DYNAMICAL FRACTIONAL ORDER MODELING AND CONTROL WITH APPLICATIONS

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OUTLINE

- Introduction
- Modeling by Identification
- Linear Control Strategies
- Nonlinear Control Strategies
- Conclusions
### Scenario I

<table>
<thead>
<tr>
<th>Controller/ System</th>
<th>Integer</th>
<th>Fractional</th>
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</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>Fractional</td>
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**Scenario II**

<table>
<thead>
<tr>
<th>Controller/System</th>
<th>Linear</th>
<th>Nonlinear</th>
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</thead>
<tbody>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>Nonlinear</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Scenario III: LTI Systems

- Integer
- Fractional:
  - Non-commensurate
  - Commensurate:
    - Rational
    - Non-rational
Experimental observation: time and frequency responses do not follow standard laws for LTIS: exponentials, slopes, phase tendencies, etc.

Systems and processes with: presence of memory phenomena (relaxation, diffusion, etc.)
MODELING BY IDENTIFICATION

- Model Structure:

\[ H(s, \theta) = \frac{\sum_{k=0}^{n} a_k s^{k\alpha}}{\sum_{k=0}^{m} b_k s^{k\alpha}} \quad \theta = [a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m] \]

- Cost function

\[
J = \frac{\int_{0}^{\infty} [y(t) - \tilde{y}(t)]^2 \, dt}{\int_{0}^{\infty} [h(t)]^2 \, dt} = \frac{\int_{-\infty}^{\infty} \left[ H(\omega) - \tilde{H}(\omega) \right] \frac{H(\omega) - \tilde{H}(\omega)}{\omega^2} \, d\omega}{\int_{-\infty}^{\infty} H(\omega) \overline{H(\omega)} \, d\omega}
\]
Electrochemical Process - Circuit analogies

\[ G_\Omega = \frac{1}{R_\Omega}, \quad G_{ct} = \frac{1}{R_{ct}}, \quad Y_w = a (j\omega)^{1/2}, \quad Y_{CPE} = b (j\omega C)^{\alpha}. \]

\[ i_c(t) = b_2 \tau_2^{a_2} D^{a_2} v_2(t), \]

\[ b_3 \tau_3^{a_3} D^{a_3} i_f(t) + b_4 \tau_4^{a_4} D^{a_4} i_f(t) = b_3 \tau_3^{a_3} b_4 \tau_4^{a_4} D^{a_3+a_4} v_2(t) \]

\[ i(t) = i_c(t) + i_f(t) = b_1 \tau_1^{a_1} D^{a_1} v_1(t), \]

\[ Z(s) = \frac{V(s)}{I(s)} = \frac{100s^{1.25} + 1.414s^{0.75} + 1.001 \times 10^4 s^{0.5} + 141.4}{s^{0.5} (s^{0.75} + 0.0141s^{0.25} + 0.1)} \]
MODELING BY IDENTIFICATION

- Electrochemical Process - Results

\[ Z(s) = \frac{100s^{1.25} - 1.108 \times 10^{-7}s + 1.414s^{0.75} + 1.001 \times 10^4s^{0.5} + 1.282 \times 10^{-9}s^{0.25} + 141.4}{s^{1.25} - 7.45 \times 10^{-12}s + 0.01414s^{0.75} + 0.1s^{0.5} + 1.179 \times 10^{-14}s^{0.25} - 2.3 \times 10^{-16}} \approx Z(s). \]

\[ \alpha = 1 \]

\[ \alpha = 0.25 \]
Systems Identification

The physical model of the system can be described by

\[ \frac{\partial^2}{\partial x^2} \left[ EI \frac{\partial^2 y(x, t)}{\partial x^2} - C_a V_a(x, t) \right] + \rho A \frac{\partial^2 y(x, t)}{\partial t^2} = 0 \]

\[ y(0, t) = 0, \]
\[ EI \frac{\partial^2 y(0, t)}{\partial x^2} - I_h \frac{\partial^3 y(0, t)}{\partial t^2 \partial x} + T(t) = 0, \]
\[ EI \frac{\partial^2 y(L, t)}{\partial x^2} + I_t \frac{\partial^3 y(L, t)}{\partial t^2 \partial x} = 0, \]
\[ EI \frac{\partial^3 y(L, t)}{\partial x^3} - M_t \frac{\partial^2 y(L, t)}{\partial t^2} = 0, \]

\[ G_{a,V_a}(L, s) = \frac{Y(L, s)}{V_a(s)} = \frac{s^2 N(s)}{D(s)} \]

where \( \beta^4 = -\frac{A s^2}{(EI)}. \)
Flexible structure: Results

<table>
<thead>
<tr>
<th>n, n</th>
<th>α = 2</th>
<th>α = 1</th>
<th>α = 0.5</th>
<th>n, n</th>
<th>α = 2</th>
<th>α = 1</th>
<th>α = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 5</td>
<td>1.6 × 10^{-5}</td>
<td>–</td>
<td>–</td>
<td>10, 10</td>
<td>–</td>
<td>7.2 × 10^{-5}</td>
<td>3.9 × 10^{-5}</td>
</tr>
<tr>
<td>12, 12</td>
<td>–</td>
<td>2.4 × 10^{-5}</td>
<td>1.8 × 10^{-6}</td>
<td>16, 16</td>
<td>–</td>
<td>6.4 × 10^{-5}</td>
<td>2.6 × 10^{-7}</td>
</tr>
<tr>
<td>18, 18</td>
<td>–</td>
<td>6.6 × 10^{-5}</td>
<td>8.8 × 10^{-9}</td>
<td>20, 20</td>
<td>–</td>
<td>–</td>
<td>2.4 × 10^{-9}</td>
</tr>
</tbody>
</table>
LINEAR CONTROL STRATEGIES

- Linear control:
  - Fixed controller structure + specifications: generalized control actions: FrPID
  - Plant dependent controller structure + reference model
  - State-Space based methods: pole placement for commensurate order

martes 14 de diciembre de 2010
2.3.4.2 General Characteristics

It was first proposed in [rm] and is the starting point for the CRONE cases. For this reason, it is interesting to take this system as the reference.

• Open-loop:
  - The magnitude curve has a constant slope of \(-20\alpha\ \text{dB/dec.}\)
  - The gain crossover frequency depends on \(A\).
  - The phase plot is a horizontal line of value \(-\alpha\pi/2\).
  - The Nyquist plot is a straight line which starts from the origin with argument \(-\alpha\pi/2\).

\(F(s) = \frac{A}{s^\alpha}, \ 0 < \alpha < 2\).

1. Open-loop:

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  - The Nyquist plot is a straight line which starts from the origin with argument \(-\alpha\pi/2\).

\(G(s) = \frac{A}{s^\alpha + A}, \ 0 < \alpha < 2\)

2. Closed-loop with unity negative feedback:

  - The gain margin is infinite.
  - The phase margin is constant with value \(\varphi_m = \pi (1 - \alpha/2)\), only depending on \(\alpha\).
Tip Position Control of a Flexible - Link

\[ c(\theta_m - \theta_t) = ml^2 \ddot{\theta}_t. \]

\[ KV = J\ddot{\theta}_m + \nu\dot{\theta}_m + \hat{\Gamma}_{\text{coup}} + \hat{\Gamma}_{\text{Coul}}, \]

\[ V_{\text{Coul}} = \frac{\hat{\Gamma}_{\text{Coul}}}{K} \text{sgn} (\hat{\theta}_m) \]

\[ \hat{\Gamma}_{\text{coup}} = c(\theta_m - \theta_t) \]
\( G_m(s) = \frac{K/J}{s(s + v/J)} \)
proposed that the dynamics of the inner loop can be approximated by control of the tip position. The motor position is first supposed to track the reference and friction as its value is divided by \([xxy]t\). It has also been demonstrated that in the case of motors with gears, assumption in direct driven motors, and motors with reduction gears as well, the dynamics can be made quite fast; it cannot be considered negligible in general.

For the case of a beam with only one vibrational mode, a simplifying loop is obtained by placing a strain gauge at the base of the link to control the arm. These measurements provide the system to close the loop. We actually feed back the measurements of a strain gauge to close the loop. The gain margin is infinite, depending on the damping ratio and the natural frequency of the system.

In the case of a double integrator, the dynamics of the arm has been reduced to a double integrator in the case of a beam with only one vibrational mode. A simplifying loop is placed at the base of the link to control the arm. These measurements provide the system to close the loop. We actually feed back the measurements of a strain gauge to close the loop. The gain margin is infinite.

The stability study by using Nyquist plots shows that the condition for the natural frequency of the system is not critical to get stable control systems being sufficient to close the loop. We actually feed back the measurements of a strain gauge to close the loop. The gain margin is infinite.

The block diagram for the outer loop used in this work is shown in Figure 15.4.3 Outer-loop. The stability study by using Nyquist plots shows that the condition for the natural frequency of the system is not critical to get stable control systems being sufficient to close the loop. We actually feed back the measurements of a strain gauge to close the loop. The gain margin is infinite.

Other alternative definitions can be found in [41x-1058]uq 6.1. These curves correspond to the damping ratios and the natural frequency as functions of the position of the poles. These roots are represented in Figure 15.7.2.3.4.3 Step Response and Characteristic Parameters. Other alternative definitions can be found in [41x-1058]uq 6.1. These curves correspond to the damping ratios and the natural frequency as functions of the position of the poles. These roots are represented in Figure 15.7.2.3.4.3 Step Response and Characteristic Parameters.

As observed in the schematic, an estimation of the tip position is needed to obtain the relation between the coupling torque and the motor inertia. As observed in the schematic, an estimation of the tip position is needed to obtain the relation between the coupling torque and the motor inertia.

The stability study by using Nyquist plots shows that the condition for the natural frequency of the system is not critical to get stable control systems being sufficient to close the loop. We actually feed back the measurements of a strain gauge to close the loop. The gain margin is infinite.

\[\theta_t(s) = \frac{\omega_0^2}{s^2} u(s) + \frac{1}{s^2} P(s)\]
 Outer Loop

\[ R_e(s) \frac{\frac{2}{s^2}}{\omega_0^2} \] in the form of fractional integrator with order \(2 - \alpha, 0 < \alpha < 1\)

![Control System Diagram](image)
Results

![Graph 1](image1.png)

![Graph 2](image2.png)

![Graph 3](image3.png)

![Photo](image4.png)
Several nonlinear strategies can be found in the literature:

- Adaptive: MRAC and Gain Scheduling
- Reset control
- Sliding Mode Control
NONLINEAR CONTROL STRATEGIES

- Sliding Mode Control of Power Electronic Buck Converter:
  - Fractional PID Sliding Surfaces
  - Direct Boolean Approach
  - Augmented System Approach
NONLINEAR CONTROL STRATEGIES

• Converter Model

\[
\begin{align*}
\begin{bmatrix}
\dot{v}_c \\
\dot{\hat{v}}_c
\end{bmatrix} &= \begin{bmatrix}
0 & 1 \\
-\frac{1}{LC} & -\frac{1}{RC}
\end{bmatrix}
\begin{bmatrix}
v_c \\
\hat{v}_c
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{1}{LC}
\end{bmatrix} V_g u. \\
\dot{x} &= A_0 x + B_0 V_g, \quad \text{for } u = 0, \\
\dot{x} &= A_1 x + B_1 V_g, \quad \text{for } u = 1.
\end{align*}
\]
the composition of a minimum phase function. This fact limits the values of the converterm. This one is used as a voltage compensator for canceling wave perturbations up to q ms and so a settling time less than p ms.

The input voltage variations in a buck converter can be viewed as the equivalent control force the system trajectory to follow a suitable selected surface on the phase function of the instantaneous values of the state variables in such a way as to implement a control action which exploits the inherent variable structure since its structure is periodically changed by the action of the controlled switch network.

On the other hand, since power electronic converters inherently include switch devices which exhibit a discontinuous behavior, the DC-DC buck practical implementation is required. This fact limits the values of the process gain. In this way, here we propose a method for the control of power transfer function, providing a controlled system robust to changes in the converter switches are driven as a pseudolcontinuous system using the bilinear transformation.

\[
\begin{align*}
    y(k+1) & = 0.3070 \ x(k) - 0.1100 \ x(k) + 0.1798 \\
    & \quad 0.2585 \ x(k) + 1.0571 \ x(k) \times 10^3 V_g \bar{v}(k) \\
    d(k) & = \bar{v}^{-1}(\bar{v}(k))
\end{align*}
\]
Sliding Surfaces

\[ S = K_p (v_r - v_c) + K_i \mathcal{D}^{-\lambda} (v_r - v_c) + K_d \mathcal{D}^{\mu} (v_r - v_c) \]

\[ u_{eq} = \frac{v_c}{V_g} + \frac{K_i L C}{K_d V_g} \mathcal{D}^{2-\mu-\lambda} (v_r - v_c) + \frac{L}{RCV_g} \left( i_L - \frac{v_c}{R} \right) - \frac{K_p L}{K_d V_g} \mathcal{D}^{1-\mu} \left( i_L - \frac{v_c}{R} \right) \]
18.5 Simulation and Experimental Results

18.5.1 Simulation Results

In order to show the performance of the proposed methods, both linear and sliding mode control, simulation and experimental results for all the controllers, with the specifications listed before, are shown here. In the case of linear control, the simulated system corresponds to the block diagram of Figure 18.1 where the block "table" performs the conversion between fictitious control signal and duty ratio, the block PWM provides to the filter a voltage \( V_g \) during the interval \( d_b \). The block "filter" is the LC filter plus the load resistance.

\[
\begin{align*}
v_r &= 12 \text{ V} \\
\text{controller} & \rightarrow v(k) \rightarrow \text{table} \rightarrow d(k) \rightarrow \text{PWM} \rightarrow \text{filter} \rightarrow V
\end{align*}
\]

In order to implement the calculated linear regulators in a digital processor, the first step is to find the realizable discrete equivalents of the previous controllers. This is achieved using inverse \( \omega \) transformation, and in the case of fractional-order integrators, they are approximated by the continued fraction expansion (CFE) and the Tustin's rule. Using the previous approximation, bshmmc does not satisfy the causality principle and so a modification is introduced. Since the order \( \lambda \) is bigger than \( l_f \), \( z^{-\lambda} \) is considered as composed of a pure integrator and a fractional-order integrator of order \( \lambda' \). The fractional-order integrator is approximated by the expressed method and the pure integrator using the forward rectangular rule.

Figure 18.1 displays the open-loop Bode plots of the compensated system with the described approximations. These results show that the design specification, phase margin \( \phi_m \) and gain crossover frequency \( \omega_c \) achieved in the design process are kept.

Next, the simulated step responses are presented. Figure 18.12 shows the simulated step responses obtained with the described linear controllers. An overshoot can be observed in the time response of the controlled system using a controller based on the discrete version of the Bode's ideal loop transfer function. This overshoot is due to \( \lambda > l_f \) and can be removed by changing the design specifications.

18.5.2 Experimental Results

In order to show the feasibility of the proposed methods, a real prototype of the buck converter has been built and experimental results are reported and discussed. The prototype of the converter system is shown in Figure 18.13. Figure 18.14 shows the block diagram of the controller. The controller algorithms have been implemented in a Pentium 266 MHz machine. The process interface has been carried out with a multifunction data acquisition card PCL 818 and a multifunction Counter/Timer card PCL 836, which provides three PWM channels.

\[
\begin{align*}
\text{controller} & \rightarrow v(k) \rightarrow \text{table} \rightarrow d(k) \rightarrow \text{PWM} \rightarrow \text{filter} \rightarrow V
\end{align*}
\]

Figure 18.15 shows the experimental responses obtained with the linear controllers. Figure 18.16 shows the experimental responses obtained with the sliding mode controllers.
18.5 Simulation and Experimental Results

Since there is a good agreement between simulated and experimental results in all the cases, the following comparison is only done considering simulation results.

(a) Smith predictor structure case, (b) phase-lag compensation case, (c) design based on the discrete model case, (d) sliding surface through a PID structure, (e) sliding surface through a PI structure, (f) sliding surface through a PI$\lambda$ structure, and (g) sliding surface through a PI$\lambda$Dµ structure.

Figure 18.10: Simulation results of the controlled converter for the defined sliding surfaces.

Table 18.1: SMC controller parameters

<table>
<thead>
<tr>
<th>Network</th>
<th>Kp</th>
<th>Ki</th>
<th>Kd</th>
<th>$\lambda$</th>
<th>$\mu$</th>
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<tr>
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<td>20</td>
<td>0.25</td>
<td>10</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PID 2</td>
<td>15</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PI$\lambda$Dµ</td>
<td>100</td>
<td>20</td>
<td>0.125</td>
<td>0.06</td>
<td>0.90</td>
</tr>
<tr>
<td>PI$\lambda$</td>
<td>0.2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PID</td>
<td>-</td>
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<td>-</td>
</tr>
</tbody>
</table>

Experiments: Fractional PID
NONLINEAR CONTROL STRATEGIES

- Direct Boolean Approach
- Augmented System Approach
Conclusions

- Linear and nonlinear control strategies use derivatives and integrals for control laws: Try fractional ones - integer order derivatives and integrals are only particular case.

- Huge work should be made for analytic proofs.