FRACTIONAL ORDER CONSTITUTIVE MODEL OF GEOMATERIALS UNDER THE CONDITION OF TRIAXIAL TEST

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ABSTRACT
Fractional calculus has been successfully applied to characterize the rheological property of viscoelastic materials, however, geomaterials were seldom involved in fractional order constitutive models (FOCM), and the issue of first loading and then unloading is rarely discussed through fractional calculus. It is considered that all materials are arranged in a queue and ideal solid and Newtonian fluid are located at both ends of the queue in FOCM. On the basis of FOCM, stress-strain relation under the condition of first loading and then unloading, besides creep, stress-relaxation and loading of constant strain rate are obtained. The stress-strain relation is utilized to fit triaxial test results of geomaterials under the corresponding conditions. The comparison between the test and fitting results reveals that FOCM can reasonably describe the stress-strain, stress-time or strain-time characteristics of geomaterials, which shows that fractional calculus is a good tool to constitutive model research of geomaterials.

INTRODUCTION
The primary objective of constitutive modeling is to provide a relationship between stress and strain, so that the experimentally obtained stress-strain relationship can be accurately modeled. In this context, if the material behaves linear-elasticity, then the constitutive modeling becomes very simple to calibrate. Unfortunately, most engineering materials, including soils, exhibit rather complicated mechanical property. Soil is a complicated material that behaves nonlinearity and often shows time dependent and stress-path dependent behavior when subjected to stresses. Moreover, soil behaves differently in primary loading, unloading and reloading. Because of the complex mechanical feature of soil, there exist a large variety of models which have been recommended in recent years to represent the stress-strain and failure behavior of soils. All these models inhibit certain advantages and limitations which largely depend on their application. Few basic and practical soil constitutive models such as Hooke's law, Mohr-Coulomb, Drucker-Prager, Duncan-Chang or Hyperbolic (model), (Modified) Cam Clay, Plaxis Soft Soil (Creep) and Plaxis Hardening Soil Model were discussed as summarized by Brinkgreve [1] according to the advantages and limitation of the models. From the mechanical point of view, these models were established on the basis of solid mechanics or rheology, which are two branches of continuum mechanics. The models from solid mechanics do not consider time factor, while those ones established on rheology take into account time dependent behavior. Comparing these two kinds of models, the former ones are relatively simple, while the latter is closer to reality. In order to better reflect soil properties and have a simple constitutive model, we need some new theory or method to connect these branches of continuum mechanics. Perhaps fractional calculus can help us to achieve this. The fractional calculus is a
branch of mathematics which deals with the generalization of integrals and derivatives to all real (and even complex) orders. It is found that fractional calculus can be an excellent mathematical instrument for mechanical modeling. Some scientists successfully adopted it to viscoelastic mechanics [2], electromagnetic [3], signal processing [4], chaos system [5] and so on. Among all fields, viscoelastic research is considered to be the most successful. Previously, viscoelastic materials have been characterized primarily with the Kelvin-Voigt model[6] for the constitutive relationship. Bagley and Koeller[7-10] have developed models using fractional derivatives. Padovan[11] and others[12-18] have examined various issues involved in the numerical implementation of these sorts of models. These research efforts are mainly on the rheological property of viscoelastic materials, including creep and stress-relaxation. However, few discussed the problem of first loading and then unloading through fractional calculus. Moreover, in applications of fractional order constitutive models, researchers mainly studied polymer, not much on soil using fractional calculus. Fractional derivative has been proved that it can well simulate the physical process, which shows memory and path dependent behavior. Moreover, fractional order model has advantage of having fewer parameters and a simple form. This paper aims at developing fractional order model of soils, we will derive analytical formulas under different conditions, including creep, stress-relaxation, loading and unloading. Laboratory experiments are presented in order to validate the proposed model.

Basics of fractional calculus

The question is about the notion of $d^nf/dx^n$ when $n$ is a fraction or non-integer. Starting from the Cauchy formula for the n-fold primitive of function $f(t)$[19],

$$I^nf(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \quad t > 0, \quad n \in \mathbb{N}$$

(1)

assuming that $f(t) = 0$, for $t > 0$ we get,

$$I^rf(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \quad \alpha \in \mathbb{R}^+$$

(2)

where $\Gamma$ is the gamma function. The Riemann-Liouville fractional derivative of order $\alpha$ is defined by,

$$D^\alpha f(t) = \begin{cases} 
\frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+m-1}} d\tau \right], & m-1 < \alpha < m \\
\frac{d^m}{dt^m} f(t), & \alpha = m 
\end{cases}$$

(3)

Using (2) and (3) it is found that,

$$I^{\alpha} t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}$$

$$D^{\alpha} t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \alpha > 0, \quad \gamma > -1, \quad t > 0.$$

(4)

It is pointed out that the $\alpha$th derivative of a constant function $C = constant$ is equal to

$$D^{\alpha} C = \frac{Ct^{-\alpha}}{(1-\alpha)}, \quad \alpha > 0, \quad t > 0$$

(5)

which is different from zero. Of course if $\alpha$ is an integer then $D^{\alpha} C = 0$. Furthermore,

$$D^{\alpha} f(t) = 0, \quad \text{if } f(t) = \frac{1}{t^{1-\alpha}}.$$  

(6)

Therefore, the function $t^{\alpha-1}$ yields zero $\alpha$-fractional derivative. Assuming that $f(t)$ is an absolutely continuous function in the interval $t \in [a, b]$ and with $0 < \alpha < 1$, we define

$$D^\alpha_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(t) - f(\tau)}{(t-\tau)^{\alpha}} d\tau$$

$$D^\alpha_{a-} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(t) - f(\tau)}{(t-\tau)^{\alpha}} d\tau.$$  

(7)

It may be shown that $f^\alpha(t)$ exists for certain class of functions and is equal to Riemann-Liouville fractional derivative and reads,

$$f^\alpha(a)(t) = \frac{f(t)}{\Gamma(1-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{f(\tau) - f(\tau)}{(t-\tau)^{\alpha}} d\tau.$$  

(8)

An important property known as integration by parts formula states:

$$\int_a^b z(t) D^\alpha_{a+} y(t) dt = \int_a^b y(t) D^\alpha_{a-} z(t) dt$$

(9)

when, $z(a) = z(b) = 0$.

Fractional order constitutive model

Derivation of model

It is well known that the ideal solid obeys Hooke’s law, $\sigma(t) \approx e(t)$, and Newtonian fluid satisfies Newton’s law of viscosity, $\sigma(t) \approx d^1 e(t)/dt^1$, where $\sigma$ is the stress and $e$ is the
strain. So, it is not difficult to imagine that material being intermediate between ideal solid and Newtonian fluid should follow $\sigma(t) \approx \frac{d^{\alpha} \varepsilon(t)}{dt^{\alpha}}, \ (0 \leq \alpha \leq 1)$ where $\alpha$ is the order and $d^{\alpha} \varepsilon(t)/dt^{\alpha}$ is $D^{\alpha} \varepsilon(t)$. When $\alpha = 0$, the function becomes Hooke’s law; when $\alpha = 1$, it is the Newton viscous law. Obviously, $\alpha$ is related to material property, for the example, $\alpha = 0$ and $\alpha = 1$ represent ideal solid and Newtonian fluid, respectively.

A simple fractional constitutive equation is the “intermediate model” suggested by Smit and De Vries [20]

$$\sigma(t) = E \theta^{\alpha} \frac{d^{\alpha} \varepsilon(t)}{dt^{\alpha}}, \quad (10)$$

where $E$ and $\theta$ are material constants and $0 \leq \alpha \leq 1$.

If $\alpha$ is considered as a constant during loading, which means that material property is unchanged, $E \theta^{\alpha}$ is also a constant. So, some researchers put Eq. (10) as

$$\sigma(t) = E \frac{d^{\alpha} \varepsilon(t)}{dt^{\alpha}}, \quad (11)$$

Obviously, in Eq. (11), physical meaning of $E$ is unclear, because its dimension is [stress][time]$^{\alpha}$. So we consider that Eq. (10) is better from physical point of view, and Eq. (10) will be used in the following.

The fractional order constitutive model in Eq. (10) can consider that all materials are arranged in a queue, and ideal solid and Newtonian fluid are located at both ends of the queue. It is clear that different order denotes different location in this material queue. The major innovation of the model is that most materials are seen as the matter whose behaviors are intermediate between that of the ideal solid and Newtonian fluid.

### Formula of creep and stress relaxation

Creep and stress relaxation are two important rheological behaviors. The so-called creep is when a body which is not quite an elastic solid (i.e. an inelastic solid) does not maintain a constant deformation under constant stress, but rather continues to deform with time - or “creeps”. When such a body is constrained at constant deformation, the stress required to hold it gradually diminishes-or “relaxes” with time.

Based on Riemann-Liouville definition of fractional calculus, when $\sigma(t) = c$, Eq. (10) can be rewritten as

$$\sigma(t) = \frac{c}{E \theta^{\alpha}} \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \ (0 \leq \alpha \leq 1) \quad (12)$$

where $c$ is constant. So Eq. (12) represents material creep behavior. From Fig. 1, we can observe that strain of material being intermediate between ideal solid and Newtonian fluid may increase slowly under the condition of creep, which is neither as linearly increasing as ideal solid nor unchanged like Newtonian fluid. So the fractional order constitutive model can depict the nonlinear gradual process of strain in creep. It can also be discovered from Fig. 1 that strain increases along with time when $\alpha$ does not equal 0. But lots of experimental values show that strain can not increase infinitely but gradually approaches a fixed value. The fractional order model here considers the order as a constant, which means the characteristics of the material is not variable and causes the discrepancy between experiments and the model. Generally, the mechanical characteristics of some materials may change along the whole process of creep. So the fractional order model used to simulate the creep should consider the fact that the order also changes along with time. Therefore, the variable fractional order model needs more research efforts.

By replacing a Newtonian dashpot in the classical Nishihara model with the fractional derivative Abel dashpot, a creep constitutive model was proposed on the basis of fractional derivative by Zhou [21]. Obviously, Zhou’s model is still component model which are based on a linear combination of elements such as the Newtonian dashpot, the Hooke spring and the fractional derivative Abel dashpot. It is well known that the Newtonian dashpot represents the Newtonian fluid, the Hooke spring can be considered as ideal solid and the fractional derivative Abel dashpot denotes the materials which are intermediate between Newtonian fluid and ideal solid. But it is difficult to find out the location of materials, which can be modeled by a linear combination of these elements introduced before, in the queue mentioned above. Therefore, Zhou’s model is actually a mathematical model, since the order in this model have no physical meaning.

### Formula of stress-relaxation

Stress relaxation is the other important rheological behavior, which refers to the behavior of stress reaching a peak and then decreasing or relaxing over time under a fixed level of strain. A fixed level of strain means $\varepsilon(t) = c$, so Eq. (10) may be rewritten...
\(\varepsilon(t) = \frac{1}{G(2 - \alpha)} \) \( (0 \leq \alpha \leq 1)\).

(14)

**FIGURE 2.** Curves of stress relaxation from fractional order constitutive model.

**FIGURE 3.** Stress-strain curves under constant strain rate from fractional order constitutive model.

as

\[\sigma(t) = E\theta^\alpha \frac{t^{1-\alpha}}{\Gamma(1 - \alpha)}, \quad (0 \leq \alpha \leq 1).\]

(13)

**FIGURE 4.** Stress-time curves or profile of loading and unloading.

**FIGURE 5.** The stress-strain curves of loading and unloading can show an important index property that gives an indication of overall strength and deformation of material. We presume that \(\varepsilon(t) = ct\), so Eq. (10) can be rewritten as

\[\sigma = E\theta^{\alpha} \frac{ct^{1-\alpha}}{\Gamma(2 - \alpha)}, \quad (0 \leq \alpha \leq 1).\]

(14)
(a) Salt rock with $\sigma_1 = 28.7$ MPa and $\sigma_3 = 7.18$ MPa (data after [22])

(b) Rock salt with $\sigma_1 = 21.4$ MPa and $\sigma_3 = 0.7$ MPa (data after [23])

(c) Soft soil with $\sigma_1 = 184$ kPa and $\sigma_3 = 0.1$ kPa (data after [24])

(d) Frozen soil with $\sigma_1 = 6.5$ MPa and $\sigma_3 = 2.75$ MPa (data after [25])

**FIGURE 6.** Comparison between experiment and simulation using Eq. (20) in creep.

(a) aturate Soil with $\sigma_1 = 100$ kPa (data after [26])

(b) Soft Clays with $\varepsilon_1 = 12\%$ (data after [27])

(c) Slip zone soils with $\sigma_3 = 100$ kPa and $\varepsilon_1 = 8\%$ (data after [28])

**FIGURE 7.** Comparison between experiment and simulation using Eq. (21) in Stress relaxation.

Of course, Eq. (14) can be rewritten as the strain form as follows:

$$\sigma = E(c\theta)^\alpha \frac{c\varepsilon^{1-\alpha}}{\Gamma(2-\alpha)}, \quad (0 \leq \alpha \leq 1). \quad (15)$$

Because the strain rate $c$ is included in Eq. (14) and (15), it is very clear that the fractional order constitutive model can be used to consider the problem of rate sensitivity.

Based on Fig. 3, we find that the stress-strain relationship is nonlinear when material, being intermediate between ideal solid...
FIGURE 8. Comparison between experiment and simulation using Eq. (22) in loading under constant strain rate

FIGURE 9. Comparison between experiment and simulation using Eq. (23) and (24) in first loading then unloading under constant strain rate

noted is that we can not obtain $E$ and $\theta$, so $E\theta^a$ is seen as one entity.

Formula of unloading under the constant strain and stress rate

In this subsection, we will discuss the case of first loading and unloading. Because the question mentioned before is rarely researched, two different conditions of first loading and then unloading will be considered, which are constant stress rate and constant strain rate as shown in Fig. 4.

Under the condition of loading, the strain or stress-time relationship is

$$\sigma = a_1 t \text{ or } \varepsilon = a_1 t, \ 0 \leq t \leq t_1. \quad (16)$$

The strain or stress-time relationship of unloading is

$$\sigma = a_2 - a_3 t \text{ or } \varepsilon = a_2 - a_3 t, \ t_1 \leq t \leq t_2, \quad (17)$$

where $a_1$, $a_2$ and $a_3$ are constants and $a_1 t_1$ is equal to $a_2 - a_3 t_1$. 

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When \( t \in (0, t_1) \), substitution of Eq. (16) into fractional order constitutive model gives

\[
E \theta^a \varepsilon = \frac{a_1 t^{1+a}}{\Gamma(2 + \alpha)} \quad \text{or} \quad \sigma = E \theta^a \frac{a_1 t^{1-a}}{\Gamma(2 - \alpha)} \\
0 \leq t \leq t_1, \quad 0 \leq \alpha \leq 1.
\]

Similarly, when \( t = (t_1, t_2) \), based on Eq. (2), Eq. (4) and Eq. (5), fractional order constitutive model can be written as

\[
E \theta^a \varepsilon = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} a_1 \tau d\tau \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} (a_2 - a_3 \tau) d\tau \\
= -\frac{a_1 t^{1+a}}{\Gamma(2 + \alpha)} + \frac{a_1 t^{1-a}}{\Gamma(2 - \alpha)}.
\]

The stress-strain curves of first loading and then unloading in Fig. 5.

\[
\sigma = E \theta^a \frac{(a_1 + a_2)(t - t_1)^{1-a}}{\Gamma(2 - \alpha)} + \frac{a_1 t^{1-a}}{\Gamma(2 - \alpha)},
\]

\( t_1 \leq t \leq t_2, \quad 0 \leq \alpha \leq 1 \).

According to Eqs. (16) - Eq. (19), we can describe the stress-strain curves of first loading and then unloading in Fig. 5. From the purely theoretical curves in Fig. 5, it is shown that the material, being intermediate between ideal solid and Newtonian fluid, produces non-reversible deformation (plastic deformation) after unloading, so we think that the fractional order constitutive model can depict not only viscoelasticity but also viscoplasticity. Moreover, the stress-strain curves of unloading are non-linear, which is very close to experimental results of some materials.

Based on these curves shown in Fig. 5 - Fig. 9 of fractional order stress-strain relationship, it is obvious that axial stress-strain relationship is nonlinear in uniaxial stress state, and this theory can reflect the plasticity and rheology of material. These conclusions reflect the advantages of fractional order stress-strain relationship, and it can not be derived from Hooke law in solid mechanics.

**Model validation**

Triaxial test is a common laboratory testing method widely used for obtaining shear strength parameters for a variety of soil types under drained or undrained condition. The test involves subjecting a cylindrical soil sample to radial stresses (confining pressure) and controlled increases in axial stresses or axial displacements. In case of consolidated test, the test is carried out by a first stage of applying confining pressure in the pressure chamber and allowing drainage of the sample. This stage corresponds to the consolidation of the sample. The deviatoric load is then applied through the vertical axis. The deviatoric stress is indeed the difference between the vertical stress \( \sigma_1 \) and the confining stress \( \sigma_3 \). If we assume that the vertical strain \( \varepsilon_1 \) is only from deviatoric stress \( \sigma_1 - \sigma_3 \), the formula for creep, relaxation, and loading and unloading in triaxial compression of geomaterials can be obtained through substituting \( \sigma_1 - \sigma_3 \) and \( \varepsilon_1 \) for \( \sigma \) and \( \varepsilon \) in Eqs. (10)-(19).

Here we think that soil is a kind of material being intermediate between ideal solid and Newtonian fluid and located at a certain position of queue mentioned above. Because the order \( \alpha \) is a constant, our model assumes that material feature of soil is constant. Of course, this presumption may only apply to some geomaterials or some transient loading process, we believe that variable order fractional model can resolve the problem, which will be accounted for separately in a sequel.

**Simulation of creep**

As mentioned before, the creep formula (Eq. 12) under the condition of triaxial test may be written as:

\[
\varepsilon_1(t) = \frac{c}{E \theta^a \Gamma(1 + \alpha)}, \quad 0 \leq \alpha \leq 1
\]

where \( c \) means constant and \( \sigma_1 - \sigma_3 = c \).

A series of triaxial creep experiments on salt rock, rock salt, soft soil and frozen soil have been completed by Yang[21], Cristescu[22], Zhu[23] and Zhang[24]. In order to validate the fractional order constitutive model under creep, Eq. (20) is used to fit these creep tests [21-24].

Fig. 6 shows the comparison between fitting and test result. And the orders are shown in figure(Fig. 7 - Fig. 9 also contain orders named \( \alpha \), too). It is seen from Fig. 6 that Eq. (20) can give relatively good simulation for the measured stress time relations in creep for these geomaterials. Therefore, we believe that the fractional order constitutive model can describe creep behavior of these geomaterials.

**Simulation of stress-relaxation**

Under the condition of triaxial test, the stress-relaxation formula is:

\[
\sigma_1 - \sigma_3 = E \theta^a \frac{t^{\alpha}}{\Gamma(1 - \alpha)}, \quad 0 \leq \alpha \leq 1
\]

where \( c \) means that the strain is at a fixed level. In order to validate our fractional order model under stress-relaxation, we utilize Eq. (21) to fit three stress-relaxation tests, which were completed on saturated soil, soft clays and slip zone soils by Yin[25], Li[26] and Wang[27]. Fig. 7 shows the comparison between fitting and test result of stress-relaxation. It is seen from this figure that the fitting results from fractional order model agree well with the reported test results.
Simulation of loading under constant strain rate
If the triaxial test is carried out under constant strain rate, Eq. (15) can be rewritten as

\[ \sigma_1 - \sigma_3 = E (c^\theta)^a \frac{e^{1-\alpha}}{\Gamma(2-\alpha)}, \quad (0 \leq \alpha \leq 1) \] (22)

where \( c \) is the constant strain rate. A series of triaxial experiments under constant strain rate on Wuhan clay, solid waste soils and soil mass completed by Zhou [28], Feng [29] and Yang [30]. Eq. (22) is used to fit these test results. Fig. 8 compares experiments and simulation using Eq. (22) under loading condition of constant strain rate. 50kPa, 100kPa and so on in figure refer to confining stress \( s \). It is seen from Fig. 8 that Eq. (22) can give relatively good simulation for the measured stress-strain relations under constant strain rate loading. So we think that fractional order constitutive model can describe non-linear stress-strain relationship.

Simulation of unloading under constant strain rate
There are a number of test results of first loading then unloading under the condition of constant strain rate, which we will focus in this subsection.
Under triaxial test of constant strain rate, Eqs. (18) and (19) can be written as

\[ \sigma_1 - \sigma_3 = E [\theta^a \frac{a_1}{\Gamma(2-\alpha)}] \quad (0 \leq t \leq t_1, \quad 0 \leq \alpha \leq 1), \] (23)

\[ \sigma_1 - \sigma_3 = E [\theta^a \frac{a_1}{\Gamma(2-\alpha)}] + \frac{a_1}{\Gamma(2-\alpha))} \quad (t_1 \leq t \leq t_2, \quad 0 \leq \alpha \leq 1). \] (24)

If we take account of \( \varepsilon = a_1 \theta \) \((0 \leq t \leq t_1)\) and \( \varepsilon = a_2 - a_3 t \) \((t_1 \leq t \leq t_2)\), the stress-strain relationship can be obtained. Sun [31] and Ho [32] have completed a series of first loading and then unloading tests on sandy soils and unsaturated soils, respectively. Based on Eqs. (23) and (24), we fit their test results, and the best fitting results are revealed in Fig. 8. It can be seen from Fig. 8 that the fitting results agree relatively well with the test results.
Therefore, from the above comparisons, the fractional order constitutive model can reasonably describe the stress-strain, stress-time or strain-time characteristics of geomaterials which are displayed in triaxial tests under the conditions of creep, stress relaxation, loading and first loading then unloading.

Conclusions
In this paper, the fractional order constitutive model is presented, which considers all materials arranged in a queue where ideal solid and Newtonian fluid are located at both ends of the queue. We derive the formulas under the conditions of creep, stress-relaxation, loading of constant strain rate and first loading then unloading. From the theoretical curves of first loading then unloading, it is shown that the material, being intermediate between ideal solid and Newtonian fluid, produces non-reversible deformation (plastic deformation) after unloading. In order to validate whether the fractional order model can be applied to describe mechanical feature of geomaterials, the model is utilized to fit triaxial test results of geomaterials under the condition of creep, stress-relaxation, loading of constant strain rate and first loading then unloading. The comparison between test and fitting results reveals that the fractional order constitutive model can reasonably describe the stress-strain, stress-time or strain-time characteristics of geomaterials, which means that fractional order calculus is a good tool for research constitutive model research of geomaterials as well.

But lots of experimental values show that strain can not increase infinitely but gradually approaches a fixed value. The fractional order model here considers the order as a constant, which means the characteristics of the material is not variable and causes the discrepancy between experiments and the model. Generally, the mechanical characteristics of some materials may change along the whole process of creep. So the fractional order model used to simulate the creep should consider the fact that the order also changes along with time. Therefore, the variable fractional order model needs more research efforts.

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