Fractional-order Calculus, Fractional-order Filter and Fractional-order Control: : An Overview and Some Recent Developments

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Topics:

1. Fractional-order Calculus (A bit history, definitions, examples)
2. Fractional-order ODE and Laplace Transform
3. Fractional-order Dynamic Filtering
4. Fractional-order Modeling and Control
5. Implementation Techniques
6. To Probe Further
7. Concluding Remarks: “in-between thinking”.

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Leibniz introduced the notation $d^n y/dx^n$.

In a letter to L’Hospital in 1695, Leibniz raised the following question: *Can the meaning of derivatives with integer order $d^n y(x)/dx^n$ be generalized to derivatives with non-integral orders; so that in general $n \in \mathbb{C}$?* The story goes that L’Hospital was somewhat curious about that question and replied by another question to Leibniz. What if $n = 1/2$? Leibniz in a letter dated September 30, 1695 replied: *It will lead to a paradox, from which one day useful consequences will be drawn.*
Some special functions

Euler’s Gamma function:

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad (1) \]

Special case when \( x = n \):

\[ \Gamma(n) = (n - 1) (n - 2) (\ldots) (2) (1) \Gamma(1) = (n - 1)! \quad (2) \]
Some special functions (Continued)

Mittag-Leffler function in two parameters:

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0). \quad (3) \]

It is a generalization of exponential function:

\[ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \quad (4) \]
Mittag-Leffler function: (Continued)

More particular cases:

\[ E_{2,1}(z) = \cosh(\sqrt{z}), \quad E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}, \quad (5) \]

\[ E_{1/2,1}(\sqrt{z}) = \frac{2}{\sqrt{\pi}} e^{-z} \text{erfc}(-\sqrt{z}). \quad (6) \]
Operator \( aD^\alpha_t \)

A generalization of differential and integral operators:

\[
aD^\alpha_t = \begin{cases} 
d^\alpha/dt^\alpha & \Re(\alpha) > 0, \\
1 & \Re(\alpha) = 0, \\
\int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0. 
\end{cases}
\]  

(7)

There are two commonly used definitions for the general fractional order differentiation and integral, i.e., the \textbf{Grünwald-Letnikov definition} and the \textbf{Riemann-Liouville definition}.

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Gr"unwald-Letnikov definition

\[
\frac{d}{dt} f(t) \equiv f'(t) = \lim_{h \to 0} \frac{f(t) - f(t - h)}{h}. \quad (8)
\]

\[
\frac{d^2}{dt^2} f(t) \equiv f''(t) = \lim_{h \to 0} \frac{f'(t) - f'(t - h)}{h} = \lim_{h \to 0} \frac{1}{h} \left\{ \frac{f(t) - f(t - h)}{h} - \frac{f(t - h) - f(t - 2h)}{h} \right\} = \lim_{h \to 0} \frac{f(t) - 2f(t - h) + f(t - 2h)}{h^2}. \quad (9)
\]

\[
\frac{d^3}{dt^3} \equiv f'''(t) = \lim_{h \to 0} \frac{f(t) - 3f(t - h) + 3f(t - 2h) - f(t - 3h)}{h^3}. \quad (10)
\]
In general,

\[ \frac{d^n}{dt^n} f(t) \equiv f^{(n)}(t) = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(t -jh). \tag{11} \]

\[ \binom{n}{j} = \frac{n(n-1)(n-2) \ldots (n-j+1)}{j!} = \frac{n!}{j!(n-j)!}. \tag{12} \]

\[ D_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t -jh). \tag{13} \]

\[ \binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha - j)!} = \frac{\Gamma(\alpha + 1)}{\Gamma(j + 1) \Gamma(\alpha - j + 1)}, \tag{14} \]

where \( \binom{\alpha}{0} = 1. \)
Riemann-Liouville definition

\[
\begin{aligned}
\int_a^t \int_a^{t_1} \cdots \int_a^{t_1} f(t_1) dt_1 dt_2 \cdots dt_n &= \frac{1}{\Gamma(n)} \int_a^t f(\tau) (t - \tau)^{1-n} d\tau,
\end{aligned}
\]

for \( n \in N, n > 0 \).

\[
aI_t^\alpha f(t) \equiv aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_a^t f(\tau) (t - \tau)^{\alpha+1} d\tau,
\]

for \( \alpha, a \in \mathbb{R}, \alpha < 0 \).

\[
aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t f(\tau) (t - \tau)^{\alpha - n+1} d\tau,
\]

for \( n-1 < \alpha < n \),

where \( a \) and \( t \) are the limits of \( aD_t^\alpha f(t) \).
Properties of $a D_t^\alpha$

1. If $f(z)$ is an analytic function of $z$, the derivative $a D_z^\alpha f(z)$ is an analytic function of $z$ and $\alpha$.

2. The operation $a D_z^\alpha$ gives the same result as the usual differentiation of (integer) order $n$.

3. The operator of order $\alpha = 0$ is the identity operator.

4. Fractional operators are linear:

$$a D_z^\alpha \{a f(z) + bh(z)\} = a a D_z^\alpha f(z) + b a D_z^\alpha h(z). \quad (18)$$

5. For fractional integrations of arbitrary order $\alpha > 0, \beta > 0, (\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0)$ the additive index law (semigroup property) holds:

$$a D_z^{-\alpha} a D_z^{-\beta} f(z) = a D_z^{-(\alpha+\beta)} f(z) \quad (19)$$
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Laplace Transformation of $aD_t^\alpha$

From the **Riemann-Liouville definition**:

$$
\int_0^\infty e^{-st} 0D_t^\alpha f(t) \, dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k 0D_t^{\alpha-k-1} f(t) \bigg|_{t=0},
$$

(20)

for $(n - 1 < \alpha \leq n)$ where $F(s) = \mathcal{L}[f(t)]$ is the normal Laplace transformation.

From the **Gr"unwald-Letnikov definition**:

$$
\int_0^\infty e^{-st} 0D_t^\alpha f(t) \, dt = s^\alpha F(s)
$$

(21)
Laplace Transform of Mittag-Leffler function

Applying the above definition for Laplace transformation
\[
\int_0^\infty e^{-st} t^{\alpha k + \beta - 1} E_{\alpha, \beta}^{(k)}(\pm at^\alpha) dt = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}, \quad (\text{Re}(s) > |a|^{1/\alpha}). \tag{22}
\]

The particular case of (22) for \( \alpha = \beta = 1/2 \)
\[
\int_0^\infty e^{-st} t^{k-1/2} E_{1, 1/2}^{(k)}(\pm a\sqrt{t}) dt = \frac{k!}{(\sqrt{s} \mp a)^{k+1}}, \quad (\text{Re}(s) > a^2). \tag{23}
\]
is useful for solving semidifferential equations.
A typical $n$-term linear FODE in time domain is given by
\[ a_n D_t^{\beta_n} y(t) + \cdots + a_1 D_t^{\beta_1} y(t) + a_0 D_t^{\beta_0} y(t) = 0 \]  
(24)
where $a_k (k = 0, 1, \cdots, n)$ are constant coefficients of the FODE; $\beta_k, (k = 0, 1, 2, \cdots, n)$ are real numbers. Without loss of generality, assume that $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 \geq 0$. 
Analytical Time Domain Solution of FODEs

\[ y(t) = \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0+k_1+\cdots+k_{n-2}=m} (m; k_0, k_1, \ldots, k_{n-2}) \]

\[ \prod_{i=0}^{n-2} \left( \frac{a_i}{a_n} \right)^{k_i} t^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2} (\beta_{n-1}-\beta_j)k_j-1} \]

\[ E_{\beta_n-\beta_{n-1},+\beta_n+\sum_{j=0}^{n-2} (\beta_{n-1}-\beta_j)k_j}^{(m)} \left( -\frac{a_n-1}{a_n} t^{\beta_n-\beta_{n-1}} \right), \]

where \( E_{\lambda,\mu}(z) \) is the Mittag-Leffler function in two parameters (4) and

\[ E_{\lambda,\mu}^{(n)}(y) = \frac{d^n}{dy^n} E_{\lambda,\mu}(y) = \sum_{j=0}^{\infty} \frac{(j + n)! y^j}{j! \Gamma(\lambda j + \lambda n + \mu)}, \]

for \( (n = 0, 1, 2, \ldots) \).

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Give me a break! Too heavy math!!

Show me the pictures or some cartoons!!!
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Fractional derivatives of the Heaviside function.

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Fractional derivatives of sine function.

Fractional derivatives of function $y = \sin(t)$

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Fractional derivatives of ramp function.
Solving FODE: Laplace method

Consider a simple FODE

$$0D_t^{1/2}f(t) + af(t) = 0, \quad (t > 0); \quad 0D_t^{-1/2}f(t) \bigg|_{t=0} = C \quad (25)$$

Applying the Laplace transform we obtain

$$F(s) = \frac{C}{s^{1/2} + a}, \quad C = 0D_t^{-1/2}f(t) \bigg|_{t=0}$$

and the inverse transform with a help of (23) gives the solution of (25):

$$f(t) = Ct^{-1/2}E_{\frac{1}{2}, \frac{1}{2}}(-a\sqrt{t}). \quad (26)$$

Using series expansion (4) of $E_{\alpha, \beta}(t)$, it is easy to check that for $a = 1$ solution (26) is identical to an alternative solution

$$f(t) = C(1/\sqrt{\pi t} - e^t\text{erfc}(\sqrt{t}))$$

obtained by a more complicated procedure.
Solving FODE: Numerical Solution

**Basis:**

\[
D^\alpha f(t) \simeq \Delta_h^\alpha f(t)
\]

\[
\Delta_h^\alpha f(t) = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(kh - jh)
\]

For example:

\[
D^\alpha y(t) + by(t) = q(t), \quad t > 0, \quad n - 1 < \alpha \leq n
\]

\[
y^{(k)} = 0, \quad k = 0, 1, ..., n - 1
\]

\[
h^{-\alpha} \sum_{j=0}^{k} w_j^{(\alpha)} y_{k-j} + by_k = q_k
\]
where

\[ w^{(\alpha)}_j = (-1)^j \binom{\alpha}{j}, \quad j = 0, 1, 2, \ldots \]  \hfill (31)

\[ t_k = kh, \quad y_k = y(t_k), \quad y_0 = 0, \quad q_k = q(t_k), \quad k = 0, 1, 2, \ldots \]

Algorithm

\[ y_i = 0, \quad i = 1, 2, \ldots, n - 1 \]  \hfill (32)

\[ y_k = -bh^{\alpha}y_{k-1} - \sum_{j=1}^{k} w^{(\alpha)}_j y_{k-j} + h^{\alpha} q_k, \quad k = n, n + 1, \ldots \]  \hfill (33)

Short Memory Principle:

\[ D^{\alpha} f(t) \simeq_{t-L} D^{\alpha} f(t), \quad t > L \]  \hfill (34)

\[ \varepsilon(t) = |D^{\alpha} f(t) -_{t-L} D^{\alpha} f(t)| \leq \frac{ML^{-\alpha}}{\Gamma(1 - \alpha)}, \quad L \leq t \leq t_1 \]  \hfill (35)
\[ \varepsilon(t) < \epsilon, \quad L \leq t \leq t_1 \Rightarrow L \geq \left( \frac{M}{\epsilon |\Gamma(1 - \alpha)|} \right)^{\frac{1}{\alpha}} \]  \hspace{1cm} (36)

**Computation of coefficients**

\[ w_0^{(\alpha)} = 1; \quad w_k^{(\alpha)} = \left( 1 - \frac{\alpha + 1}{k} \right) w_{k-1}^{(\alpha)}, \quad k = 1, 2, \ldots \]  \hspace{1cm} (37)

\[ (1 - z)^{\alpha} = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k = \sum_{k=0}^{\infty} w_k^{(\alpha)} z^k \]  \hspace{1cm} (38)

With \( z = e^{-j\omega} \)

\[ (1 - e^{-j\omega})^{\alpha} = \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-jk\omega} \]  \hspace{1cm} (39)

**Fourier Transform**

\[ w_k^{(\alpha)} = \frac{1}{2\pi} \int_{0}^{2\pi} (1 - e^{-j\omega})^{\alpha} e^{jk\omega} \, d\omega \]  \hspace{1cm} (40)
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Dynamic Models: d.e.’s

Continuous Models:

\[ a_n D^{\alpha_n} y(t) + \cdots + a_0 D^{\alpha_0} y(t) = b_m D^{\beta_m} u(t) + \cdots + b_0 D^{\beta_0} u(t) \]  

\(41\)

Commensurate order systems

\[ \alpha_k, \beta_k = k\alpha, \quad \alpha = \frac{1}{q}, \quad q \in \mathbb{Z}^+ \]  

\(42\)

\[ \sum_{k=0}^{n} a_k D^{k\alpha} y(t) = \sum_{k=0}^{m} b_k D^{k\alpha} u(t) \]  

\(43\)

Discrete Models

\[ a_n \Delta^{\alpha_n}_h y(t) + \cdots + a_0 \Delta^{\alpha_0}_h y(t) = b_m \Delta^{\beta_m}_h u(t) + \cdots + b_0 \Delta^{\beta_0}_h u(t) \]  

\(44\)
Input-Output Representations (transfer function)

\[
G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} \quad (45)
\]

\[
G(z) = \frac{b_m \left( \omega \left( z^{-1} \right) \right)^{\beta_m} + b_{m-1} \left( \omega \left( z^{-1} \right) \right)^{\beta_{m-1}} + \cdots + b_0 \left( \omega \left( z^{-1} \right) \right)^{\beta_0}}{a_n \left( \omega \left( z^{-1} \right) \right)^{\alpha_n} + a_{n-1} \left( \omega \left( z^{-1} \right) \right)^{\alpha_{n-1}} + \cdots + a_0 \left( \omega \left( z^{-1} \right) \right)^{\alpha_0}} \quad (46)
\]
State-Space Representation

\[ D^{\alpha_k}x = Ax + Bu; \quad y = Cx + Du \]

\[
\begin{bmatrix}
D^{\alpha}x_1 \\
D^{\alpha}x_2 \\
\vdots \\
D^{\alpha}x_n
\end{bmatrix} =
\begin{bmatrix}
-b_{n-1} & -b_{n-2} & \cdots & -b_0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} u
\]

\[ y = \begin{bmatrix}
a_m & a_{m-1} & \cdots & a_1 & a_0 \\
\vdots & \ddots & \cdots & \vdots & \vdots \\
& & & & \\
& & & & \\
& & & & x_n
\end{bmatrix} \]
State Transition Matrix

\[ x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{(s^{\alpha}I - A)^{-1}BU(s) + (s^{\alpha}I - A)^{-1}x(0)\right\} \]

(47)

By defining \( \Phi(t) \equiv \mathcal{L}^{-1}\left\{(s^{\alpha}I - A)^{-1}\right\} \), \( t \geq 0 \)

\[ x(t) = \Phi(t)x(0) + \Phi(t) \ast [Bu(t)] = \Phi(t)x(0) + \int_{0}^{t} \Phi(t - \tau)Bu(\tau)d\tau \]

(48)

\[ x(t) = \left[ I + \frac{Ax(0)}{\Gamma(1 + \alpha)}t^\alpha + \frac{A^2x(0)}{\Gamma(1 + 2\alpha)}t^{2\alpha} + \ldots + \frac{A^kx(0)}{\Gamma(1 + k\alpha)}t^{k\alpha} + \ldots \right] \]

\[ = \left( \sum_{k=0}^{\infty} \frac{A^kt^{k\alpha}}{\Gamma(1 + k\alpha)} \right)x(0) = E_{\alpha}(At^{\alpha})x(0) = \Phi(t)x(0) \]

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General Stability Condition

\[ \exists M, \quad |G(s)| \leq M, \quad \forall s \setminus \text{Re}(s) \geq 0 \quad (49) \]

For commensurate order systems:

\[ |\arg(\lambda_i)| > \alpha \frac{\pi}{2} \quad (50) \]
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Modeling: heat transfer

\[
\frac{\partial^2 y(x, t)}{\partial x^2} = k^2 \frac{\partial y(x, t)}{\partial t},
\]
\[t > 0, \quad 0 < x < \infty\]  

\[y(0, t) = m(t)\]  
\[y(x, 0) = 0\]  
\[\left| \lim_{x \to \infty} y(x, t) \right| < \infty\]

Transfer function:

\[
\frac{d^2 Y(x, s)}{dx^2} = k^2 s Y(x, s)
\]
\[Q(0, s) = M(s)\]
\[\left| \lim_{x \to \infty} Y(x, s) \right| < \infty\]
\[ Y(x, s) = A(s)e^{-kx\sqrt{s}} + B(s)e^{kx\sqrt{s}} \quad (58) \]

\[ A(s) = Y(0, s) = M(s) \quad (59) \]

\[ B(s) = 0 \quad (60) \]

\[ Y(x, s) = M(s)e^{-kx\sqrt{s}} \quad (61) \]

\[ G(s) = \frac{Y(x, s)}{M(s)} = e^{-kx\sqrt{s}} \quad (62) \]

**think about transfer function** \( e^{-\sqrt{s}} \)!
Fractional order speed control of DC motor

System transfer function \( G(s) = \frac{k}{Js(Ts+1)} \) \( J \) being the payload inertia. Phase margin of controlled system:

\[
\Phi_m = \arg \left[ C(j\omega)G(j\omega) \right] + \pi
\]

Controller: \( C(s) = k_1 \frac{k_2 s + 1}{s^\alpha} \), \( k_2 = T \) giving a constant phase margin:

\[
\Phi_m = \arg \left[ C(j\omega)G(j\omega) \right] + \pi = \arg \left[ \frac{k_1 k}{(j\omega)^{1+\alpha}} \right] + \pi
\]

\[
= \arg \left[ (j\omega)^{-1+\alpha} \right] + \pi = \pi - (1 + \alpha) \frac{\pi}{2}
\]

Step response:

\[
y(t) = \mathcal{L}^{-1} \left\{ \frac{k k_1 / J}{s (s^{1+\alpha} + k k_1 / J)} \right\} = \left( \frac{k k_1}{J} \right)^{1+\alpha} E_{1+\alpha, 2+\alpha} \left( -\frac{k k_1}{J} t^{1+\alpha} \right)
\]

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Note the iso-damping (similar overshoot!)

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PID from point to *plane*

\[ u(t) = k_p e(t) + k_i D^{-\lambda} e(t) + k_d D^\mu e(t) \]

\[ C_f(s) = K_p + \frac{k_i}{s^\lambda} + k_d s^\mu = k_i \left( \frac{s}{\omega_f} \right)^{\lambda+\mu} + 2\delta_f s^\lambda / \omega_f + 1 \]

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Temperature Control System

\[ G(s) = \frac{1}{31.61s^{1.26} + 0.598} \]

PD controller: \[ D_1(s) = 64.47 + 12.39s \]

Fractional PD controller: \[ D_2(s) = 64.47 + 48.99s^{0.5} \]
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**Performance comparison for 10 mins. run**

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“3-param. tunable tilt-integral-deriv. controller”

US05371670 on TID by B. J. Lurie, 1994

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**Oustaloup’s CRONE Control**

CRONE: French abbreviation for “Contrôle Robuste d’Ordre Non Entier” (non-integer order robust control) (Since 1981)

Based on concept of “Fractal Robustness”: the isodamping and the vertical sliding form of frequency template in the Nichols chart.

Given plant “$G(s)$”, how to design “$C(s)$”? The ideal situation is to make “$G(s)C(s) = (\tau s)^\alpha$” so that the characteristic equation is: $1 + (\tau s)^\alpha = 0$, which is “Fractal Robust”.

Real life applications: car suspension control, flexible transmission, hydraulic actuator etc.
Isodamping half-straight lines

\[ s_+ = \omega_u e^{+j\pi/\alpha} \]

\[ s_- = \omega_u e^{-j\pi/\alpha} \]

Isodamping half-straight lines

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Robustness in Nichols chart

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So simply the controller is: \[ C(j\omega) = \frac{(j\omega)^\alpha}{G_0(j\omega)} \]

\[ G_0(j\omega) \] is the nominal plant model.

Frequency specifications of the open loop behavior (for the nominal plant) will be given such as

- the accuracy specifications at low frequencies;
- the vertical template around unit gain frequency \( \omega_u \);
- the input sensitivity specifications at high frequencies.
For a stable minimum phase plant, it turns out that the behavior thus defined can be described by a transmittance based on the frequency-limited real non integer differentiator, i.e.,

\[
\beta(s) = \left[ K_b \left( \frac{\omega_b}{1} + 1 \right) \right]^{n_b} \left( \frac{\sqrt{1+(\omega_u/\omega_b)^2} \left( 1+s/\omega_h \right)^\alpha}{1+(\omega_u/\omega_h)^2 \left( 1+s/\omega_b \right)} \right) \left( \frac{K_h}{1+s/\omega_h} \right)^{n_h}
\]

with

\[
K_b = \left( 1 + \left( \frac{\omega_b}{\omega_u} \right)^2 \right)^{-1/2} \quad \text{and} \quad K_h = \left( 1 + \left( \frac{\omega_u}{\omega_h} \right)^2 \right)^{1/2}.
\]
In the particular case where transitional frequencies $\omega_b$ and $\omega_h$ are sufficiently distant from frequency $\omega_u$, around this frequency (i.e. $\omega_b \ll \omega \ll \omega_h$), $\beta(s)$ can be reduced to transmittance

$$\beta(s) = (\omega_u/s)^\alpha,$$

which is the same as that described by the template.

The order $\alpha$ transmittance of relation (64) describes the frequency truncation of the template defined by the transitional frequencies $\omega_b$ and $\omega_h$. This transmittance results from the substitution of the part raised at power $\alpha$ for the transmittance $\omega_b/p$ which is used in the description of the template between frequencies $\omega_A$ and $\omega_B$. 
Topics:

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5. Implementation Techniques
6. To Probe Further
7. Concluding Remarks: “in-between thinking”.

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Analog $1/\sqrt{s}$ using op-amps.
Phase plot (deg. vs. rad./sec.)

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Magnitude plot (dB vs. rad./sec.)
**Digital Implementation: indirect method**

Approximate in continuous-time domain and then discretize it.

**Oustaloup’s method:** Given a frequency range of practical interest $\omega \in [\omega_L, \omega_H]$, we can immediately get a rational transfer function of finite order which is a fit to the given fractional order differentiator. In practice, we need set the transitional frequency range much larger than the $[\omega_L, \omega_H]$, for example, $[0.1\omega_L, 10\omega_H]$. Using this transitional frequency range and order of approximation $2N + 1$, the following formulae are the so-called Oustaloup-Recursive-Approximation (ORA):

$$\lim_{N \to \infty} D_N(s) = D(s) = s^r, \quad (67)$$

where

$$D_N(s) = \left(\frac{\omega_u}{\omega_H}\right)^r \prod_{k=-N}^{N} \frac{1 + s/\omega'_k}{1 + s/\omega_k}, \quad (68)$$
and

\[ \omega_u = \sqrt{\omega_H \omega_L}, \]  
\[ \omega'_k = \omega_L \left( \frac{\omega_H}{\omega_L} \right)^{(k+N+0.5-0.5*r)/(2N+1)}, \]  
\[ \omega_k = \omega_L \left( \frac{\omega_H}{\omega_L} \right)^{(k+N+0.5+0.5*r)/(2N+1)}. \]
The 9th order Outaloup approximation can be obtained

\[ G_9(s) = \frac{31.62(s + 3164)(s + 680.8)(s + 146.8)(s + 31.62)(s + 6.816)(s + 1.467)(s + 0.3165)(s + 0.06809)(s + 0.01468)}{(s + 6812)(s + 1469)(s + 316)(s + 66.4)(s + 20.53)(s + 0.107)(s + 0.03222)(s^2 - 0.281s + 2.128)} \]

The Prewarp discretization of \( G_9(s) \) at \( T = 0.001 \) sec. can also be obtained as

\[ G_9(z) = \frac{12.1986(z - 0.9995)^4(z - 0.9922)(z - 0.9641)(z - 0.8436)(z - 0.4348)(z + 0.2929)}{(z - 0.9997)^3(z - 0.9964)(z - 0.9832)(z - 0.9242)(z - 0.6909)(z + 0.5951)(z - 0.08188)} \]
Bode plot of $\sqrt{s}$ (9/9 approximation)

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Nichols chart of $\sqrt{s}$ (9/9 approximation)

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Digital Implementation: direct method (FIR)

Generating function $s = \omega(z^{-1})$. Backward difference rule $\omega(z^{-1}) = (1 - z^{-1})/T$, performing the power series expansion (PSE) of $(1 - z^{-1})^{\pm r}$ gives the discretization formula for Grünwald-Letnikov formula. By using the short memory principle, the discrete equivalent of the fractional-order integro-differential operator, $\omega(z^{-1})^{\pm r}$, is given by

$$
(\omega(z^{-1}))^{\pm r} = T^{\mp r} \sum_{j=0}^{[L/T]} (-1)^j \binom{\pm r}{j} z^{[L/T]-j},
$$

(72)

where $T$ is the sampling period, $L$ is the memory length, $[\cdot]$ is the flooring operator and $(-1)^j \binom{\pm r}{j}$ are binomial coefficients $c_j^{(r)}$, $(j = 0, 1, \ldots)$ where

$$
c_0^{(r)} = 1, \quad c_j^{(r)} = \left(1 - \frac{1 + (\pm r)}{j}\right) c_{j-1}^{(r)}.
$$

(73)
Digital Implementation: direct method (IIR)

Recursive formula! The trapezoidal (Tustin) rule as a generating function \((\omega(z^{-1}))^{\pm r} = \left( \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \right)^{\pm r} \)

\[
(\omega(z^{-1}))^{r} = \left( \frac{2}{T} \right)^{r} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)^{r} = \left( \frac{2}{T} \right)^{r} \lim_{n \to \infty} \frac{A_{n}(z^{-1}, r)}{A_{n}(z^{-1}, -r)}
\]  

(74)

where

\[
A_{0}(z^{-1}, r) = 1, \quad A_{n}(z^{-1}, r) = A_{n-1}(z^{-1}, r) + c_{n} z^{-n} A_{n-1}(z, r),
\]

(75)

and

\[
c_{n} = \begin{cases} 
  r/n & \text{if } n \text{ is odd;} \\
  0 & \text{if } n \text{ is even.}
\end{cases}
\]

(76)

\[
s^{r} \approx \left( \frac{2}{T} \right)^{r} \frac{A_{n}(z^{-1}, r)}{A_{n}(z^{-1}, -r)}.
\]
Bode plot of $\sqrt{s}$ (n=1 approximation)

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Bode plot of $\sqrt{s}$ (n=3 approximation)

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Bode plot of $\sqrt{s}$ (n=5 approximation)

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Bode plot of $\sqrt{s}$ (n=7 approximation)

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Bode plot of $\sqrt{s}$ (n=9 approximation)
Direct Discretization Using Al-Alaoui Operator

Al-Alaoui operator: weighted sum of rectangular rule or Euler operator (0.25) and the trapezoidal rule (0.75). (FIR and IIR above).

\[
(\omega(z^{-1}))^{\pm r} = \left( \frac{8}{7T} \frac{1 - z^{-1}}{1 + z^{-1}/7} \right)^{\pm r}.
\]

(77)

Clearly, (77) is an infinite order of rational discrete-time transfer function. To approximate it with a finite order rational one, continued fraction expansion (CFE) is an efficient way. In general, any function \( G(z) \) can be represented by continued fractions in the form of

\[
G(z) \simeq a_0(z) + \frac{b_1(z)}{a_1(z) + \frac{b_2(z)}{a_2(z) + \frac{b_3(z)}{a_3(z) + \ldots}}}
\]

where the coefficients \( a_i \) and \( b_i \) are either rational functions of the
variable $z$ or constants. By truncation, an approximate rational function, $\hat{G}(z)$, can be obtained.

The resulting discrete transfer function, approximating fractional-order operators, can be expressed as:

$$D^{\pm r}(z) \approx \left( \frac{8}{7T} \right)^{\pm r} \text{CFE} \left\{ \left( \frac{1 - z^{-1}}{1 + z^{-1}/7} \right)^{\pm r} \right\}_{p,q}$$

$$= \left( \frac{8}{7T} \right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})}$$

(78)

where CFE\{u\} denotes the continued fraction expansion of $u$; $p$ and $q$ are the orders of the approximation and $P$ and $Q$ are polynomials of degrees $p$ and $q$. Normally, we can set $p = q = n$. 

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In MATLAB Symbolic Toolbox, by the following script, for a given $n$ we can easily get the approximate direct discretization of fractional order derivative:

```matlab
syms x r;
maple('with(numtheory)');

aa = ((1-x)/(1+x/7))^r;
n = 5; n2 = 2*n;

maple(['cfe := cfrac(' char(aa) ',x,n2);'])

maple('P_over_Q := nthconver','cfe',n2)

maple('P := nthnumer','cfe',n2)

maple('Q := nthdenom','cfe',n2)
```
Bode plot of $\sqrt{s}$ (p=q=1 approximation)
Bode plot of $\sqrt{s}$ (p=q=3 approximation)

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Bode plot of $\sqrt{s}$ (p=q=5 approximation)

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Bode plot of $\sqrt{s}$ ($p=q=7$ approximation)

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Bode plot of $\sqrt{s}$ ($p=q=9$ approximation)

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We have a brilliant solution! All we have to do now is to find the problem!
All around $s^\alpha$ where $\alpha \in \mathbb{R}$!

- its definition in terms of "fractional calculus"
- implications (possible benefits) in filtering and control (iso-damping, constant phase margin etc.)
- realization techniques: analog, indirect digital, direct digital (FIR, IIR etc.)

- **In brief**: *we got a new tuning knob $s^\alpha$ in our engineering practice! How to better use this knob is your business!*
Topics:

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To Probe Further

- Books:
  - Nishimoto, K. *Fractional Calculus*. New Haven, CT:

- **URLs:**
  - [http://mechatronics.ece.usu.edu/foc/](http://mechatronics.ece.usu.edu/foc/) (events, news, researchers, tutorials, MATLAB scripts, \LaTeX\ BiBTeX library for FOC research etc.)
  - [http://xxx.lanl.gov](http://xxx.lanl.gov) (some online papers with \LaTeX\ source on FOC)
  - [http://www.emath.fr/Maths/Proc/Vol.5/contents.htm](http://www.emath.fr/Maths/Proc/Vol.5/contents.htm) (an online proceedings on FOC - some papers in French!)

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Topics:

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Concluding Remarks: “in-between thinking”

Examples of “in-between thinking”:

**Fuzzy logic**: “in-between thinking” about binary logic 0 and 1
**Fractional splines**: “in-between thinking” about splines of integer orders
(see [http://bigwww.epfl.ch/demo/fractsplines/index.html](http://bigwww.epfl.ch/demo/fractsplines/index.html))

So, with **Fractional Order Calculus**, you may be able to extend a lot of things...
Good luck!
Some Recent Developments

Refer to http://mechatronics.ece.usu.edu/foc/. Some highlighted developments:

• Special Issue on FOC, *Nonlinear Dynamics*, 2002.
• First IEEE CDC Tutorial Workshop on FOC, Las Vegas, 2002.
• First IFAC Symposium on FOC, France, 2004.
• First funding on FOC from National Research Council, 2003-2005.
I’d like to take this chance to thank Blas Vinagre and Ivo Petras for their great help and discussions.

Thanks also go to Sabatier Jocelyn, A. Oustaloup, Igor Podlubny, M. A. Al-Alaoui, Yoichi Hori, and Dinh Nho Hao for many useful reference papers in either soft or hard copies.
Thank you!

Q/A Session.

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FOC web: http://mechatronics.ece.usu.edu/foc/
• Some results from CSOIS Group
• On $D^\alpha$-type ILC results.
Some results from CSOIS Group

Done:

- Analytical stability bound for a class of delayed fractional-order dynamic systems (for IEEE Trans. AC.);
- Discretization Schemes for Fractional Order Differentiators and Integrators (for IEEE Trans. CAS1);
- Direct Discretization of Fractional Order Derivative Using Al-Alaoui Operator (for 2001 IEEE CDC);
- On $D^\alpha$-type Iterative Learning Control (for 2001 IEEE CDC).

To do:

- make a TODO list. At least, have a look at QFT and Mollification techniques ...
angular position tracking error vs. ILC iter. #
angular velocity tracking error vs. ILC iter. #

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