equation \( [31, 44, 7, 61, 9] \):
   \[
   S + A^T S + SA = C^T C. \tag{30}
   \]
   It can be shown that the solution of (30) is symmetric positive definite \( [31, 44, 7, 61, 9] \).

3. \( \forall \xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{R}^n \), set \( \tilde{\xi} = D^{1/2} \tilde{\xi} \), \( K(\tilde{\xi}) = [k_1(\tilde{\xi}_1), 0, \ldots, 0]^T \in \mathbb{R}^n \) be a vector of smooth or non-smooth functions satisfying
   \[
   \forall \xi \in \mathbb{R}^n : \tilde{\xi}^T K(\tilde{\xi}) \geq \frac{1}{2} \xi^T C^T C \xi. \tag{31}
   \]

To estimate the tracking error dynamics (16), the following more general observer is proposed:

\[
\dot{\hat{e}} = [A - BK^T c] \hat{e} + \lambda D^{-1/2} S^{-1/2} K(\tilde{\xi}),
\]

\[
\hat{e} = C \hat{e}. \tag{32}
\]

If we define the observation error vector as \( \tilde{e} = \hat{e} - \hat{x} = \hat{\xi} - x \), and subtracting (32) from (16), we get the following observation error dynamics:

\[
\dot{\tilde{e}} = A \tilde{e} + B[-K^T c \tilde{e} - g(x) \tilde{\theta}^T \psi(\hat{x}) + g(x) u_r + w] - \lambda D^{-1/2} S^{-1/2} K(\tilde{\xi}),
\]

\[
\tilde{e} = C \tilde{e}. \tag{33}
\]

To make the stability analysis easy, let us define a state transformation as

\[
\tilde{z} = A \tilde{e}.
\]

(34)

It can be shown that the vector \( \tilde{z} \) has the following useful properties:

(a) \( \|z\| \leq \|\tilde{e}\| \leq \lambda^{n-1} \|\hat{e}\| \).

(b) \( Cz = z = C\tilde{e}. \tag{35} \)

Since \( \lambda D A^{-1} = \lambda A \) and the fact that \( K(\tilde{e}) = K(z) \), Eq. (33) can be written in terms of \( \tilde{z} \) as follows:

\[
\dot{\tilde{z}} = \lambda A \tilde{z} - \lambda S^{-1} K(z) + \lambda B[-K^T c \tilde{e} - g(x) \tilde{\theta}^T \psi(\hat{x}) + g(x) u_r + w],
\]

\[
z = C \tilde{z}. \tag{37}
\]

Now, to establish the stability of the proposed adaptive approach, the following lemma is used.

**Lemma 1.** If Assumptions 1 and 2 are satisfied, then there exist positive constants \( c_1, c_2, c_3, c_4 \) and \( c_5 \) such as

(a) \( |w| \leq c_1. \)

(b) \( |g(x) \tilde{\theta}^T \psi(\hat{x}) + g(x) u_r + w| \leq c_2 \|\tilde{\theta}\| + c_3. \)

(c) \( |w_1| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(d) \( |w_2| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(e) \( |w_3| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(f) \( |w_4| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(g) \( |w_5| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(h) \( |w_6| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(i) \( |w_7| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(j) \( |w_8| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(k) \( |w_9| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(l) \( |w_10| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(m) \( |w_11| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(n) \( |w_12| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(o) \( |w_13| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(p) \( |w_14| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(q) \( |w_15| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(r) \( |w_16| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(s) \( |w_17| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(t) \( |w_18| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(u) \( |w_19| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(v) \( |w_20| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(w) \( |w_21| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(x) \( |w_22| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(y) \( |w_23| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(z) \( |w_24| \leq c_4 \|\tilde{\theta}\| + c_5. \)

(\(A \subseteq B\))
Proof. (a) From Assumptions 1 to 2 and (14), we have
\[ |w| = |g(x)(u^* - \theta^T \psi(x)) + g(x)\theta^T [\psi(x) - \hat{\psi}(\hat{x})] - d(t)| \]
\[ \leq |g(x)\delta(x)| + |g(x)\theta^T [\psi(x) - \hat{\psi}(x)]| + |d(t)| \]
\[ \leq g_H c_0 + g_H M_\theta \sup_{\tau} \|\psi(\hat{x}) - \psi(x)\| + D \]
\[ = c_1 \]
posing \( c_1 = g_H c_0 + g_H M_\theta \sup_{\tau} \|\psi(\hat{x}) - \psi(x)\| + D \).

(b) Based on Assumptions 1 and 2, and the boundedness of \( u_r \) and \( w \), we have
\[ \| -g(x)\hat{\theta}^T \psi(\hat{x}) + g(x)u_r + w\| \leq \| -g(x)\hat{\theta}^T \psi(\hat{x})\| + |g(x)u_r| + |w| \]
\[ \leq g_H \sup_{\tau} \|\psi(\hat{x})\| \|\hat{\theta}\| + g_H |u_r| + c_1 \]
\[ = c_2 \|\hat{\theta}\| + c_3, \]
where \( c_2 = g_H \sup_{\tau} \|\psi(\hat{x})\| \) and \( c_3 = g_H |u_r| + c_1 \).

(c) \( w_1 \) can be arranged as follows:
\[ w_1 = -u_r + L(s)[g(x)u_r] + \tilde{\theta}^T g(x) + L(s)[-g(x)\hat{\theta}^T \psi(\hat{x})] + L(s)w. \]

Since \( L(s) \) is a stable filter and \( u_r, g(x), \psi(\cdot), \) and \( w \in L_\infty \), it is very clear that there exist positive constants \( c_6, c_7 \) and \( c_4 \) such as [35,56,27,34]
\[ \| -u_r + L(s)[g(x)u_r]\| \leq c_6, \]
\[ \|\tilde{\theta}^T \psi(\hat{x}) + L(s)[-g(x)\hat{\theta}^T \psi(\hat{x})]\| \leq c_4 \|\hat{\theta}\|, \]
\[ |L(s)w| \leq c_7. \]

Using the above inequalities, \( w_1 \) can be bounded by
\[ |w_1| \leq \| -u_r + L(s)[g(x)u_r]\| + \|\tilde{\theta}^T \psi(\hat{x}) + L(s)[-g(x)\hat{\theta}^T \psi(\hat{x})]\| + |L(s)w| \]
\[ \leq c_6 + c_4 \|\hat{\theta}\| + c_7 \]
\[ \leq c_4 \|\hat{\theta}\| + c_5, \]
where \( c_5 = c_7 + c_6. \)

4.3. Stability analysis

The following results conclude the stability of the closed-loop system (16), (24) and (37).

Theorem. Consider system (1) with its observer (32), and its controller (9), (15) and (28). Let the parameter vector \( \theta \) be adjusted by the update law (27), and Assumptions 1–2 be true. Then, we have the following property:

(1) If the design parameters are selected such as: \( \rho \geq c_5, \lambda > \overline{\lambda}/\lambda_{\min}(S), K > \overline{K}, \) and \( \sigma > \overline{\sigma} \), where \( \overline{\lambda}, \overline{K} \) and \( \overline{\sigma} \) are constants defined later and if the design function \( K(\overline{e}) \) is chosen so that the condition (31) is always satisfied, then all involved signals are uniformly ultimately bounded (UUB), i.e., \( \overline{\xi}, \overline{e}_f, \overline{e}, \overline{\hat{e}}, 0 \), and \( u \in L_\infty \).

Proof. Consider the following Lyapunov function candidate:
\[ V = V_1 + \beta V_2, \]
where \( \beta = 1/\lambda^{2n-2} \), and
\[ V_1 = \overline{\xi}^T S \overline{\xi}, \]
and
\[ V_2 = \frac{1}{2} e_f^2 + \frac{1}{2\gamma} \hat{\theta}^T \hat{\theta}. \]  

(47)

Differentiating \( V_1 \) along of solution (37) and using (30) yields
\[ \dot{V}_1 = \dot{z}^T S_z \dot{z} + \dot{z}^T S_z \dot{z} = -\lambda \dot{z}^T S_z \dot{z} - 2\lambda \dot{z}^T C \dot{z} - 2z^T \dot{z}^T C \dot{z} + 2z^T S A_{\dot{x}} B [ -K \dot{z} - g(x) \dot{\theta}^T \psi(\dot{x}) + g(x) u_r + w]. \]  

(48)

If \( K(z) \) is selected so that condition (31) is always satisfied, (48) becomes
\[ \dot{V}_1 \leq -\lambda \lambda_{\min} \left\| z \right\|^2 + 2\lambda \left\| S A_{\dot{x}} B \right\| [ -K \dot{z} - g(x) \dot{\theta}^T \psi(\dot{x}) + g(x) u_r + w]. \]  

(49)

Since \( \| S A_{\dot{x}} B K \| = \sqrt{\beta} \| S B K \| \), \( \| S A_{\dot{x}} B \| = \sqrt{\beta} \| S B \| \) and \( \sqrt{\beta} \| z \| \leq \| z \| \), and using (39), (49) can be arranged as follows:
\[ \dot{V}_1 \leq -\lambda \lambda_{\min} \left\| z \right\|^2 + c_8 \left\| z \right\|^2 + 2\sqrt{\beta} c_9 \| z \| \| \dot{\theta} \| + 2\sqrt{\beta} c_{10} \| z \|. \]  

(50)

where \( c_8 = 2 \| S B K \|, c_9 = c_2 \| S B \|, \) and \( c_{10} = c_3 \| S B \| \).

The differentiation of (47) along of the solution (24) gives
\[ \dot{V}_2 = e_f \dot{e}_f + \frac{1}{\gamma} \dot{\theta}^T \hat{\theta} = e_f [-K e_f + \dot{\theta}^T \psi(\dot{x}) + u_r + w_1] + \frac{1}{\gamma} \dot{\theta}^T \hat{\theta} = -K e_f^2 + e_f u_r + e_f w_1 + \frac{1}{\gamma} \dot{\theta}^T \left[ \dot{\theta} - \gamma e_f \dot{\theta} \right]. \]  

(51)

where \( \hat{\theta} = \dot{\theta} - \dot{\theta} = \dot{\theta} \).

Using (27)–(28) and (40), (51) becomes
\[ \dot{V}_2 = -K e_f^2 + e_f u_r + e_f w_1 - \sigma \dot{\theta}^T \theta \]
\[ \leq -K e_f^2 - \rho |e_f| + |e_f| (c_4 \| \dot{\theta} \| + c_5) - \sigma \dot{\theta}^T \theta \]
\[ \leq -K e_f^2 + c_4 \| \dot{\theta} \| |e_f| - \sigma \dot{\theta}^T \theta \]  

(52)

choosing \( \rho \geq c_5 \).

Using the following inequality \( 2\dot{\theta}^T \theta \geq \| \dot{\theta} \|^2 - \| \theta^* \|^2 \), (52) can be rewritten as follows:
\[ \dot{V}_2 \leq -K e_f^2 + c_4 \| \dot{\theta} \| |e_f| - \frac{\sigma}{2} \| \dot{\theta} \|^2 + \frac{\sigma}{2} \| \theta^* \|^2. \]  

(53)

From (50) and (53), the time derivative of (45) can be bounded by
\[ \dot{V} \leq -\lambda \lambda_{\min} \left\| z \right\|^2 + c_8 \left\| z \right\|^2 + 2\sqrt{\beta} c_9 \| z \| \| \dot{\theta} \| + 2\sqrt{\beta} c_{10} \| z \| - \beta K e_f^2 + \beta c_4 \| \dot{\theta} \| |e_f| \]
\[ - \frac{\beta \sigma}{2} \| \dot{\theta} \|^2 + \frac{\beta \sigma}{2} \| \theta^* \|^2. \]  

(54)
Substituting the following inequalities into (54)
\[ 2\sqrt{\beta c_9} \|\tilde{z}\| \|\tilde{\theta}\| \leq \frac{c_9}{\alpha_1} \|\tilde{z}\|^2 + \beta \alpha_1 \|\tilde{\theta}\|^2, \]
\[ 2\sqrt{\beta c_{10}} \|\tilde{z}\| \leq \frac{1}{\alpha_2} \|\tilde{z}\|^2 + \beta \alpha_2 \gamma_{10}^2, \]
\[ \beta c_4 \|\tilde{\theta}\| |e_f| \leq \frac{\beta c_4}{4 \alpha_3} e_f^2 + \beta \alpha_3 \|\tilde{\theta}\|^2, \]
where \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) are positive constants, yields
\[ \hat{V} \leq -(\lambda \lambda_{\text{min}}(S) - \tilde{\lambda}) \|\tilde{z}\|^2 - \beta (K - \bar{K}) e_f^2 - 0.5 \beta (\sigma - \bar{\sigma}) \|\tilde{\theta}\|^2 + \beta \bar{m}, \]
where
\[ \tilde{\lambda} = \frac{c_9}{\alpha_1} + \frac{1}{\alpha_2} + c_8, \quad \bar{K} = \frac{c_4}{4 \alpha_3}, \quad \bar{\sigma} = 2(\lambda_1 + \lambda_3) \quad \text{and} \quad \bar{m} = \frac{\sigma}{2} |\theta^*|^2 + \alpha_2 \gamma_{10}^2. \]
Choosing \( \lambda > \tilde{\lambda}/\lambda_{\text{min}}(S), K > \bar{K} \) and \( \sigma > \bar{\sigma} \), we can guarantee that \( \hat{V} \) is negative as long as \( \tilde{z} \) is outside the compact set \( \Omega_{\tilde{z}} \) defined as
\[ \Omega_{\tilde{z}} = \left\{ \tilde{z} \|\tilde{z}\| \leq \sqrt{\frac{\bar{m}}{\lambda \lambda_{\text{min}}(S) - \tilde{\lambda}}} \right\}. \]
According to standard Lyapunov theorem [19,23], we conclude that \( \tilde{z} \) is bounded and will converge to \( \Omega_{\tilde{z}} \). Moreover the radius of the set can be made arbitrarily small if \( \lambda \) is chosen to be sufficiently large. Similarly, the signal \( e_f \) is bounded and will converge to \( \Omega_{e_f} \) defined as
\[ \Omega_{e_f} = \left\{ |e_f| |e_f| \leq \sqrt{\frac{\bar{m}}{K - \bar{K}}} \right\}, \]
whose radius can be made also arbitrarily small if \( K \) is selected to be sufficiently large. The parameter error vector \( \tilde{\theta} \) is also bounded and converges to \( \Omega_{\tilde{\theta}} \) which is defined as
\[ \Omega_{\tilde{\theta}} = \left\{ \tilde{\theta} \|\tilde{\theta}\| \leq \sqrt{\frac{2\bar{m}}{\sigma - \bar{\sigma}}} \right\}. \]
Because of Assumption 2, i.e. \( \theta^* \in L_\infty \), the boundedness of \( \tilde{\theta} \) can guarantee that of \( \theta \). Since (21) is in a bounded input bounded output (BIBO) form, i.e. \( e = [(s + K)/z_0(s + z)]e_f \), we can conclude that the boundedness of the output of tracking error \( e \) follows that of the signal \( e_f \). From (16), the fact that the term \( \tilde{u} = -g(x)\tilde{\psi}(\tilde{x}) + g(x)u_r + w \) is bounded (since \( \tilde{u}, \tilde{\psi}(\tilde{x}), u_r, g(x), \) and \( w \in L_\infty \)), we can easily show the boundedness of the tracking error vector \( e \). In other words, (16) is in a BIBO form, then the boundedness of \( e \) follows that of \( \tilde{u} \) (also, to show the boundedness of \( e \), we can use the same argument as in [19,62]). Since \( e, \tilde{e} \in L_\infty \), and \( \tilde{e} = \tilde{e} - \tilde{e} \) then \( \tilde{e} \in L_\infty \). Finally, observe that \( \psi(\tilde{x}), u_r, \) and \( \theta \in L_\infty \). Hence, \( u \in L_\infty \). This ends the proof of theorem. \( \square \)

To summarize, Fig. 2 shows the overall scheme of the fuzzy adaptive control based on any observer, proposed in this paper.

**Remark 1.** (1) By giving the particular expressions to \( K(\tilde{e}) \) satisfying condition (31), HG observers and sliding mode observers can be derived. Table 1 summarizes the observers obtained according to the choice of \( K(\tilde{e}) \). Note that the appellation “sliding mode observer” of the observers 2–8 in Table 1 is inspired from [8,20,5,22].
Table 1
High-gain and sliding mode observers obtained according to the choice of $K(\bar{e})$

<table>
<thead>
<tr>
<th>$K(\bar{e})$</th>
<th>Observer derived</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $K_{HG}(\bar{e}) = C^T(C\bar{e})$</td>
<td>HG observer [21,47,46,39]</td>
</tr>
<tr>
<td>(2) $K_{SM}(\bar{e}) = C^T(C\bar{e}) + lC^T(C\text{sign}(\bar{e}))$</td>
<td>Sliding mode observer (nonsmooth) [20,5,22]</td>
</tr>
<tr>
<td>(3) $K_{SM}(\bar{e}) = C^T(C\bar{e}) + lC^T(C\tanh(k_0\bar{e}))$</td>
<td>Sliding mode observer (smooth) [8]</td>
</tr>
<tr>
<td>(4) $K_{SM}(\bar{e}) = C^T(C\bar{e}) + lC^T(C\text{Sinh}(k_0\bar{e}))$</td>
<td>Sliding mode observer (smooth)</td>
</tr>
<tr>
<td>(5) $K_{SM}(\bar{e}) = C^T(C\bar{e}) + lC^T(C\text{arctan}(k_0\bar{e}))$</td>
<td>Sliding mode observer (smooth)</td>
</tr>
<tr>
<td>(6) $K_{SM}(\bar{e}) = C^T(C\bar{e}) + lC^T(C\text{Sat}(\bar{e}))$</td>
<td>Sliding mode observer (smooth)</td>
</tr>
<tr>
<td>(7) $K_{SM}(\bar{e}) = C^T(C\bar{e}) + l(C\bar{e}/(v +</td>
<td>C\bar{e}</td>
</tr>
<tr>
<td>(8) $K_{SM}(\bar{e}) = C^T(C\bar{e}) + l(</td>
<td>C\bar{e}</td>
</tr>
</tbody>
</table>

'sign' denotes the usual saturation function, $Sat$ the usual saturation function, tanh the hyperbolic tangent function, Sinh the hyperbolic sine function and arctan the inverse tangent function, with $k_0$, $v$, $p > 0$ are real numbers.

Table 2
Sliding mode observers simplified, obtained according to the choice of $K(\bar{e})$

<table>
<thead>
<tr>
<th>$K(\bar{e})$</th>
<th>Observer derived</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $K_{SM}(\bar{e}) = lC^T(C\text{sign}(\bar{e}))$</td>
<td>Sliding mode observer (nonsmooth) [1]</td>
</tr>
<tr>
<td>(2) $K_{SM}(\bar{e}) = lC^T(C\tanh(k_0\bar{e}))$</td>
<td>Sliding mode observer (smooth)</td>
</tr>
<tr>
<td>(3) $K_{SM}(\bar{e}) = lC^T(C\text{Sinh}(k_0\bar{e}))$</td>
<td>Sliding mode observer (smooth)</td>
</tr>
<tr>
<td>(4) $K_{SM}(\bar{e}) = lC^T(C\text{arctan}(k_0\bar{e}))$</td>
<td>Sliding mode observer (smooth)</td>
</tr>
<tr>
<td>(5) $K_{SM}(\bar{e}) = lC^T(C\text{Sat}(\bar{e}))$</td>
<td>Sliding mode observer (smooth)</td>
</tr>
<tr>
<td>(6) $K_{SM}(\bar{e}) = lC^T(C\bar{e}/(v +</td>
<td>C\bar{e}</td>
</tr>
<tr>
<td>(7) $K_{SM}(\bar{e}) = l(</td>
<td>C\bar{e}</td>
</tr>
</tbody>
</table>

(2) Also, the simplified sliding mode observers can be derived by giving the particular expressions to $K(\bar{e})$. Table 2 summarizes the second type of the sliding mode observers obtained. It is easy to see that the expressions given in Table 2 satisfy the condition (31) for relatively high value of $l$ (e.g. the observer $K_{SM}(\bar{e}) = lC^T(C\text{sign}(\bar{e}))$ or $K_{SM}(\bar{e}) = lC^T(C\tanh(k_0\bar{e}))$ satisfy the condition (31), if $l \geq 0.5 \sup_{\bar{e}} |\bar{e}|$). This condition imposed on the gain $l$ is very used in the literature (i.e. in the convergence proof of the sliding mode observers, e.g. see [1,20,60]).
Remark 2. The observers presented herein may exhibit a peaking phenomenon and the estimated state errors might be very large in the initial transient period. In order to avoid the peaking, we propose the following solutions:

1) either, the observer turns on time $t_1$ (where $t_1$ is the time necessary for the observer’s convergence) before the controller is put into operation. Therefore, the peaking of the controller can be avoided [11].

2) or, we introduce an estimate saturation or input saturation [39]. Thus, during the short transient period when the states estimate exhibit peaking, the saturation prevents the peaking from being transmitted to the plant.

Remark 3. (1) Linear observers designed in [27–30,33,54–56,13,50,48,58] have the following form:

\[
\dot{\hat{e}} = (A - BK_e)\hat{e} + K^e_0\hat{e}, \\
\hat{e} = C\hat{e}.
\]  

Putting $K^e_0\hat{e} = [C^1_0, \ldots, C^n_0]T\hat{e}$, where $C_i = n!/[i!(n-i)!i!])$, we find that this observer is a particular case of our set of proposed observers, where $K^e_0\hat{e} = \lambda\Delta^{-1}_e S^{-1}K_{HG}(\hat{e})$.

Remark 4. The fuzzy parameters update law (27) consists of a gradient algorithm along with the $\sigma$-modification term. The addition of this term eliminates the assumption of the persistent excitation [18] and ensures that no parameters drift takes place. Herein, the fuzzy learning (this appellation is inspired from “Neural networks learning”) takes place on-line, and no off-line training is required. In literature, several techniques and alternations were proposed to ensure boundedness of all signals (system states, control, fuzzy parameters) in the presence of the system uncertainties. These include $\sigma$-modification [18,33–35], $\epsilon$-modification [32,38], parameter projection [56,27,52,28,55], and dead-zone [24]. The idea is to modify the adaptive law so that the time derivative of the Lyapunov function used to analyze the adaptive scheme becomes negative when the adaptive parameters go beyond certain bounds. Although, $\sigma$-modification was introduced in (27) to avoid parameter drift, it has a disadvantage that even in the ideal case when there is a perfect fuzzy approximation without other disturbances, $\sigma$-modification does not drive the errors to zero. This shortcoming motivated another variation called $\epsilon$-modification, which eliminates the main drawback of $\sigma$-modification by multiplying the norm of error signal with the $\sigma$-modification term in the adaptive law. Parameter projection keeps the fuzzy parameters inside of prescribed convex set that contains the unknown ideal parameters. This approach requires a known norm bound for the fuzzy parameters, while both $\sigma$-modification and $\epsilon$-modification require no a priori information about the fuzzy parameters. Note that a comprehensive treatment of robust adaptive control can be found in [19].

Remark 5. (a) Although the choice of the parameter $\rho$ (in Eq. (28)) depends on the unknown constant $c_5$, this parameter is generally selected by trial and error (e.g. as in [56,27,28,25]), or estimated on-line as follows: $\hat{\rho} = \gamma_1(e_f) - \sigma_1\rho$ where $\gamma_1$ and $\sigma_1$ are positive design constants (as in [33–35]).

(b) In order to remedy to the chattering, the discontinuous function $\text{sign}(e_f)$ in the robust control term (28) can be replaced by an equivalent smooth function such as: $\text{Sat}(e_f)$, $\tanh(k_r e_f)$, $\arctan(k_r e_f)$, or $e_f / (e_f + |e_f|), \ldots$ where $k_r, \epsilon_r > 0$.

5. Some useful comments on the previous “observer-based fuzzy control schemes”

Several elegant schemes of the adaptive fuzzy control based on observer are proposed in literature. However, in the majority of these schemes, there are some drawbacks which are propagated from one paper to another. Thus, it is worth to quote them in order to avoid them in the future works. The comments made here will reinforce the two wise comments in [26,49]. It is important to note that the notations employed in this section are those used in the papers in question.

The comments are given in Appendix.

6. Simulation

In this simulation, we check our proposed approach on the tracking control of unknown affine nonlinear systems in order to illustrate the points made in the earlier sections.
Fig. 3. Fuzzification for the fuzzy system inputs $x_1, x_2$ in Example 1.

Example 1. Consider the Duffing forced oscillation system [52]:

$$
\dot{x}_1 = x_2,
\dot{x}_2 = -0.1x_2 - x_1^3 + 12\cos(t) + u + d(t),
y = x_1.
$$

Note that the system (60) is chaotic if $u = 0$ [52]. It is assumed that the external disturbance $d(t)$ is a square wave having an amplitude $\pm 1$ with a period of $2\pi(s)$. The control objective is to maintain the system to track the desired trajectory, $y_m = \sin(t)$, under the condition that only the system output $y$ is measurable.

We suppose that there is no a priori knowledge of the system nonlinearities. The adaptive fuzzy system used to approximate the ideal control $u^*$ has as input vector $\hat{x}_T = [\hat{x}_1, \hat{x}_2]$. As Fig. 3 shows it, for each input variable to the fuzzy system, we defined three (triangular and trapezoidal) membership functions uniformly distributed on the interval $[-3, 3]$.

The initial values of the estimated parameters are selected as $\theta(0) = 0$ (i.e. no a priori information on the fuzzy parameters). The design parameters used in this simulation are chosen as follows: $\varepsilon = 1$, $x_0 = 100$, $K = 100$, $\rho = 5$, $\gamma = 1500$, $\sigma = 0.1$. The feedback gain vector is chosen as $K_c^T = [1, 2]$, solving the matrix equation (30), we obtain the following positive matrix:

$$
S = \begin{bmatrix}
1 & -1 \\
-1 & 2
\end{bmatrix}.
$$

From Table 1 and (31)–(32), we can design the following smooth sliding mode observer [8]:

$$
\begin{align*}
\dot{\hat{e}}_1 &= \hat{e}_2 + 2\lambda \hat{e} + 2\lambda \tanh(k_0\hat{e}), \\
\dot{\hat{e}}_2 &= -K_c^T\hat{e} + \lambda^2 \hat{e} + 1\lambda^2 \tanh(k_0\hat{e})
\end{align*}
$$

with $l = 0.1$, $\lambda = 10$, $k_0 = 35$, and $\hat{e}(0) = [1.5, 1.5]^T$.

Figs. 4 and 5 show the simulation results of the adaptive fuzzy control based on sliding mode observer, for $u_r = 0$ (i.e. without the robust control term) and $u_r \neq 0$ (i.e. with the robust control term), respectively. These results show a good tracking and observation performance with all signals in the closed-loop being bounded. But, the state estimation $\hat{x}_2$ exhibits a weak peaking for a very short transient period. This peaking phenomenon is very natural in the HG and sliding mode observer [47,46]. Also, it is worth noting that the addition of the robust control term (i.e. $u_r \neq 0$) improves the tracking performance.

Example 2. Consider the famous inverted pendulum system. Let $x_1 = \theta$ be the angle of the pendulum with respect to the vertical line and $x_2 = \dot{\theta}$. The dynamic equations of such system are given by [52]

$$
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} &= \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} (f + g u + d(t)), \\
y &= \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\end{align*}
$$

(62)
where

\[ f(x_1, x_2) = \frac{mlx_2 \sin x_1 \cos x_1 - (M + m)g \sin x_1}{ml \cos^2 x_1 - \frac{4}{3}l(M + m)}, \quad g(x_1, x_2) = \frac{-\cos x_1}{ml \cos^2 x_1 - \frac{4}{3}l(M + m)}, \]

and \( g \) is the acceleration due to gravity, \( M \) is the mass of the cart, \( m \) is the mass of the pole, \( l \) is the half-length of pole and \( u \) is the applied force.

It is assumed that the external disturbance \( d(t) \) is a square wave having an amplitude \( \pm 1 \) with a period of \( 2\pi(s) \). The control objective is to maintain the system to track the desired angle trajectory, \( y_m = \sin(t) \), under the condition that only the system output \( y \) is measurable. The system parameters are given as \( M = 1 \text{ kg}, m = 0.1 \text{ kg}, l = 0.5 \text{ m}, g = 9.8 \text{ m/s}^2 \).

From Table 1 and (31)–(32), for the direct adaptive fuzzy controller (9), we design the following HG observer:

\[
\begin{align*}
\dot{\hat{e}}_1 &= \hat{e}_2 + 2\hat{e} \tilde{\epsilon}, \\
\dot{\hat{e}}_2 &= -K_\epsilon^T \hat{e} + \lambda^2 \hat{e}
\end{align*}
\]  

with \( K_\epsilon^T = [100, 20], \lambda = 100 \), and \( \hat{e}(0) = [0, 0]^T \).

We suppose that there is no \textit{a priori} knowledge of the system nonlinearities. The adaptive fuzzy system used to approximate the ideal control \( u^* \) has as input vector \( \hat{x}^T = [\hat{x}_1, \hat{x}_2] \).

As Fig. 6 shows it, for each variable input to the fuzzy system, we defined three (triangular and trapezoidal) membership functions uniformly distributed on the interval \([-2, 2]\).
The initial values of the estimated parameters are selected as $\theta(0) = 0$ (i.e. no a priori information on the fuzzy parameters). The design parameters used in this simulation are chosen as follows: $\alpha = 10, z_0 = 20, K = 20, \rho = 0, \gamma = 1500, \sigma = 0.1$. Solving the matrix equation (30), we obtain the following positive matrix:

$$ S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. $$

In order to show the effectiveness of the fuzzy adaptive control term, during all simulations of this example, we choose $\rho = 0$ (i.e. $u_r = 0$). Also, to simulate practical situations, the used measurement (i.e. the system output $y$)
has been corrupted by an additive noise. Figs. 7 and 8 illustrate the simulation results of the direct adaptive fuzzy control based on HG observer, for $\lambda = 10$ and $120$, respectively. From these results, we remark that the fuzzy control term is able to achieve the desired performance even in the presence of the measurement noise. Also, it can be seen that a low value of $\lambda$ (Fig. 7) allows to obtain almost free noise estimates. But, for a very large value of $\lambda$ (Fig. 8), the observer exhibits an important peaking and becomes noise sensitive since the level of the noise in the provided estimates is significantly high. It is very important to notice that in practice the choice of $\lambda$ is a compromise between a good estimation of the system states and a satisfactory behavior with noise rejection.

7. Conclusion

In this paper, a direct adaptive fuzzy controller based on any observer (HG, sliding mode (like) observer, etc.) for a class of SISO affine nonlinear system has been proposed. In designing the controller, neither differentiation of the system output nor exact knowledge of nonlinearities of the nonlinear affine system is required. The design of the underlying update law as well as the robust control term is based on an appropriate filtering of the output tracking error. This particularly allows to overcome the output observation error filtering or the necessity of the famous SPR condition. The performances of the proposed control system have been demonstrated in a realistic simulation framework involving an inverted pendulum system and a chaotic system.
Fig. 8. Simulation results for Example 2 with noise, for $\dot{x} = 120$. (a) $x_1$ (solid line) and $y_m$ (dot line). (b) $x_2$ (solid line) and $\dot{y}_m$ (dot line). (c) $\hat{x}_1$ (solid line) and $y_m$ (dot line). (d) $\hat{x}_2$ (solid line) and $\dot{y}_m$ (dot line)) and (f) control input $u$.

Appendix A. Comments on [27]

(1) The control law proposed in [27, Eq. (4)], that is

$$u = \frac{1}{\hat{g}(\hat{x})}[-\hat{f}(\hat{x}) + y^{(n)} + K^T e + y^{(n)} - u_c]$$

(A.1)

can be singular.

Indeed, the control law in [27, Eq. (4)] is not always well-defined, i.e. $\hat{g}(\hat{x})$ can be equal to zero any moment. The projection operator proposed in [27, Eq. (31)] cannot ensure $\hat{g}(\hat{x}) \neq 0$. But, it can guarantee only the boundedness of $\theta_g$ (i.e. $\|\theta_g\| \leq m_{\theta_g}$). Solutions to this problem can be found in [52,35].

(2) In [27, Remark 1, p. 586], the inequality

$$\|e\| \leq \beta_1 \|\theta_f\| + \beta_2 \|\theta_g\|$$

(A.2)

is questionable.

In [27, Remark 1, p. 586], the inequality (A.2) is correct, if $u \in L_\infty$. Since, actually, $\|e\| \leq \beta_1 \|\theta_f\| + \beta_2 |u|\|\theta_g\|$. However, the use of the assumption $u \in L_\infty$, implicitly or explicitly, is not acceptable, before the stability analysis.
(3) The robust control term in [27, Eq. (34)], that is
\[ v = -\rho \text{sign}(\tilde{e}_1) \]  
(A.3)
is questionable.

Actually, \( v \neq -\rho \text{sign}(\tilde{e}_1) \), since it is \( v_f = L^{-1}(s)v = -\rho \text{sign}(\tilde{e}_1) \) (i.e. it is the filtered version of \( v \) which is designed from the derivative of the Lyapunov function [27, Eq. (35)], it is not \( v \), see [27, Eqs. (37) and (28)].

The problem, How to have \( v \approx v_f \)? If not how to calculate \( v \) from the expression of \( v_f \)?

Also, it is worth noting that the authors use here the same notation for the robust control term [27, Eq. (34)] and the Lyapunov function [27, Eq. (35)].

Appendix B. Comments on [56]

(1) Definition of the ideal (optimal) control in [56, Eq. (3)], that is
\[ u^* = \frac{1}{g(x)}[-f(x) + y_m^{(n)} + K^T_c \hat{e}] \]  
(B.1)
is incorrect.

Indeed, the correct definition of the ideal control is
\[ u^* = \frac{1}{g(x)}[-f(x) + y_m^{(n)} + K^T_c e]. \]  
(B.2)

If we substitute [56, Eq. (3)], i.e. (B.1), in the system equation [56, Eq. (2)], we find the following dynamics \( e^{(n)} + K^T_c \hat{e} = 0 \). From these dynamics, can we conclude on the convergence of the tracking error towards zero?

But, by using the correct definition (B.2), we get the following dynamics: \( e^{(n)} + K^T_c e = 0 \). From this latter, we can easily conclude that the tracking error converges towards zero independently of others errors (as \( \hat{e} \) or \( \tilde{e} \)). Also, in this case, the matrix \( A_c \) in the observation error dynamics [56, Eq. (17)] becomes \( A_c = A - BK^T_c - K_0 C^T \) which must be a Hurwitz matrix.

(2) The robust control term in [56, Eq. (21)]
\[ v = -\rho \text{sign}(\tilde{e}_1) \]  
(B.3)
is questionable.

Indeed, it is \( v_f = L^{-1}(s)[g(x)v] = -\rho \text{sign}(\tilde{e}_1) \) (see [56, Eq. (A.4), p. 356, Eq. (16)]). Then, \( v \neq -\rho \text{sign}(\tilde{e}_1) \). The authors design a filtered version \( v_f \) from the Lyapunov function, but they use in [56, Theorem 1, p. 347] the nonfiltered version, i.e. \( v \). It is the same error as in [27]. See the previous comments on [27].

Appendix C. Comments on [55]

(1) Definition of the optimal control in [55, Eq. (17)], that is
\[ u^* = \frac{1}{g(x)}[-f(x) + y_m^{(n)} + K^T_c \hat{e}] \]  
(C.1)
is questionable.

In fact, the correct definition is
\[ u^* = \frac{1}{g(x)}[-f(x) + y_m^{(n)} + K^T_c e]. \]  
(C.2)

The same error is present in [56], see the comments on [56]. The authors used the incorrect definition of the optimal control in [55, Eq. (17)] in order to make easier the controller’s design and the stability analysis. Note that if we use the correct definition (C.2), the fitness function [55, Eq. (41)] and the supervisory control term [55, Eq. (45)] become functions of the unavailable term \( K^T_c e \).
Appendix D. Comments on [28]

(1) The robust control term in [28, Eq. (26)],
\[ v = -\rho \text{sign}(\tilde{e}_1) \quad \text{with} \quad u_s = v \]  
(D.1)
is incorrect.

Indeed, it is \( v_f = L^{-1}(s)[g(x)v] = -\rho \text{sign}(\tilde{e}_1) \), and not \( v = -\rho \text{sign}(\tilde{e}_1) \) (see [28, Eqs. (21) and (A.4), p. 859]). It is the same error as in [27]. See the comments on [27].

Appendix E. Comments on [50]

(1) The robust control terms in [50, Eqs. (28)–(29) and (56)–(57)] are questionable.

Tong et al. [50] designed the filtered versions (see [50, Eqs. (28), (56), (29), (57)]), i.e.
\[ u_{a1} = L(s)u_a = -\frac{1}{r} \hat{e}, \]  
(E.1)
\[ u_{s1} = L(s)u_s = -K_T P_1 \hat{\varepsilon}, \]  
(E.2)
respectively, instead of \( u_a \) and \( u_s \).

Then, how to calculate \( u_a \) and \( u_s \) (i.e. the original terms) from the expressions of \( u_{a1} \) and \( u_{s1} \), respectively? See the comments on [27].

(2) Definition of the ideal (optimal) control in [50, below Eq. (47)], i.e.
\[ u^* = \frac{1}{b}[-f(\hat{x}) + y_m(n) + K_T \hat{\varepsilon}] \]  
(E.3)
is incorrect.

Indeed, the correct definition is \( u^* = (1/b)[-f(\hat{x}) + y_m(n) + K_T \hat{\varepsilon}] \). The same error is present in [56]. See the comments on [56].

(3) The control law proposed in [50, Eq. (15)], that is
\[ u = \frac{1}{g(\hat{x}/\hat{\theta}_f)}[-\hat{f}(\hat{x}/\hat{\theta}_f) + y_m(n) + K_T \hat{\varepsilon} - u_a - u_s] \]  
(E.4)
can be singular.

In fact, the control law in [50, Eq. (15)] is not always well-defined, i.e. \( \hat{g}(\hat{x}/\hat{\theta}_f) \) can be equal to zero any moment. Solutions to this problem can be found in [52,35].

Appendix F. Comments on [29]

Remark 1 in [29, p. 33] is not correct. Actually, the matrix equations in [29, Eqs. (19), (38)] do not have a solution, i.e. the observation error dynamics do not verify the SPR condition, if the matrix \( B \) is defined as in [29, below Eq. (2)].

For more details, see the comments of [26]. Thus, in [29, Remark 1], there are not two cases.
Appendix G. Comments on [13]

The observation error dynamics [13, Eq. (12)] do not verify the SPR condition (see the definition of \( b \), [13, below Eq. (10)]). Then, the matrix equation [13, Eq. (17)] does not have a solution. To prove that \( Pb \neq C \), see the comments in [26].

Appendix H. Comments on [48]

The observation error dynamics [48, Eqs. (13) and (35)] are not an SPR system, i.e. \( P_i B_i \neq C_i \) (see the vectors \( B_i \) and \( C_i \) [48, below Eq. (2)]). Then, the matrix equation [48, Eq. (15)] does not have a solution. See the comments in [26].

Appendix I. Comments on [30]

In [30], the robust control terms, [30, Eqs. (21) and (22)], respectively:

\[
\begin{align*}
\mathbf{u}_s &= -\frac{B_e(\hat{x})}{1 - \alpha_g} \text{sgn}(B^TP\tilde{e}), \\
\mathbf{u}_b &= -\frac{1}{2\alpha} B^TP\tilde{e}
\end{align*}
\]  

are not implementable.

Indeed, the signal \( B^TP\tilde{e} \) is not available for measurement. Then, the controller proposed in [30] is not realizable. The signal \( B^TP\tilde{e} \) can be available, if only if \( B^TP\tilde{e} = C^T\tilde{e} = \tilde{e} \), i.e. the dynamics of the observation error [30, Eq. (24)] are an SPR system. Unfortunately, the observation error dynamics are not SPR (see the definition of the vector \( B \) [30, Eq. (4)]). Then, the controller proposed in [30] is untenable. For more details, see [26].

Appendix J. Comments on [45]

The observation error dynamics [45, Eq. (12)] are not a SPR system. Then, the matrix equation [45, Eq. (14)] has not a solution, i.e. \( P_2B \neq C \). See the definition of \( B \) [45, below Eq. (2)] and the comments in [26]. Then, the signal \( e^TP_2B \) is unavailable and the robust control \( u_s = -k \text{sign}(e^TP_2B) \) [45, Eq. (17)] is not realizable.

Appendix K. Comments on [49]

The design of the supervisory control \( u_{s1} \) [49, Eq. (12)] is questionable. In fact, in the comments [49], the authors designed a filtered version for the supervisory controller \( u_{s1} = L^{-1}(s)[u_s] \), see [49, Eq. (12), and below Eq. (9)]. Is \( u_s \) equal to \( u_{s1} \)? If not how to calculate \( u_s \) from the expression of \( u_{s1} \)?

Appendix L. Comments on [12]

(1) The robust control term in [12, Eq. (33)], i.e.

\[ u_{si} = \eta_0 \text{sgn}(\tilde{y}_i) \]  

is questionable.

Indeed, it is \( u_{si}^f = H_i^{-1}(s)[u_{si}] = \eta_0 \text{sgn}(\tilde{y}_i) \), and it is not \( u_{si} = \eta_0 \text{sgn}(\tilde{y}_i) \) (see [12, Eqs. (32) and (33)]. It is the same error as in [27]. See the comments on [27].

(2) In expression of [12, Eq. (16)], there are some drawbacks (examine with attention the passage of [12, Eq. (14)] towards [12, Eqs. (16)–(17)]. The authors used to carry out this passage, implicitly, the following incorrect expressions:

\[ H_i^{-1}[\phi_i \psi_i^\pi \tilde{x}] = (H_i^{-1}[\phi_i])\psi_i^\pi \tilde{x}. \]
or
\[ H_i^{-1}(\phi_i \hat{x}) = \phi_i^T \hat{x}, \]
\[ H_i^{-1}([\phi_i (\hat{x} + \tilde{v}_i \omega)]) = (H_i^{-1}(\phi_i))(\hat{x} + \tilde{v}_i \omega), \]
or
\[ H_i^{-1}([\phi_i (\hat{x} + \tilde{v}_i \omega)]) = \phi_i^T (\hat{x} + \tilde{v}_i \omega), \]

where \( H_i^{-1} \) is a stable filter. Indeed, \( \hat{x} \) and \( \hat{x} + \tilde{v}_i \omega \) are not constant. Then, the control scheme proposed in [12] is questionable.

Appendix M. Comments on [17]

The estimation error dynamics [17, Eq. (8)] are not an SPR system, i.e. \( B^T P \neq C \). Then, the expression \( e^T P B = e^T C^T = Ce \) [17, below Eq. (15)] is incorrect and the adaptive fuzzy observer proposed in [17] is not realizable. See the comments in [26].

Appendix N. Comments on [38]

(1) The parameter vector update law [38, Eq. (15)], i.e.
\[ \dot{\theta} = -\Gamma \phi e^T - k_u \Gamma \| r \| \theta \]  \quad \text{(N.1)}

is not realizable.

The expression [38, Eq. (15)], i.e. (N.1), can be rewritten as follows:
\[ \theta = -\Gamma \int \phi e^T dt - k_u \Gamma \int \| r \| \theta dt \]
\[ = -\Gamma \int \phi (\dot{e} + Ae) dt - k_u \Gamma \int \| r \| \theta dt \]
\[ = -\Gamma \int \phi \dot{e} dt - \Gamma \int \phi Ae dt - k_u \Gamma \int \| r \| \theta dt \]
\[ = -\Gamma \phi e + \Gamma \int \phi \dot{e} dt - \Gamma \int \phi Ae dt - k_u \Gamma \int \| r \| \theta dt. \] \quad \text{(N.2)}

Then, this update law uses an unavailable signal \( \| r \| \), where \( r = \dot{e} + Ae \) is the filtered tracking error. Thus, the controller in [38] is untenable. But, we can easily overcome this drawback by using the following update law: \( \dot{\theta} = -\Gamma \phi e^T - k_u \Gamma \theta \).

(2) In [38, Proposition 2], the authors say that the fuzzy basis vector \( \phi \) is constant, whereas \( \phi \) is variable with time.

References

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