Discrete Fractional Order State-Space System – Properties, Estimation and Control

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Outline

- Introduction
- DFOSS properties
- State feedback control
- State estimation methods: observer, FKF, EFKF
- Simulink Toolkit FSST
- Application
- Conclusions
Fractional Difference

Fractional order difference (discrete Grünwald-Letnikow definition):

\[ \Delta^n x_k = \sum_{j=0}^{k} (-1)^j \binom{n}{j} x_{k-j} \]

where

\[
\binom{n}{j} = \begin{cases} 
  1 & \text{for } j = 0 \\
  \frac{n(n-1)\ldots(n-j+1)}{j!} & \text{for } j > 0
\end{cases}
\]

\( n \in \mathbb{R} \) - fractional order.
For example

\[ n = 1 \quad , \quad \Delta^1 x_k = 1 \cdot x_k - 1 \cdot x_{k-1} + 0 \cdot x_{k-2} - 0 * \cdot k_{-3} \ldots \]

\[ n = -1 \quad , \quad \Delta^{-1} x_k = 1 \cdot x_k + 1 \cdot x_{k-1} + 1 \cdot x_{k-2} + 1 \cdot x_{k-3} \ldots \]

\[ n = 0.5 \quad , \quad \Delta^{0.5} x_k = x_k - 0.5x_{k-1} - 0.125x_{k-2} - 0.0625x_{k-3} \ldots \]

\[ n = -0.5 \quad , \quad \Delta^{-0.5} x_k = x_k + 0.5x_{k-1} + 0.375x_{k-2} + 0.3125x_{k-3} \ldots \]

\[ n = 1.5 \quad , \quad \Delta^{1.5} x_k = x_k - 1.5x_{k-1} + 0.375x_{k-2} + 0.0625x_{k-3} \ldots \]

\[ n = -1.5 \quad , \quad \Delta^{-1.5} x_k = x_k + 1.5x_{k-1} + 1.875x_{k-2} + 2.1875x_{k-3} \ldots \]
Discrete F.O. State-space system intro

The traditional integer order discrete system

\[ x_{k+1} = Ax_k + Bu_k \]  \hspace{1cm} (2)
\[ y_k = Cx_k \]  \hspace{1cm} (3)

by subtraction \( x_k \) from both sides of the equation (2), we can rewrite it into the following form:

\[ \Delta^1 x_{k+1} = A_dx_k + Bu_k \]  \hspace{1cm} (4)
\[ x_{k+1} = \Delta^1 x_{k+1} + x_k \]  \hspace{1cm} (5)
\[ y_k = Cx_k \]  \hspace{1cm} (6)

where \( A_d = A - I \) oraz \( \Delta^1 x_{k+1} \) is a first order difference.
Equation (5) originated from the difference definition.
By generalization for fractional order we can obtain the Discrete Fractional Order State-Space System.
Definition 1  \textit{Discrete Fractional Order system in state space representation is given as follows:}

\[ \Delta^n x_{k+1} = A_d x_k + B u_k \]
\[ x_{k+1} = \Delta^n x_{k+1} \]
\[ \sum_{j=1}^{k+1} (-1)^j \binom{n}{j} x_{k-j+1} \]
\[ y_k = C x_k + D u_k \]

where \( N \) - number of state equations
Solution of DFOSS:

\[ x_k = \Phi(k)x_0 + \sum_{j=0}^{k-1} \Phi(k - j - 1) Bu_j \]  

where \( \Phi(k) \) - transition matrix

\[ \Phi(k + 1) = (A_d + I{\left( \begin{array}{c} n \\ 1 \end{array} \right)})\Phi(k) - \sum_{j=2}^{k+1} (-1)^j \left( \begin{array}{c} n \\ j \end{array} \right) \Phi(k + 1 - j) \]  

Practical implementation needs to reduce number of samples

\[ x_{k+1} = \begin{cases} 
\Delta^nx_{k+1} - \sum_{j=1}^{L} (-1)^j \left( \begin{array}{c} n \\ j \end{array} \right) x_{k-j+1} & \text{for } k > L \\
\Delta^nx_{k+1} - \sum_{j=1}^{k+1} (-1)^j \left( \begin{array}{c} n \\ j \end{array} \right) x_{k-j+1} & \text{for } k \leq L 
\end{cases} \]  

where \( L \) - is a memory (buffer) length
The DFOSS can be rewritten as an infinite dimensional integer order model

\[
\begin{bmatrix}
  x_{k+1} \\
  x_k \\
  x_{k-1} \\
  \vdots
\end{bmatrix} =
\begin{bmatrix}
  x_k \\
  x_{k-1} \\
  x_{k-2} \\
  \vdots
\end{bmatrix} + Bu_k
\]  \hspace{1cm} (10)

\[
y_k = C
\begin{bmatrix}
  x_k \\
  x_{k-1} \\
  x_{k-2} \\
  \vdots
\end{bmatrix} \hspace{1cm} (11)
\]
where

\[
A = \begin{bmatrix}
(A_d + I\alpha) & -(-1)^2(\frac{\alpha}{2}) & -(-1)^3(\frac{\alpha}{3}) & \ldots \\
I & 0 & 0 & \ldots \\
0 & I & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
B \\
0 \\
0 \\
\vdots
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
C & 0 & 0 & \ldots
\end{bmatrix}
\]
Reachability

**Theorem 1** System given by Definition 1 is reachable (in N steps) iff one of following conditions is satisfied [8]:

\[
\text{rank} \begin{bmatrix}
B & \Phi(1)B & \ldots & \Phi(N-1)B
\end{bmatrix} = N
\]

\[
\text{rank} \begin{bmatrix}
B & (A_d + In)B & \ldots & (A_d + In)^{N-1}B
\end{bmatrix} = N
\]

\[
\text{rank} \begin{bmatrix}
B & A_dB & A_d^2B & \ldots & A_d^{N-1}B
\end{bmatrix} = N
\]

Proof: (sketch) Solution for \(x_0 = 0\) and final state \(x_N\): \[
x_N = \sum_{j=0}^{N-1} \Phi(N-j-1)Bu_j = \begin{bmatrix}
B & \Phi(1)B & \ldots & \Phi(N-1)B
\end{bmatrix} \begin{bmatrix}
u_{N-1} \\
u_{N-2} \\
\vdots \\
u_0
\end{bmatrix}
\]
Reachability (2)

using the following property

\[ \Phi(k) = (A_d + In)^k + f_{k,k-1}(A_d + In)^{k-1} + \cdots + f_{k,1}(A_d + In) + f_{k,0}I \]

we can rewrite the first condition to

\[ \text{rank} \left[ B \ (A_d + In)B \ (A_d + In)^2B \ \cdots \ (A_d + In)^{N-1}B \right] = N \]

using another property that

\[ (A_d + nI)^j = A_d^j + w_{j-1}A_d^{j-1} + \cdots + w_1A_d + w_0I \]

we can transform the second condition into

\[ \text{rank} \left[ B \ A_dB \ A_d^2B \ \cdots \ A_d^{N-1}B \right] = N \]
**Theorem 2** System given by Definition 1 is controllable (in \( N \) steps) iff one of following conditions is satisfied [8]:

\[
\begin{align*}
\text{rank} \begin{bmatrix} B & \Phi(1)B & \ldots & \Phi(N - 1)B \end{bmatrix} &= N \\
\text{rank} \begin{bmatrix} B & (A_d + In)B & \ldots & (A_d + In)^{N-1}B \end{bmatrix} &= N \\
\text{rank} \begin{bmatrix} B & A_dB & A_d^2B & \ldots & A_d^{N-1}B \end{bmatrix} &= N
\end{align*}
\]

or

\[
\Phi(N) = 0 \quad (12)
\]

*in this case for* \( u_j = 0 \) *where* \( j = 0, \ldots, N \) *the* \( x_N = 0 \) *for any* \( x_0 \).

*Where* \( x(N) = \Phi(N)x_0 \)
Observability

**Theorem 3** System given by definition 7 is observable (in N steps) iff one of following condition is satisfied:

\[
\text{rank } \begin{bmatrix}
C \\
C\Phi(1) \\
\vdots \\
C\Phi(N-1)
\end{bmatrix} = N, \quad \text{rank } \begin{bmatrix}
C \\
C(A_d + In) \\
\vdots \\
C(A_d + In)^{N-1}
\end{bmatrix} = N,
\]

\[
\text{rank } \begin{bmatrix}
C \\
CA_d \\
\vdots \\
C A_d^{N-1}
\end{bmatrix} = N
\]
Stability

**Theorem 4** [3] The system given by the Definition 1 is stable if (but not iff)

\[ |A_d + \binom{n}{1} I| < r(k, n) \]

where

\[ r(k, n) = \begin{cases} 
2 - \frac{\Gamma(k+2-n)}{\Gamma(1-n)\Gamma(k+2)} - n & ; n \in \langle 2, 1 \rangle \\
\frac{\Gamma(k+2-n)}{\Gamma(1-n)\Gamma(k+2)} + n & ; n \in (1, 0) \\
1 & ; n = 0
\end{cases} \]

The \( r(n, k) \) is a stability radius of the system, i.e., it is a radius of the disk in which stable eigenvalues (but not all stable eigenvalues) of the system are placed.
Example of stability regions

\[ n = 0.1, 0.5 \text{(upper)}, \ n = 1.2, 1.8 \text{(down)}, \ L = 40 \]
Discrete transfer function of DFOSS

By applying Z transform to the DFOSS

\[ z\Delta^\alpha(z)X(z) = AX(z) + BU(z) \]
\[ Y(z) = CX(z) \]

This immediately gives for a Single-Input-Single-Output case the relation

\[ G(z) = \frac{Y(z)}{U(z)} = C(I(z\Delta^\alpha(z)) - A)^{-1}B = \frac{b_{N-1}z^{N-1}\Delta^\alpha(N-1)(z) + \cdots + b_0}{z^N\Delta^\alpha N(z) + \cdots + a_1z\Delta^\alpha(z) + a_0} \]

where \( z^d\Delta^{d\alpha}(z) \) is a polynomial of \( z \) given as follows:

\[ z^d\Delta^{d\alpha}(z) = \sum_{j=0}^{k+d} (-1)^j \binom{\alpha}{j} z^{-j+d} \]
Then, the Fractional Difference Equation is as follows:

\[
\Delta^{\alpha N} y_k + a_{N-1} \Delta^{\alpha(N-1)} y_{k-1} + \cdots + a_0 y_{k-N} = \\
b_{N-1} \Delta^{\alpha(N-1)} u_{k-1} + \cdots + b_0 u_{k-N}
\]

what can be rewritten in the form

\[
\begin{bmatrix}
Y_k \\
Y_{k-1} \\
\vdots
\end{bmatrix}
=
\begin{bmatrix}
\varphi_k \\
\varphi_{k-1} \\
\vdots
\end{bmatrix}
\theta
\]  

where

\[
\varphi_k = \begin{bmatrix} -\Delta^{\alpha(N-1)} y_{k-1} & \cdots & -y_{k-N} & \Delta^{\alpha(N-1)} u_{k-1} & \cdots & u_{k-N} \end{bmatrix}
\]

\[
\theta^T = \begin{bmatrix} a_{N-1} & \cdots & a_0 & b_{N-1} & \cdots & b_0 \end{bmatrix}, 
Y_k = \begin{bmatrix} \Delta^{\alpha N} y_k \end{bmatrix}
\]
Identification example

Let us assume the following continuous time system

\[ 0D_t^{0.5} x(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t) \]

The transfer function of the system has the form:

\[ G(s) = \frac{3s^{0.5} + 2}{s^1 + 3s^{0.5} + 2} = \frac{-1}{s^{0.5} + 1} + \frac{4}{s^{0.5} + 2} \]

The step response of the system is

\[ y(t) = -t^{0.5} E_{0.5,1}(-t^{0.5}) + 4t^{0.5} E_{0.5,1}(-t^{0.5}), \]

where \( E_{\alpha,\beta} \) is a Mittag-Leffler function.
Identification example (2)

In parametric identification of this system the following discrete transfer function form is assumed:

\[
G(z) = \frac{b_1 z \Delta^{0.5}(z) + b_0}{z^2 \Delta^1(z) + a_1 z \Delta^{0.5}(z) + a_0}
\]  \hspace{1cm} (14)

which is rewritten in the form

\[
\varphi_k = \left[ -\Delta^{0.5}y_{k+1}, -y_k, \Delta^{0.5}u_{k+1}, u_k \right]
\]  \hspace{1cm} (15)

\[
\theta^T = \left[ a_1 \quad a_0 \quad b_1 \quad b_0 \right], \quad Y_k = \left[ \Delta^1 y_k \right]
\]  \hspace{1cm} (16)

Solving the equation (13) we obtain the following parameters:

\[
a_1 = -0.5867 \quad a_0 = -0.1705 \\
b_1 = 0.6181 \quad b_0 = 0.1677
\]
Identification example (3)

Step response of continuous system and indentified discrete model
Estimation Introduction

General estimation scheme.

where:

- $x_k$ is state vector,
- $y_k$ is system output,
- $u_k$ is system input.
- $\hat{x}_k$ is estimate of state variable $x_k$ reconstructed from data $u_j, y_j$ for $j = 0 \ldots k$

Output functions represents signal transfer from state variables to system output – measurements.
Estimation Introduction (2)

Estimator general scheme.

where: $\hat{y}_k$ is an output estimate,
$e_k$ is an output estimation error,
The block System + error correction contain the original system with a part of state estimate correction based on the output estimation error.
Remark: the estimator strongly based on the estimated system.
That is why we need new estimation algorithms for fractional systems.
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Observer

**Theorem 5** [7] *The state observer for Discrete Fractional State-Space Systems (called Discrete Fractional Order Observer (DFOO)) is given by the following equations*

\[
\Delta^n \hat{x}_{k+1} = F_d \hat{x}_k + G u_k + H y_k
\]

\[
\hat{x}_{k+1} = \Delta^n \hat{x}_{k+1} - \sum_{j=1}^{k+1} (-1)^j \binom{n}{j} \hat{x}_{k-j+1}
\]

*where \( \hat{x}_k \) is an estimate of the state variable \( x_k \), \( F_d = A_d - HC \) and \( G = B \).*

The estimation error equation is

\[
\Delta^n e_{k+1} = F_d e_k
\]
Stochastic DFOSS

Definition 2  The General Stochastic Discrete Fractional Order State-Space System is given by the following set of equations

\[ \Delta^\gamma x_{k+1} = A_d x_k + B u_k + \omega_k \]  
(17)

\[ x_{k+1} = \Delta^\gamma x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j x_{k-j+1} \]  
(18)

\[ y_k = C x_k + \nu_k \]  
(19)

where \( \omega_k \) is an internal noise and \( \nu_k \) is an output noise and \( n_1 \ldots n_N \) are orders of system equations

\[ \Upsilon_k = \text{diag} \left[ \begin{array}{c} (n_1)_k \\ \vdots \\ (n_N)_k \end{array} \right] \Delta^\gamma x_{k+1} = \begin{bmatrix} \Delta^{n_1} x_{1,k+1} \\ \vdots \\ \Delta^{n_N} x_{N,k+1} \end{bmatrix} \]
Fractional Kalman Filter

Fractional Kalman Filter [2,5]:

\[
\Delta^\gamma x_{k+1} = A_d \hat{x}_k + Bu_k
\]

\[
\tilde{x}_{k+1} = \Delta^\gamma \tilde{x}_{k+1} + \sum_{j=1}^{k+1} (-1)^j \gamma_j \tilde{x}_{k-j+1}
\]

\[
\tilde{P}_k = (A_d + \gamma_1) P_{k-1} (A_d + \gamma_1)^T + Q_{k-1} + \sum_{j=2}^{k} \gamma_j P_{k-j} \gamma_j
\]

\[
K_k = \tilde{P}_k C^T (C \tilde{P}_k C^T + R_k)^{-1}
\]

\[
\hat{x}_k = \tilde{x}_k + K_k (y_k - C \tilde{x}_k)
\]

\[
P_k = (I - K_k C) \tilde{P}_k
\]
The Fractional Kalman Filter was obtained by minimization of the following cost function

$$\hat{x}_k = \arg \min_x [(\tilde{x}_k - x)\tilde{P}_k^{-1}(\tilde{x}_k - x)^T + (y_k - Cx)R_k^{-1}(y_k - Cx)^T]$$

and was derived under the following assumptions:

1. 

$$E[x_{k+1-j}, z^*_k] \cong E[x_{k+1-j}, z^*_{k+1-j}]$$

for $$i = 1 \ldots (k + 1)$$

2. 

$$E[(\hat{x}_l - x_l)(\hat{x}_m - x_m)^T]$$ are equal to zero when $$l \neq m$$

The first assumption means that the only one, the newest, state vector is updated.

The second assumption means that there is no correlation between past state vectors.

ExFKF — algorithm with less restrictive assumptions, see [2]
Estimation example

\[ A_d = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ C = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} 0.7 & 1.2 \end{bmatrix}^T \]

\[ E[\nu_k \nu_k^T] = 0.01, \quad E[\omega_k \omega_k^T] = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix} \]

Fractional Kalman Filter parameters used in the example are:

\[ P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix} \]

\[ R = \begin{bmatrix} 0.01 \end{bmatrix} \]
Figure 1: Input and output signals from the plant
Estimation example (3)

Figure 2: Estimated and original state variables
Figure 3: Estimation errors
Definition 3 The Non-linear Stochastic Discrete Fractional Order System in state-space representation is given by the following set of equations

\[
\begin{align*}
\Delta^\gamma x_{k+1} &= f(x_k, u_k) + \omega_k \quad (20) \\
x_{k+1} &= \Delta^\gamma x_{k+1} \\
&\quad - \sum_{j=1}^{k+1} (-1)^j \gamma_j x_{k-j+1} \\
y_k &= h(x_k) + \nu_k \quad (22)
\end{align*}
\]

where \(\omega_k\) is an internal noise and \(\nu_k\) is an output noise.
Extended Fractional Kalman Filter

**Theorem 6** [5] For the nonlinear discrete fractional order stochastic system in state-space representation the Extended Fractional Kalman Filter is given as follows

\[
\Delta^\gamma \tilde{x}_{k+1} = f(\tilde{x}_k, u_k) \tag{23}
\]

\[
\tilde{x}_{k+1} = \Delta^\gamma \tilde{x}_{k+1}
\]

\[
- \sum_{j=1}^{k+1} (-1)^j \gamma_j \tilde{x}_{k+1-j} \tag{24}
\]

\[
\tilde{P}_k = (F_{k-1} + \gamma_1) P_{k-1} (F_{k-1} + \gamma_1)^T + Q_{k-1} + \sum_{j=2}^{k} \gamma_j P_{k-j} \gamma_j^T \tag{25}
\]

\[
\hat{x}_k = \tilde{x}_k + K_k [y_k - h(\tilde{x}_k)] \tag{26}
\]

\[
P_k = (I - K_k H_k) \tilde{P}_k \tag{27}
\]
Extended Fractional Kalman Filter (2)

with initial conditions

\[ x_0 \in \mathbb{R}^N, \quad P_0 = E[(\hat{x}_0 - x_0)(\hat{x}_0 - x_0)^T] \]

where

\[ K_k = \tilde{P}_k H^T_k (H_k \tilde{P}_k H^T_k + R_k)^{-1} \]

\[ F_{k-1} = \left[ \frac{\partial f(x, u_{k-1})}{\partial x} \right]_{x=\hat{x}_{k-1}} \]

\[ H_k = \left[ \frac{\partial h(x)}{\partial x} \right]_{x=\tilde{x}_k} \]

and noises \( \nu_k \) and \( \omega_k \) are assumed to be independent and with zero expected value.
Nonlinear estimation example

System for parameter $a_1$ joint estimation

\[
\begin{align*}
x_k^w &= [x_k^T \ a_1]^T \\
\Delta \gamma x_{k+1}^w &= f(x_k^w, u_k) + \omega_k \\
x_{k+1}^w &= \Delta \gamma x_{k+1}^w - \sum_{j=1}^{k+1} (-1)^j \gamma_j x_{k+1-j}^w \\
y_k &= h(x_k^w) + \nu_k
\end{align*}
\]

where

\[
f(x_k^w, u_k) = \begin{bmatrix} x_{2,k}^w \\
-a_0 x_{1,k}^w - a_1 x_{2,k}^w + u_k \\
0
\end{bmatrix}
\]

\[
h(x_k) = \begin{bmatrix} b_0 x_{1,k}^w + b_1 x_{2,k}^w \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} n_1 & n_2 & 1 \end{bmatrix}
\]
Nonlinear estimation example (2)

Linearized matrices for EFKF are defined as

\[ F_k = \left[ \frac{\partial F(x, u_k)}{\partial x} \right]_{x = \hat{x}_k^w} = \begin{bmatrix} 0 & 1 & 0 \\ -a_0 & -a_1 & -\tilde{x}_{2,k}^w \\ 0 & 0 & 0 \end{bmatrix} \]

\[ H_k = \left[ \frac{\partial H(x)}{\partial x} \right]_{x = \tilde{x}_k^w} = \begin{bmatrix} b_0 & b_1 & 0 \end{bmatrix} \]

Parameters of the Extended Fractional Kalman Filter used in the example are:

\[ P_0 = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0004 & 0 & 0 \\ 0 & 0.0004 & 0 \\ 0 & 0 & 0.0001 \end{bmatrix}, \quad R = \begin{bmatrix} 0.003 \end{bmatrix} \]
Figure 4: Estimated and original state variables of the plant
Figure 5: Estimation of a parameter $a_1$

- "Fractional Order Difference"
- "Fractional Order State-Space System"
- "Fractional Order Stochastic State-Space System"
- "Fractional Kalman Filter"
Use of Fractional Order Difference Block

The block (realized by C-MEX S-function fodif.c) has the following parameters:

- $N$ is a matrix of orders,
- $Ts$ is a sample time,
- $Nbuf$ is a width of a circular buffer of past states vectors (memory length $L$).
Use of Discrete Fractional Order State-Space System Block

The block (C-MEX S-function fsim_x0.c) has parameters:

- $A_d, B, C, N$ are the system matrices, where $A_d \in \mathbb{R}^{N_x \times N_x}$, $B \in \mathbb{R}^{N_x \times N_u}$, $C \in \mathbb{R}^{N_y \times N_x}$, $N \in \mathbb{R}^{N_x}$.

- $T_s$ is the sampling time, $N_{buf}$ is the memory length, $x_0$ is the vector of initial conditions.
Use of the Discrete Stochastic Fractional Order State-Space System Block

The block (C-MEX S-function fsim_x0_st.c) has the following parameters: $A_d, B, C, N, Ts, Nbuf, x0$ (the same as for DFOSS block). It has one additional input $\omega$ for the system noise $\omega_k$. 

The block (C-MEX S-function fkf.c), parameters are $A, B, C, N$ are the system matrices, $P, Q, R, x_0$ are the FKF matrices where $P \in \mathbb{R}^{N_x \times N_x}$, $Q \in \mathbb{R}^{N_x \times N_x}$, $R \in \mathbb{R}^{N_y \times N_y}$, $x_0 \in \mathbb{R}^{N_x}$, $T_s$ is the sampling time, $N_{buf}$ is the memory length.
Fractional Kalman Filter Block (2)

Diagram of the state variables estimation using Fractional Kalman Filter
Diagram of the state feedback control with FKF as an estimator

The state feedback control law is given as follows [1,4,9]

\[ u_k = K x_k \]
State feedback control with FKF (2)
Experimental setup scheme connected to DS1104 Control Card

Obtained continuous time model of ultracapacitor 0.22F (for higher frequency):

\[
\frac{d^{0.45} x(t)}{dt^{0.45}} = [a_{0,0}] x(t) + [b_0] u(t) \\
y = [c_0] x(t) + [d_0] u_k
\]
Discrete modeling (2)

Assumed discrete model:

\[ \Delta^{0.2} \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = A_d \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + B u_k \]

\[ y_k = C x_k \]

Obtained discrete model parameters:

\[ A_d = \begin{bmatrix} 0 & 1 \\ 0.0353 & 0.0018 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ C = \begin{bmatrix} -0.0186 & 0.1884 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix} \]

\[ \alpha = 0.2, T_s = 0.1 \]
Figure 6: Results of discrete time modeling
State feedback control (1)

System eigenvalues

$$\text{eig}[A_d + \text{diag}(N)] = [0.013 \quad 0.389]$$

Stability radius for $L = \infty$ is equal to 0.2.

Observer eigenvalues

$$\text{eig}[F + \text{diag}(N)] = [0.117 \quad 0.0039]$$

Regulator matrix:

$$K = [0.05 \quad 0.05]$$

what implies following system eigenvalues

$$\text{eig}(A_d + BK + nI) =$$

$$\begin{bmatrix}
  0.2268 + 0.1634i \\
  0.2268 - 0.1634i
\end{bmatrix}$$
Figure 7: Results of state feedback control (system output)
Advances of Ultracapacitors

The model presented previously is only for restricted range of frequencies (higher) and local. The continuous model for wide range of frequencies

\[ G_{uc}(s) = R + \frac{(Ts + 1)^\alpha}{Cs} \]  

(28)

Measured and theoretical Bode diagrams of ultracapacitor 0.33F
Ultracapacitor is a nonlinear system

Ultracapacitors have fractional order nonlinear dynamics. Fractional order modelling can improve design methods. Fractional control algorithms can improve efficiency in charging/discharging process, safe energy, allows better fault detection.
Conclusion

- basic identification, discretization and control algorithms for discrete fractional order systems was presented
- advanced estimation algorithms for DFOSS was presented
- application to modeling, estimation and control of the system with ultracapacitor was presented

Future work and open problems:

- Identification method needs to be improved (more robust to noise algorithm)
- Discretization algorithm
- Optimal control algorithms
- Robust control algorithms
References

Thank you for your attention