INITIALIZATION OF Riemann-Liouville AND Caputo Fractional Derivatives

Trigeassou Jean-Claude
IMS LAPS
University of Bordeaux, France

Maamri Nezha
LAII
University of Poitiers, France

Oustaloup Alain
IMS LAPS
University of Bordeaux, France

ABSTRACT

Riemann-Liouville and Caputo fractional derivatives are fundamentally related to fractional integration operators. Consequently, the initial conditions of fractional derivatives are the frequency distributed and infinite dimensional state vector of fractional integrators. The paper is dedicated to the estimation of these initial conditions and to the validation of the initialization problem based on this distributed state vector. Numerical simulations applied to Riemann-Liouville and Caputo derivatives demonstrate that the initial conditions problem can be solved thanks to the estimation of the initial state vector of the fractional integrator.

Keywords: Riemann-Liouville derivative, Caputo derivative, fractional integration operator, initial conditions, initialization of fractional derivatives

1. INTRODUCTION

Initialization of fractional derivatives remains an open domain, in spite of a large number of contributions (see for example Lorenzo 2001, Hartley 2002, Ortigueira 2003, Lorenzo 2008, Sabatier 2008). Particularly, the concept of the initialization function has been introduced by Lorenzo and Hartley and applied to different situations.

An other approach is based on the infinite dimensional state vector of the fractional integrator which provides a straightforward interpretation of initial conditions. This new concept has been applied to the initialization of Fractional Differential Equations: it has been possible to estimate these initial conditions thanks to an observer and then to initialize correctly the corresponding FDE (Trigeassou 2011).

In a recent paper (Trigeassou 2010), this concept has been generalized to the interpretation of the initial conditions of Riemann-Liouville and Caputo fractional derivatives. A new formulation of the initial conditions of the Laplace Transforms of these derivatives has been proposed. The main result is the presence of the infinite dimensional state vector of the integrator in these initial conditions.

In this paper, our objective is to validate this new interpretation of fractional derivatives initial conditions. More specifically, we want to demonstrate that the integrator state vector is a good solution for the initialization of fractional derivatives. The main difficulty of the initialization problem is to estimate these initial conditions. Assuming that the exact formulation of the fractional derivative is available on a ‘past’ interval, we use an observer technique to provide an accurate estimation of the state vector at an instant $t_0$. Though the problem is fundamentally the same for the two derivatives, numerical simulations show that there are some significant differences due to integer order differentiation.

After a reminder of fractional integration and the definition of implicit fractional differentiation in section 2, we present the fractional integration operator in section 3 and the initial conditions of Riemann-Liouville and Caputo fractional derivatives in section 4. The proposed initialization technique is applied to the Caputo derivative in section 5 and to the Riemann-Liouville derivative in section 6.
2. FRACTIONAL INTEGRATION AND DIFFERENTIATION

2.1 Fractional integration

Consider the function \( f(\mu) \) and its repeated integrals:

\[
I_n(f(t)) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau
\]

where \( I_0 = f(t) \)

Using integration by parts, we get (Matignon, 1994), (Podlubny, 1999):

\[
I_n(f(t)) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau
\]

where \( n \) is an integer number.

Consider now that \( n \) is a real positive number: thus the factorial function \((n-1)!\) has to be replaced by the gamma function \( \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \)

(3)

Then, the \( n \)th fractional order Riemann-Liouville integral \((n \text{ real positive})\) of the function \( f(t) \) is defined by the relation (Miller, 1993), (Oldham, 1974), (Oustaloup, 1995):

\[
I_n(f(t)) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau
\]

(4)

\[
I_n(f(t)) \text{ is the convolution of the function } f(t) \text{ with the impulse response:}
\]

\[
h_n(t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau
\]

(5)

of the fractional integration operator whose Laplace transform is:

\[
I_n(s) = L[h_n(t)] = \frac{1}{s^n}
\]

(6)

2.2 Implicit fractional differentiation

Fractional differentiation is the dual operation of the fractional integration.

Consider the fractional integration operator \( I_n(s) \) whose input and output are respectively \( x(t) \) and \( y(t) \).

Then:

\[
y(t) = I_n(x(t)) \quad (7) \quad \text{or} \quad Y(s) = \frac{1}{s^n} X(s)
\]

(8)

Reciprocally, \( x(t) \) is the \( n \)th order fractional derivative of \( y(t) \) defined as:

\[
x(t) = D_n(y(t)) \quad (9) \quad \text{or} \quad X(s) = s^n Y(s)
\]

(10)

where \( s^n \) represents the Laplace transform of the fractional differentiation operator (for initial conditions equal to zero).

Thus, this fractional derivative definition is based on the operator \( I_n(s) \), without analytical formulation of \( D_n(y(t)) \): so it is an implicit definition of the fractional derivative.

3. FRACTIONAL INTEGRATION OPERATOR

3.1 Fractional integration operator

The fractional order integrator is an infinite dimensional system (Heleschewitz, 1998), (Montseny, 1998), (Trigeassou, 1999). Its state-space model is given by:

\[
\begin{align*}
\frac{d z(\omega,t)}{dt} &= -\omega z(\omega,t) + v(t) \\
x(t) &= \int_0^\infty \mu_0(\omega) z(\omega,t) d\omega
\end{align*}
\]

(11)

\[
\mu_0(\omega) = \sin(n\pi) \frac{\omega^{-n}}{\pi}
\]

(12)

where: \( v(t) \) : input, \( x(t) \) : output

\( z(\omega,t) \) : continuously distributed state

\( x(t) \) is the weighted sum of the \( z(\omega,t) \) internal state variables of the integrator.

3.2 State of the fractional integrator

The state \( z(\omega,t) \) of the operator \( I_n(s) \) is an infinite dimensional distributed state. Let \( z(\omega,t_0) \) be the frequency distributed state at instant \( t_0 \). This state represents the initial condition (or initialization function) of the integrator: it summarizes all the past behavior for \( t < t_0 \).

The solution of system (11) excited by \( v(t) \) for \( t \geq t_0 \), with the initial condition \( z(\omega,t_0) \) is given by (Sabatier, 2008):

\[
z(\omega,t) = z(\omega,t_0) e^{-\omega(t-t_0)} + \int_{t_0}^t e^{-\omega(t-\tau)} v(\tau) d\tau
\]

(13)

Consequently, the output \( x(t) \) of the fractional integrator (11) is composed of a free response term caused by the initial condition \( z(\omega,t_0) \) and of a forced response term caused by \( v(t) \), like all linear systems (Kailath, 1980).

Remark: For an integer order integrator \( I(s) = \frac{1}{s} \), we have \( \mu_i(\omega) = \delta(\omega) \), i.e. \( x(t) = z(t) \)

(14)

which means that \( x(t) \) and \( z(t) \) are the same variable and that the output of the integer order integrator is also the state variable of the integer order integrator, located in \( \omega = 0 \).

At the opposite, for \( I_n(s) \), the output \( x(t) \), which is the integral of \( z(\omega,t) \) weighted by the function \( \mu_n(\omega) \), is only a pseudo state variable: this means that \( x(t_0) \) is unable to summarize the past behavior of \( I_n(s) \) for \( t < t_0 \). Thus, the initialization
function $z(\omega,t_0)$ is equivalent to the initial state $x(t_0)$ of the integer order integrator $I(s)$.

3.3 Discrete approximation of the operator

The continuously distributed integrator model is not adapted to practical applications, like simulation. A discrete frequency approximation of this operator has been proposed in Trigeassou 2009. $J+1$ cells, ranging from 0 to $J$, provide a modal state space model of the integrator. See this reference, particularly for the definition of the different modes $\omega_j$ and of their weights $c_j$.

$$Z(t) = \begin{bmatrix} z_0 \\ c_1 \\ \vdots \\ z_J \end{bmatrix}$$

$$\frac{d}{dt}Z(t) = A_l Z(t) + B_r v(t)$$

$$x(t) = C^T_t Z(t)$$

with:

$$A_l = \begin{bmatrix} 0 & 0 \\ -\omega_1 & \ddots \\ 0 & \ddots & -\omega_J \end{bmatrix}; \quad B_r = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C^T_t = [c_0 \ c_1 \ \cdots \ c_J]$$

This discrete model has been used to initialize successfully a FDE (refer to Trigeassou 2011 for more details).

4. INITIAL CONDITIONS OF FRACTIONAL DERIVATIVES

4.1 Differentiation and convolution

The relation

$$I_n(s) D_n(s) = \frac{1}{s^n} s^n = 1$$

corresponds in the time domain to the convolution relation:

$$h_n(t) * d_n(t) = \delta(t)$$

where $d_n(t)$, impulse response of the fractional differentiator, is the convolution inverse of $h_n(t)$ (Matignon, 1994).

So we get: $d_n(t) = \frac{t^{n-1}}{\Gamma(-n)}$ where $n > 0$

4.2 Explicit formulations of the fractional derivative

Assume that the fractional order $n$ is situated between the two integer numbers $N-1$ and $N$ : $N-1 < n \leq N$

We can write $D_n(s) = \frac{s^n}{s^{N-n} s^n} = \frac{1}{s^{N-n}} s^n$ (21)

where $\frac{1}{s^{N-n}}$ represents the fractional integration $I_{N-n}(-)$ and $s^n$ the integer order differentiation $\frac{d^n}{dt^n}$.

Then $L\{D_n(f)\} = D_n(s) F(s) = \frac{1}{s^{N-n}} s^n F(s)$ (22)

(with zero initial conditions)

and using the inverse Laplace transform, we get two expressions for $D_n(f)$:

The first one corresponds to:

$$D_n(f) = h_{N-n}(t) * f(t)$$

and the second one to:

$$D_n(f) = \frac{d^n}{dt^n} \left( h_{N-n}(t) * f(t) \right)$$

This first expression is known as the Caputo derivative (Caputo, 1969):

$$D_n(f) = \frac{1}{\Gamma(N-n)} \int_0^t (t-\tau)^{N-n-1} \frac{d^n f(\tau)}{d\tau^n} d\tau$$

while the second one is the Riemann-Liouville derivative (Podlubny, 1999):

$$D_n(f) = \frac{d^n}{dt^n} \left[ \frac{1}{\Gamma(N-n)} \int_0^t (t-\tau)^{N-n-1} f(\tau) d\tau \right]$$

where $\Gamma(N-n)$ is correctly defined because $N-n > 0$

4.3 Initial conditions for $0 < n < 1$

Refer to Trigeassou 2010 for a complete presentation of the initial conditions problem.

4.3.1 Implicit derivative

Consider the integrator $I_n(s)$, with initial condition $z_i(\omega,0)$, where $\omega$ ranges from 0 to $\infty$.

Because:

$$L\left[ \frac{\partial z(\omega,t)}{\partial t} \right] = s Z(w,s) - z_i(w,0)$$

the Laplace transform of equation (11) is:

$$Z(\omega,s) = \frac{V(s) + z_i(\omega,0)}{s+\omega}$$

(28)
Because \( \int_0^\infty \frac{\mu_n(\omega)}{s+\omega} \, d\omega = \frac{1}{s^n} \) (29)

we get finally the Laplace transform of the implicit fractional derivative :

\[
L\{D_n(f)\} = s^n - s^n \int_0^\infty \frac{\mu_n(\omega) z_I(\omega,0)}{s+\omega} \, d\omega
\]

where the second term is based on the distributed initial condition \( z_I(\omega,0) \) (and with \( x(t) = f(t) \) and \( v(t) = D_n(f(t)) \)).

4.3.2 Caputo derivative \( 0<n<1 \)

Expression (23) can be written as:

\[
D_n(f) = h_{-n}(t) + \frac{df}{dt} = I_{-n}\left( \frac{df}{dt} \right)
\]

Consider the Laplace transform of the Caputo derivative, which is defined as :

\[
L\{D_n(f(t))\} = L\left\{ I_{-n}\left( \frac{df}{dt} \right) \right\}
\]

Then, using the results related to the implicit derivative (30), and replacing \( n \) by \( 1-n \), we get :

\[
L\{D_n(f)\} = L\left\{ \frac{df}{dt} \right\} - s F(s) - f(0)
\]

\[
= \frac{1}{s^{1-n}} L\left\{ \frac{df}{dt} \right\} + \int_0^\infty \frac{\mu_{1-n}(\omega) z_C(\omega,0)}{s+\omega} \, d\omega
\]

where \( z_C(\omega,0) \) is the initial state of integrator \( I_{-n}(s) \).

Moreover : \( L\left\{ \frac{df}{dt} \right\} = s F(s) - f(0) \) (34)

where \( f(0) \) is the initial state of integer order integrator \( I(s) \).

Finally, we get the Laplace transform of the Caputo derivative with initial conditions \( f(0) \) and \( z_C(\omega,0) \) :

\[
L\{D_n(f)\} = s^n F(s) - s^{n-1} f(0) + \int_0^\infty \frac{\mu_{1-n}(\omega) z_C(\omega,0)}{s+\omega} \, d\omega
\]

4.3.3 Riemann-Liouville derivative \( 0<n<1 \)

Expression (24) can be written as :

\[
D_n(f) = \frac{d}{dt} [h_{-n}(t) * f(t)] = \frac{d}{dt} [I_{-n}(f(t))]
\]

Using the same technique as previously, it is straightforward to get the Laplace transform of the Riemann-Liouville derivative, with initial conditions \( (I_{-n}(f))_0 \) and \( z_{RL}(\omega,0) \) :

\[
L\{D_n(f)\} = s^n F(s) - (I_{-n}(f))_0 + s \int_0^\infty \frac{\mu_{1-n}(\omega) z_{RL}(\omega,0)}{s+\omega} \, d\omega
\]

4.4 Initial conditions, general case

4.4.1 Caputo derivative \( N-1<n\leq N \)

The general formulation of the Caputo derivative is given by:

\[
D_n(f) = h_{N-n}(t) * \frac{d^N f(t)}{dt^N} = I_{N-n}\left( \frac{d^N f(t)}{dt^N} \right)
\]

and its Laplace transform is:

\[
L\{D_n(f)\} = s^n F(s) - \sum_{i=1}^{N} s^{n-i} \left( \frac{d^{N-i} f(t)}{dt^{N-i}} \right)_0
\]

\[
+ s \int_0^\infty \frac{\mu_{1-n}(\omega) z_C(\omega,0)}{s+\omega} \, d\omega
\]

(Caputo state)_0 = \left\{ f(0),...,\left( \frac{d^{N-1} f(t)}{dt^{N-1}} \right)_0, z_C(\omega,0) \right\}

4.4.2 Riemann-Liouville derivative \( N-1<n\leq N \)

The general formulation of the Riemann-Liouville derivative is given by:

\[
D_n(f) = \frac{d^N}{dt^N} [h_{N-n}(t) * f(t)] = \frac{d^N}{dt^N} [I_{N-n}(f(t))]
\]

and its Laplace transform is :

\[
L\{D_n(f)\} = s^n F(s) - \sum_{i=1}^{N} s^{N-i} \left( \frac{d^{N-i} I_{N-n}(f)}{dt^{N-i}} \right)_0
\]

\[
+ s \int_0^\infty \frac{\mu_{1-n}(\omega) z_{RL}(\omega,0)}{s+\omega} \, d\omega
\]

(R.L. state)_0 = \left\{ \left( \frac{d^N I_{N-n}(f)}{dt^N} \right)_0, z_{RL}(\omega,0) \right\}

5. INITIALIZATION OF THE CAPUTO DERIVATIVE

5.1 Introduction

Our objective is to verify that the initial conditions defined in part 4 are able to initialize fractional derivatives. Indeed, the more general situation would be to consider an unknown
function \( f(t) \) and to define its initial conditions at any instant \( t_0 \) in order to calculate its derivative \( D_n(f) \).

Practically, we restrict our attention to well known functions and to their corresponding derivatives. This is why we consider the case of the sine function:

\[
f(t) = \sin(t) \quad \text{(Oldham 1974)}
\]

whose fractional derivative is:

\[
D_n(f) = \sin(t + n \frac{\pi}{2}) \quad \text{(44)}
\]

Remark: in Oldham 1974, \( D_n(f) \) includes additive terms corresponding to transients. So, equation (44) represents the "steady-state" fractional derivative, which has to be correctly initialized.

Because our verification is fundamentally related to the initial state \( z(\omega, t_0) \) of the fractional integrator, we consider only the case \( 0 < n < 1 \).

5.2 Problem statement

The Caputo derivative is defined by:

\[
D_n(f) = I_{1-n} \left[ \frac{df(t)}{dt} \right]
\]

where \( I_{1-n}(s) \) is the fractional integrator, with order \( 1 - n \).

Practically, this integral is calculated using the infinite dimensional system:

\[
\frac{\partial z(\omega, t)}{\partial t} = -\omega z(\omega, t) + \frac{df(t)}{dt}
\]

\[
D_n(f) = \int_{0}^{\infty} \mu_{1-n}(\omega) z(\omega, t) d\omega
\]

Let \( z(\omega, t_0) \) be the initial state at \( t = t_0 \) of \( I_{1-n}(s) \).

Then, for \( t > t_0 \):

\[
z(\omega, t) = z(\omega, t_0) e^{-\omega(t-t_0)} + e^{-\omega(t-t_0)} \frac{df(t)}{dt}
\]

where \( e^{-\omega(t-t_0)} \) is the impulse response of system (46) \( z(\omega, t) \) is the sum of a free response, initialized by \( z(\omega, t_0) \) and \( e^{-\omega(t-t_0)} \) is the sum of a forced response, corresponding to the input \( \frac{df(t)}{dt} \).

The Caputo derivative will be correctly initialized if we have knowledge of the initial state: practically, this state is unknown and we have to estimate it.

5.3 Estimation of \( z(\omega, t_0) \)

Let \([0, t_0] \) represent the ‘past’ of \( I_{1-n}(s) \) and \( t > t_0 \) its ‘future’. Moreover, we assume that we have access to informations on \( f(t) \) and \( D_n(f) \) on this ‘past’ interval.

5.3.1 Natural estimation

Let \( y(\omega, t) \) be the state of an estimator. We desire that:

\[
y(\omega, t_0) = z(\omega, t_0)
\]

At \( t = 0 \), we consider that \( y(\omega, 0) = 0 \).

Then, \( y(\omega, t) \) is equal to the forced response

\[
y(\omega, t) = e^{-\omega t} \frac{df(t)}{dt}
\]

and

\[
x(t) = \int_{0}^{\infty} \mu_{1-n}(\omega) y(\omega, t) d\omega
\]

Then, if \( t_0 \to \infty \), \( x(t) \to D_n(f) \) and \( y(\omega, t) \to z(\omega, t) \)

Unfortunately, because of very slow modes (long range memory), this ‘natural’ approach is unable to provide a realistic solution in a finite time interval.

5.3.2 Forced estimation

We propose to speed up the convergence of \( y(\omega, t) \) to \( z(\omega, t) \) using a modified input for \( I_{1-n}(s) \) based on a closed loop solution, comparable to an observer.

Assume that we know the exact solution \( D_n(f) \) and consider the modified input of the integrator:

\[
v(t) = \frac{df(t)}{dt} + K (D_n(f) - x(t))
\]

where \( K \) is an adaptation gain and

\[
x(t) = \int_{0}^{\infty} \mu_{1-n}(\omega) [e^{-\omega t} \ast v(t)] d\omega
\]

Notice that this closed loop forces \( x(t) \) to converge to \( D_n(f) \), but we have no guaranty on the convergence \( y(\omega, t) \to z(\omega, t) \), particularly at \( t = t_0 \).

First, we have to check the stability of this estimator. Because \( y(\omega, 0) = 0 \), we get:

\[
X(s) = \frac{1}{s^{1-n}} V(s)
\]

and it is straightforward to get:

\[
X(s) = \frac{\int [\frac{df(t)}{dt} + K L D_n(f)]}{K + s^{1-n}}
\]

Because \( 0 < n < 1 \), this closed loop is stable for \( K > 0 \).

Then we have to determine a proper value of \( K \).

Notice that \( x(t) \) converges slowly to \( D_n(f) \) if \( K \to 0 \). On the other hand, if \( K \) is large, \( x(t) \) converges quickly to \( D_n(f) \), but we have no guaranty on the convergence of \( y(\omega, t) \).

Because we have no information on \( z(\omega, t_0) \), we can only appreciate the accuracy of this estimation through the response of \( x(t) \) for \( t > t_0 \), with \( K = 0 \).

Indeed, because \( v(t) = \frac{df(t)}{dt} \) for \( t > t_0 \), we get:

\[
y(\omega, t) = y(\omega, t_0) e^{-\omega(t-t_0)} + e^{-\omega(t-t_0)} \frac{df(t)}{dt}
\]

Consider the quadratic criterion:
\[ J = \int_{t_0}^{t_0+T} \left[ D_n(f) - x(t) \right]^2 dt \]  

where \( T \) is an observation interval used to test the influence of \( y(\omega,t_0) \) on the convergence of \( x(t) \) towards \( D_n(f) \) for \( t > t_0 \).

A straightforward solution is to calculate this criterion for different values of \( K \) and to choose the value \( K_{opt} \) corresponding to a minimal value of \( J \). Indeed, \( K_{opt} \) depends on \( t_0 \), \( T \) and on the fractional order \( n \).

Practically, we use the discrete frequency approximation (15) (16) (17) of the integrator for all the calculations.

5.4 Numerical simulations

5.4.1 Experiment conditions

Remind that \( f(t) = \sin(t) \) and \( D_n(f) = \sin(t + n \frac{\pi}{2}) \).

All experiments have been performed with \( n = 0.5 \).

The fractional integrator has been frequency discretized into \( J + 1 = 21 \) cells, ranging from \( \omega_1 = 10^{-4} \text{rad/s} \) to \( \omega_20 = 10^{2} \text{rad/s} \), with \( \omega_0 = 0 \text{rad/s} \). For time simulation, we have used the sampling period \( T_e = 2.5 \text{ms} \).

5.4.2 Optimal value of \( K \)

We present fig n°1 \( f(t) \), \( D_n(f) \) and \( x(t) \) initialized by \( y(\omega,0) = 0 \). Indeed, convergence is very slow as expected and it is necessary to accelerate it.

Then, we present fig n°2 the comparison between \( D_n(f) \) and the forced response \( x(t) \) with \( K_{opt} = 0.6 \) (and with \( K = 0 \) for \( t > t_0 \)). We can conclude that \( x(t) \) is a good approximation of \( D_n(f) \) for \( t > t_0 \), and indirectly that \( y(\omega,t_0) \) has converged to \( z(\omega,t_0) \).

5.4.3 Initialization of \( D_n(f) \)

We have performed an other estimation of \( z(\omega,t_0) \) at \( t_0 = 4s \) which has been used to initialize the fractional derivative. We present also the non initialized derivative in fig n°3. There is perfect accordance between the exact derivative and the initialized one, while the non initialized derivative converges very slowly, because of a long range free response.

Finally, we present fig n°4 four of the twenty one discrete state variables, initialized at \( t_0 = 4s \), corresponding to \( \omega_3 = 0.0027 \text{rad/s} \), \( \omega_0 = 0.0841 \text{rad/s} \), \( \omega_4 = 1.334 \text{rds} \) and \( \omega_8 = 21.13 \text{rds} \). The sum of all these state variables represent the free response (which is equal to the difference between the initialized and non initialized derivatives).
6. INITIALIZATION OF THE RIEMANN-LIOUVILLE DERIVATIVE

6.1 Problem statement

The Riemann-Liouville derivative of a function \( f(t) \) is defined as:

\[
D_n(f) = \frac{d}{dt} [I_{1-n}(f)]
\]

Practically, the fractional integral is calculated using the infinite dimensional system:

\[
\frac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + f(t)
\]

\[
I_{1-n}(f) = \int_{0}^{\infty} \mu_{1-n}(\omega) z(\omega,t) d\omega
\]

where \( z(\omega,t) \) has to be initialized at \( t = t_0 \). Thus, our objective is the same as previously, i.e. we have to estimate \( z(\omega,t_0) \).

6.2 Forced estimation

Let \( y(\omega,t) \) be the state of an estimator. Then,

\[
y(\omega,t) = e^{-\omega t} * f(t)
\]

and

\[
x(t) = \int_{0}^{\infty} \mu_{1-n}(\omega) y(\omega,t) d\omega
\]

The transfer function between \( f(t) \) and its derivative \( D_n(f) \) is \( s^n \), so it is not possible to use a closed loop to compare \( D_n(f) \) and \( \frac{dx(t)}{dt} \). But it is possible to compare \( I_{1-n}(f) \) and \( x(t) \), because

\[
I_{1-n}(f) = I_1[D_n(f)]
\]

Then, \( f(t) \) is replaced by

\[
f(t) + K(I_{1-n}(f) - x(t))
\]

and it is possible to speed up the convergence of \( x(t) \) to \( I_{1-n}(f) \), and thus of \( \frac{dx(t)}{dt} \) to \( D_n(f) \).

Notice that:

\[
I_1[D_n(f)] = \int_{0}^{t} D_n(f(\tau)) d\tau = -\cos(t + \frac{n\pi}{2})
\]

As for the Caputo derivative, it is necessary to tune the adaptation gain \( K \) using a modified quadratic criterion

\[
J = \int_{t_0}^{t} [D_n(f) - \frac{dx(t)}{dt}]^2 dt
\]

to test the convergence of \( \frac{dx(t)}{dt} \) to \( D_n(f) \) for \( t > t_0 \).

6.3 Numerical simulations

We present fig n°5 the initialization of the Riemann-Liouville derivative at \( t_0 = 5s \) like previously, there is a perfect accordance between the initialized derivative and the exact one.

On figure n°6, we present the non initialized derivative and the corresponding free response: we can conclude that the Riemann-Liouville derivative is more sensitive to initial conditions than the Caputo derivative (refer to fig n°3).

Finally, on fig n°7, we consider four of the twenty one state variables initialized at \( t_0 = 5s \), corresponding to \( \omega_k = 0.0027 \text{rad/s} \), \( \omega_{k0} = 0.0841 \text{rad/s} \), \( \omega_{k1} = 1.334 \text{rad/s} \) and \( \omega_{k2} = 21.13 \text{rad/s} \): this graph is similar to that of fig n°4 and it seems that there is a contradiction between the previous
impulsive free response and the behavior of these state variables.

In fact, the free response is not equal to the sum of these state variables, but to the sum of their integer order derivatives: it is the reason of this impulsive behavior (see fig n°6) and the main difference between the initialization of the two fractional derivatives.

CONCLUSION

In this paper, it has been demonstrated that the initial conditions of fractional derivatives correspond to the initial state vector of the associated fractional integrator $I_{1-n}(s)$. The validation of this concept has been possible thanks to the estimation of this initial state vector using an observer technique. Though the initialization of the two derivatives is governed by the same principles, it appears that the Riemann-Liouville derivative is more sensitive to initial conditions than the Caputo derivative.

In future works, some points will deserve more investigation. The relation between the initialization function of Lorenzo and Hartley and the integrator state vector will have to be analyzed. The initial vector has been estimated with a numerical algorithm: is it possible to formulate an analytical solution? Moreover, it will be necessary to investigate initial state vector estimation in the case where only the ‘past’ of the considered function is available.

REFERENCES