A NEW FRACTIONAL ORDER OBSERVER DESIGN FOR FRACTIONAL ORDER NONLINEAR SYSTEMS

Sara Dadras
Tarbiat Modares University
Tehran, Tehran, Iran

Hamid Reza Momeni
Tarbiat Modares University
Tehran, Tehran, Iran

ABSTRACT
In this paper, a class of fractional order systems is considered and simple fractional order observers have been proposed to estimate the system’s state variables. By introducing a fractional calculus into the observer design, the developed fractional order observers guarantee the estimated states reach the original system states. Using the fractional order Lyapunov approach, the stability (zero convergence) of the error system is investigated. Finally, the simulation results demonstrate validity and effectiveness of the proposed scheme.

INTRODUCTION
In recent years, fractional calculus has attracted increasing interests and there has been a rapid grow in the number of applications where fractional calculus has been used [1-3]. Lately, this technique has been applied to physics and engineering science problems and fractional order systems has been widely studied in different fields of science [4, 5]. It has become apparent that a large number of physical phenomena can be modeled by fractional order models [6-8] and many real world physical systems are better characterized by fractional order differential equations.

For the fractional order systems, there are many theories and criterions regarding the controllability, observability and stability, including the linear and nonlinear systems [9-15]. Besides, the design of different controllers for fractional order systems has received a great deal of attention recently and many important and significant results have been reported [16-18]. The problem of state observers that predicts the present system state is clearly central for the design of state feedback controllers. However, there are a few results regarding the state estimation for the fractional-order system [19-21]. Despite the importance of system state estimation in many practical engineering applications, to the best of the authors’ knowledge, there is no work in which the problem of designing fractional order observer for fractional order systems is investigated.

In this work, we first propose a new fractional order observer for fractional order linear systems and then, an extension of the previously mentioned observer for fractional order nonlinear systems is presented. The considered class of fractional order systems has separable nonlinearity and the nonlinear part is assumed to satisfy the Lipschitz condition. Many physical systems can be expressed or transformed into this form. To prove the convergence to zero of the estimation error, we use the fractional order Lyapunov approach. In other words, by using Lyapunov approach, a sufficient condition for the asymptotic stability of the error system is given. Finally, giving numerical examples, it is shown that we extend successfully the observer design method to cope with state estimation problem for fractional order systems.

The remainder of the paper is organized as follows: In Section 2, some basic concepts of fractional calculus is described. In Section 3, a class of fractional-order systems is introduced. In Section 4, two novel fractional order observers are presented and stability analysis of the fractional-order error systems when the proposed observers are applied is given. In Section 5, numerical simulations are given to confirm the effectiveness of the proposed observers. Finally, some concluding remarks are presented.

FRACTIONAL-ORDER CALCULUS
Fractional-order integration and differentiation is the generalization of the integer-order ones. Efforts to extend the specific definitions of the traditional integer-order to the more general arbitrary order context led to different definitions for fractional derivatives [22]. In this section, two commonly used definitions are presented.
Definition 1. [2] One of the basic functions of the fractional calculus is Euler's Gamma function which is defined by
\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt
\]
which converges in the right half of the complex plane.

Definition 2. [2] The Caputo fractional derivative of order \( \alpha \) of a continuous function \( f : \mathbb{R}^+ \to \mathbb{R} \) is defined as follows
\[
c_0^\alpha D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n
\]
where \( n \) is the first integer larger than \( \alpha \), i.e. \( n-1 < \alpha < n \) and \( \Gamma \) is the Gamma function. The notation \( f^{(n)}(t) \) represents the \( n \)-th order derivative of the function \( f(t) \).

Definition 3. [2] The \( \alpha \)-th order Riemann-Liouville fractional derivative of function \( f(t) \) with respect to \( t \) and the terminal value 0 is given by
\[
_0^\alpha D_t^\alpha f(t) = \frac{d^m f(t)}{dt^m} + \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{m-\alpha+1}} d\tau
\]
where \( m \) is the first integer larger than \( \alpha \), i.e. \( m-1 < \alpha < m \) and \( \Gamma \) is the Gamma function.

Property 1. [1] Between the two definitions (Riemann-Liouville and Caputo fractional derivative), there are following relations
\[
_0^\alpha D_t^\alpha f(t) = c_0^\alpha D_t^\alpha f(t) + \sum_{k=0}^{m-1} \frac{t^k}{\Gamma(k-\alpha+1)} f^{(k)}(0^+),
\]
where \( m-1 < \alpha < m \), \( m \in \mathbb{N} \).

Property 2. [1] Let us consider the Riemann-Liouville fractional derivative of order \( \alpha \). Then, we have
\[
_0^\alpha D_t^\alpha \left( t^a \right) = \frac{a t^{a-\alpha}}{\Gamma(1-\alpha)}
\]
where \( a \) is a positive constant.

Lemma 1. [10] Let \( A \in \mathbb{R}^{n \times n} \) be a deterministic real matrix without uncertainty. Then a necessary and sufficient condition for the asymptotical stability of \( _0^\alpha D_t^\alpha x(t) = Ax(t) \) is
\[
\left| \arg(\text{spec}(A)) \right| > \frac{q \pi}{2}
\]
where \( q \in (0,1) \).

The stability region is illustrated in figure 1.

Fig. 1. Stability region of linear fractional-order system with order \( q \) [13].

Theorem 1. [23] Let \( x=0 \) be an equilibrium point for the non-autonomous fractional-order system
\[
_0^\alpha D^\alpha_t x(t) = f(x, t)
\]
where \( q \in (0,1) \). Assume that there exist a Lyapunov function \( V(t, x(t)) \) and arbitrary positive constants \( \alpha_1, \alpha_2, \alpha_3 \) satisfying
\[
\alpha_1 \|x\|_\beta^\alpha, \alpha_2 \|x\|_\beta^\alpha, \alpha_3 \|x\|_\beta^\alpha
\]
where \( \beta \in (0,1) \). Then the equilibrium point of system (7) is Mittag-Leffler stable.

Remark 1. [23] For both Riemann-Liouville and Caputo fractional derivative definition, if the conditions of Theorem 1 are satisfied, then it can be concluded system (7) is Mittag-Leffler stable.


DESCRIPTION OF SYSTEM MODEL
Consider the nonlinear system described by
\[
_0^\alpha D^\alpha_t x = Ax + f(x, t) + Bu
\]
\[
y = Cx
\]
where \( x, u, y \) are the state variables, input and output, respectively. \( A \) is a constant matrix and \( B \) is a constant input weighting vector.

Assumption 1. There exists a known positive scalar for the known nonlinear term \( f(x) \) to satisfy the Lipschitz condition; i.e.
\[ \| f(x) - f(\hat{x}) \| \leq \eta \| x - \hat{x} \| \quad (10) \]

**Assumption 2.** The matrix pair \((A, C)\) is observable.

It follows from Assumption 2 that there exist a matrix \(L\) such that \(A-LC\) is stable, and thus for any \(Q>0\), the Lyapunov equation
\[ (A-LC)^T P + P(A-LC) = -Q \quad (11) \]
has a unique solution \(P>0\).

**Remark 2.** Consider a fractional order system given by the following linear state space form with finite dimension \(n\):
\[
\begin{cases}
0 \quad 0 \quad tD^\alpha x = Ax + Bu, \\
y = Cx
\end{cases} \quad x(0) = x_0 \quad (12)
\]
System (12) is observable on \([0, 1]\) if and only if
\[
\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (13)
\]
has rank \(n\).

**FRACTIONAL-ORDER OBSERVER DESIGN**

**Theorem 2.** The state estimation error of the fractional-order linear system (9) \((f(x)=0)\) asymptotically tends toward zero due to the following observer design
\[
\begin{align*}
\dot{e} &= A\hat{x} + Bu + L(y - C\hat{x}) \\
\hat{x} &= (A-LC)e
\end{align*} \quad (14)
\]
if the observer design constant, \(L\), is selected such that
\[
\arg(\text{spec}(J)) > \frac{\alpha \pi}{2} \quad \text{in which } J = A-LC.
\]

**Proof.** Let \(e = \hat{x} - x\). From (9) and (14), we have
\[
\begin{align*}
\dot{e} &= \dot{\hat{x}} - \dot{x} \\
&= A\hat{x} + Bu + L(y - C\hat{x}) - (A-LC)e \\
&= (A-LC)e
\end{align*} \quad (15)
\]
which asymptotically converges to zero according to lemma 1, if \(\arg(\text{spec}(J)) > \frac{\alpha \pi}{2}\). This implies that using the observer (14), one can estimate the original system internal variables with a good accuracy. The proof is complete. \(\square\)

**Theorem 3.** The state estimation error of the fractional-order nonlinear system (9) asymptotically tends toward zero due to the following observer design
\[
\begin{align*}
\dot{e} &= A\hat{x} + f(\hat{x}) + Bu + L(y - C\hat{x}) \\
\hat{x} &= (A-LC)e
\end{align*} \quad (16)
\]
if the observer design constant, \(L\), is selected such that
\[
\lambda_{\min}[(A-LC)^T + (A-LC)] > 2(\eta + \rho).
\]

**Proof.** Defining \(e = \hat{x} - x\) and using Eq. (9) and (16), we have
\[
\begin{align*}
\dot{e} &= \dot{\hat{x}} - \dot{x} \\
&= A\hat{x} + Bu + L(y - C\hat{x}) + f(\hat{x}) - f(x) \\
&= (A-LC)e + f(\hat{x}) - f(x)
\end{align*} \quad (17)
\]
Now, consider a Lyapunov candidate function \(V = 2e^T e\). The fractional derivative of \(V\) is given by
\[
\begin{align*}
\dot{V} &= 2e^T e - \left(2e^T e\right)^{(\alpha)}(0^+) \sum_{k=0}^{\infty} \frac{2}{(1+k)\Gamma(1-k+\alpha)} (D^\alpha e)(D^{\alpha-k} e) \\
&= 2e^T e - \left(2e^T e\right)^{(\alpha)}(0^+) \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} (D^k e)(D^{\alpha-k} e) \\
&= 2e^T e - \left(2e^T e\right)^{(\alpha)}(0^+) \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} (D^k e)(D^{\alpha-k} e) \\
&= e^T ((A-LC)^T + (A-LC)) e \\
&\leq 2e^T e \quad (18)
\end{align*}
\]
Using Eq. (5), Eq. (18) can be modified as follows
\[
\begin{align*}
\dot{V} &= (D^\alpha e)^T e + e^T (D^\alpha e) - 2D^\alpha (e^T e)(0^+) \\
&= 2e^T e - \left(2e^T e\right)^{(\alpha)}(0^+) \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} (D^k e)(D^{\alpha-k} e) \\
&= e^T ((A-LC)^T + (A-LC)) e \\
&\leq 2e^T e \quad (19)
\end{align*}
\]
where
\[
\sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} (D^k e)(D^{\alpha-k} e) = Y, \quad \|Y\| \leq \rho e^2. \quad (20)
\]
Since
\[
\begin{align*}
2\|e^T (0^+)\|^2 &\geq \frac{t^{-\alpha}}{(1-\alpha)} \geq 0 \quad (21)
\end{align*}
\]
without loss of generality, one can easily conclude that
\[
\begin{align*}
\dot{V} &\leq e^T (\dot{e}^T) e + 2e^T e + 2\|e\|^2 \\
&\leq e^T (\dot{e}^T) e + 2e^T e \quad (22)
\end{align*}
\]
From Assumption 2, it follows that
\[
\dot{V} \leq -e^T Qe + 2e^T (f(\hat{x}) - f(x)) + 2\|e\|^2. \quad (23)
\]
From the know lemma \(\lambda_{\min} \|e\|^2 \leq e^T Qe \leq \lambda_{\max} \|e\|^2\), Eq. (23) can be rewritten as
\[
\dot{V} \leq -\lambda_{\min} \|e\|^2 + 2\|e\|^2 \|f(\hat{x}) - f(x)\| + 2\|e\|^2 \quad (24)
\]
Using Eq. (10) and (20), it yields
\[ c_0D^\alpha_tv \leq -\lambda_{\min} \|e\|^2 + 2\eta \|e\|^2 - \lambda I + 2\rho \|e\|^2 \]
\[ = -\lambda_{\min} \|e\|^2 + 2\eta \|e\|^2 + 2\rho \|e\|^2 \]
\[ = -(\lambda_{\min} + 2\eta + 2\rho) \|e\|^2 \]
\[ (25) \]

Consequently, we have
\[ c_0D^\alpha_tv \leq -\lambda \|e\|^2 \]
\[ (26) \]

which asymptotically converges to zero according to Theorem 1, if the observer parameter \( L \) is chosen appropriately. Therefore, it can be concluded that the estimated trajectories attain to the original system trajectories. The proof is complete.

Remark 3. Using LMI technique, a parameter adjustment scheme can be proposed for the presented observer in Theorem 3, i.e.
\[ [(A-\lambda)L] + (A-\lambda)I - 2(\eta + \rho)I > 0 \]
where \( I \) is the identity matrix. Consequently, one can easily calculate the observer gain, \( L \), using LMI Toolbox in MATLAB.

Remark 4. In the proof of Theorem 3, if the Riemann-Liouville definition is used instead of the Caputo definition, Eq. (17) will be replaced by
\[ \int_{0}^{\tau} f(t) \frac{\Gamma(\alpha + 1)}{\Gamma(1 + \alpha)} \left( \frac{\Gamma(1 + \alpha) \Gamma(1 - k + \alpha)}{\Gamma(1 + k) \Gamma(1 - k + \alpha)} \right) \left( \frac{\Gamma(1 + \alpha) \Gamma(1 - k + \alpha)}{\Gamma(1 + k) \Gamma(1 - k + \alpha)} \right) x(t) dt \]
\[ (28) \]
and the rest of the proof is straightforward according to the proof of Theorem 3.

ILLUSTRATION

In this part, we consider two fractional order systems, linear and nonlinear, to show the effectiveness of the proposed observers.

For the linear system, the heat transfer system is taken into consideration and the new observer (14) is applied to the fractional order model of the mentioned system proposed by Melchior et. al. [25]. The model dynamics are as follows
\[ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.0601251 & -0.42833 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \]
\[ (29) \]

To include the effect of the thermal flux in the system, we generate a sinusoidal signal as the input \( u \) to such model.

According to theorem 2, we have to select \( L \) such that
\[ \arg(\text{spec}(A-L)) < \frac{\alpha\pi}{2} \]
\[ \Rightarrow 0.5 < \frac{2}{\pi} \tan^{-1} \frac{\text{Im}(\lambda_{\text{+LC}})}{\text{Re}(\lambda_{\text{+LC}})} \]
\[ (30) \]

Choosing \( L = [1 \ 0 \ 0]^T \), it is easy to check that the above mentioned condition is satisfied.
\[ A-\lambda LC = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -0.601251 & -0.42833 \end{bmatrix} \]
\[ (31) \]

Substituting Eq. (31) into Eq. (30) yields
\[ \frac{2}{\pi} \tan^{-1} \frac{\text{Im}(\lambda_{\text{+LC}})}{\text{Re}(\lambda_{\text{+LC}})} = 0.9875 > 0.5 \]
\[ (32) \]

Using the proposed linear observer (14) with the above settings, we obtained the simulation results given in figures 2 and 3.

Fig. 2. Estimate of state evolution with fractional order linear observer (14).
Figure 2 shows the time evolution of three system internal variables and their estimation with the linear fractional order observer (14). Figure 3 illustrates the time history of the estimation error. It can be seen from the simulations that the observer (14) make the state estimations approach the actual states precisely.

For the effectiveness of state estimation via observer (16), the fractional-order Lu system [26] with the following representation is considered:

\[
\begin{align*}
\rho \frac{d^{\rho}}{dt^{\rho}} x = \begin{bmatrix} -\rho & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
\end{align*}
\]

(33)

where \((\rho, \mu, \nu) = (35, 3, 28)\).

The plots in figure 4 compare the evolution of the three components of the system state with their respective estimates.
provided by the observer (16) with the above settings. Figure 5 reports the evolution of the estimation error. The plot clearly shows that the state estimation error converges to zero with a fast rate and an acceptable performance is achieved using the proposed nonlinear observer.

CONCLUSION
This paper describes two simple observers for linear and nonlinear systems, respectively. The presented fractional order observers can be used for fractional order systems. The proposed observers guarantee that the state estimation error converges to zero asymptotically. Stability of the origin for the error system is analyzed using fractional order Lyapunov approach. Simulation results have illustrated the effectiveness of the proposed fractional order observers.

ACKNOWLEDGMENTS
The first author would like to thank National Elite Foundation for the valuable financial support.

REFERENCES