FRACTIONAL ORDER SIGNAL PROCESSING: TECHNIQUES AND APPLICATIONS

by

Rongtao Sun

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Approved:

Dr. YangQuan Chen
Major Professor

Dr. Scott Budge
Committee Member

Dr. Jacob Gunther
Committee Member

Dr. Anhong Zhou
Committee Member

Dr. Byron R. Burnham
Dean of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah
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Abstract

Fractional Order Signal Processing: Techniques and Applications

by

Rongtao Sun, Master of Science
Utah State University, 2007

Major Professor: Dr. YangQuan Chen
Department: Electrical and Computer Engineering

This thesis first presents a brief overview of three existing fractional order signal processing (FOSP) techniques: autoregressive fractional integrated moving average (ARFIMA) model, Hurst parameter estimation, and fractional Fourier transform (FrFT). The developments in the mathematical communities are introduced, relationship between the fractional operator and long-range dependence is demonstrated, and fundamental properties of each technique and some of its applications are summarized.

A fractional Fourier transform-based tool has been proposed to analyze the long-range dependence in time series. The degree of long-range dependence is characterized by the Hurst parameter. The proposed FrFT-based estimation of Hurst parameter can be implemented efficiently for a very large data set. The new estimator can process very long experimental data series locally to achieve a reliable local estimation of the Hurst parameter. Fractional Gaussian noise, with a known Hurst parameter, which typically possesses long-range dependence, is used to test the accuracy and robustness of the FrFT-based Hurst parameter estimation. The proposed estimation method has been compared with some commonly used methods as well.
We have applied several fractional order signal processing techniques to analyze the bioelectrochemical corrosion noises and the water-surface-elevation data of the Great Salt Lake. The long-range dependence is reported and characterized in both applications.

(92 pages)
This work is dedicated to
my parents, Bingye Sun and Cuiyu Han.
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Rongtao Sun
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<tr>
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<td>autoregressive fractional integral moving average</td>
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<td>CE</td>
<td>counter electrode</td>
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<td>CSOIS</td>
<td>Center for Self-Organizing and Intelligent Systems</td>
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<td>FBM</td>
<td>fractional Brownian motions</td>
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<td>FD</td>
<td>fractal dimension</td>
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<td>FFGN</td>
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<td>WE</td>
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<td>zero resistance ammeter</td>
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Chapter 1

Introduction

1.1 Background

The concept of a fractional order operator has been investigated extensively in recent years involved with various signal processing theories and techniques. The first reference to fractional operator appeared during 1695 in a letter from Leibnitz to L’Hospital, where he formulated a question about the meaning of a non-integer order derivative [1]. In the letter Leibnitz raised the following question: “Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?” L’Hospital was somewhat curious about that question and replied by another question to Leibnitz: “What if the order will be 1/2?” Within this correspondence, the idea of fractional calculus was born. Leibnitz responded to the question, “This is an apparent paradox from which, one day, useful consequences will be drawn.” [2]. Since then, fractional order calculus (FOC) evolved through the contributions of many famous mathematicians such as Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Leibnitz, Grunwald, and Letnikov [3]. Nevertheless, it was not until the 1920s that FOC was applied to engineering [3]. Furthermore, only in the last three decades has the application of FOC attracted attention [4]. It was motivated by the works of Mandelbrot on fractals, which led to a significant impact in several scientific areas. Recently, more fractional order signal processing (FOSP) [5] techniques have appeared, such as fractional Brownian motion [6], fractional linear systems [7, 8], autoregressive fractional integrated moving average (ARFIMA) model [9], 1/f noise [10], Hurst parameter estimation [11], fractional Fourier transform (FrFT) [12], fractional linear transform [13], and fractional splines and wavelets [14]. FOSP techniques have been found to be very useful in analyzing long-range dependence (LRD) in time series.
1.2 Long-Range Dependence and Research Motivations

The first model for long-range dependence was introduced by Mandelbrot and Van Ness (1968) in terms of fractional Brownian motions (FBM) [15], and has since been extensively developed. Long-range dependent processes are characterized by their autocorrelation functions. Consider a second order stationary time series $Y = f(u)$ with zero mean as an example. The time series $Y$ is said to be long-range dependent provided its autocorrelation $r_Y(\tau) = \mathbb{E}[f(\tau)f(0)]$ decays slowly as a power law function of the lag $\tau$, so that the series $\sum_\tau r_Y(\tau)$ is not summable [16].

The Hurst parameter [17] $H$ characterizes the degree of LRD. The LRD is typically modeled by supposing a power law decay of the spectral density and $H$ is equal to $(1 + \alpha)/2$ with $\alpha$ being the power of the decay function [18]. A process is said to have long-range dependence when $0.5 < H < 1$ [11]. Many methods have been proposed for Hurst parameter estimation like rescaled range (R/S) analysis [19], aggregated variance method [20], absolute value method [21], variance of residuals method [22], local Whittle method [23], periodogram method [24], and wavelet-based method [16]. All existing estimation methods have their limitations in terms of estimation accuracy and efficiency. On the other hand, Stoev, Taqqu, Park, and Marron (2004) have shown that nonstationarity effects such as abrupt shifts in the mean and some other contaminations may affect the above estimation methods and result in overestimating the Hurst parameter [25]. In practice, it is quite important to estimate the Hurst parameter of LRD time series with low bias and high efficiency. Besides, the algorithm should be designed to be robust to nonstationarity. Therefore, we proposed to use local fractional Fourier transform (FrFT) method for Hurst parameter estimation. FrFT may be considered as a fractional power of the classic Fourier transform. The first idea of fractional power of the Fourier operator appeared in 1929 [26]. Just as the complex exponentials are the basic functions in Fourier analysis, the chirps (signals sweeping all frequencies in a certain interval) are the basis in fractional Fourier analysis. In long-range dependence applications, it is possible to improve performance by the use of the FrFT [27]. It is shown that FrFT has a strong relationship with wavelet
transform which is very suitable for analyzing LRD [28]. Since FrFT has been shown to have a computational complexity comparable to the wavelet transform, the performance improvements may come without additional cost. Furthermore, our FrFT-based Hurst parameter estimation method implements a set of windows over the spectrum to process the data locally for some time series long enough to have partial nonstationarity. Local deviation from long range dependence could be visualized by the local analysis. Note that, the wavelet-based method in this thesis also applies local analysis to deal with nonstationarity. In the experimental parts of this thesis, most commonly used Hurst parameter estimators are implemented for comparison.

Before applying the proposed method to real applications, it is important to test it with some standard stochastic processes with known Hurst exponents like white noise and fractional Gaussian noise [20]. White noise has a Hurst parameter of 0.5 [29], while fractional Gaussian noise is a random noise that has deterministic Hurst parameters. FGN may be derived from fractional Brownian motion (FBM) [15]. FBM is a long-range dependent Gaussian process with a Hurst parameter $H$ and with stationary increments. It plays a fundamental role in modeling long-range dependence. In practice, it is its increments that are used in modeling. The increment time series $G_H(k) := B_H(k) - B_H(k - 1); k \in \mathbb{Z}$ of the FBM process $B_H$ is called FGN [16].

Recent research has shown that, some data series such as financial data [30], network traffic data [18, 31], and video traffic data [32] may exhibit long range dependence which may affect the conventional signal processing methods. We explored the long-range dependence in biocorrosion signals [33, 34]. In this thesis, biocorrosion experiments in different artificial saliva will be done in the purpose of detecting the diseases form them. The corresponding biocorrosion signals will be collected. However, it is very difficult to characterize the differences between the biocorrosion signals when they possess LRD. Therefore, we will implement the FOSP techniques in this thesis to analyze their LRD.

The Great Salt Lake (GSL) is the fourth largest terminal lake in the world and has a large impact on the people and land of the intermountain west. Streams contribute about
66% of the annual inflow to the lake, precipitation is estimated to be about 31% and 3% is groundwater contribution [35,36]. Most of the surface flow inputs come from the Bear, Weber, and Jordan rivers and the lake drains three states: Utah, Idaho, and Wyoming (fig. 1.1 is adopted [37]). There was a record-breaking rise of GSL surface levels in the 1980s. The lake levels rose to a new historic high level of 4211.85 ft in 1986, 12.2 ft of this increase occurring after 1982. The rise had caused 285 million U.S. dollars worth of damage to lakeside industries, roads, railroads, wildfowl management areas, recreational facilities, and farming that had been established on the exposed lake bed [37].

Conventional time series analysis methods and models were found to be insufficient to describe adequately this dramatic rise and fall of GSL levels. One reason for such inadequacy may be the fact that there is not enough information prior to this event within just one-dimensional time series output of the system [37]. This opened up the possibility of investigating whether there is long-range dependence in GSL levels. In order to precisely model and predict it, the FOSP techniques will also be applied to the water-surface-elevation data from Great Salt Lake.
1.3 Thesis Contribution

This thesis reviewed three FOSP techniques: ARFIMA, Hurst parameter estimation, and FrFT.

Besides, an improved Hurst parameter estimator based on FrFT is introduced to analyze the long range dependence in time series. The efficiency and accuracy has been examined by implementing it into fractional Gaussian noise.

The LRD in biocorrosion signals is explored in this thesis. Our FrFT-based Hurst parameter estimator is applied to analyze the biocorrosion data with comparison to some other popular Hurst parameter estimators.

Also, some of the FOSP techniques are applied to the GSL water-surface-elevation data including both LRD analysis and ARFIMA modeling.

1.4 Organization of the Thesis and Chapter Previews

The rest of this thesis is organized as follows. In Chapter 2, a brief overview of the three fractional order signal processing techniques is presented with some simulations. In Chapter 3, an improved Hurst parameter estimation method which is based on fractional Fourier transform is proposed. Chapter 4 presents extensive tests and validations with fractional Gaussian noise (FGN) for our FrFT-based Hurst parameter estimator. In Chapter 5, the long-range dependence analysis is applied to biocorrosion signals. In Chapter 6, LRD analysis and ARFIMA modeling are applied to Great Salt Lake water-surface-elevation data. The conclusions and future works will be discussed in Chapter 7.
Chapter 2

Three FOSP Techniques: ARFIMA, Hurst Parameter Estimation, and FrFT - A Brief Overview

In recent years, fractional order signal processing (FOSP) is becoming an active research area, due to the demand on analysis of long-range dependence/self-similarity in time series, such as financial data, communications networks data, and biocorrosion noise. FOSP is based on the idea of fractional order calculus (FOC). FOC is a generalization of the differential and integral operators [3]. It is the root of the fractional systems described by fractional order differential equations. The simplest fractional order dynamic systems include the fractional order integrators and fractional order differentiators. The autoregressive fractional integrated moving average (ARFIMA) model is a typical fractional order system. It is a generalization of autoregressive moving average (ARMA) model [38]. The traditional models can only capture short-range dependence; for example, Poisson processes, Markov processes, autoregressive (AR), moving average (MA), autoregressive moving average (ARMA), and autoregressive integrated moving average (ARIMA) processes [39]. For time series that possesses long-range dependence, ARFIMA models give a good fit. In LRD processes, there is a strong coupling between values at different times [40]. This indicates that the decay of the autocorrelation function is hyperbolic and decays slower than exponential decay, and that the area under the function curve is infinite. We can also say that their autocorrelation functions are power-law distributed. $1/f$ noise is a signal that possesses long-range dependence. The power or square of some variable associated with the random process, measured in a narrow bandwidth, is roughly proportional to reciprocal frequency. In this chapter, we will illustrate that the $1/f$ noise could be the output of a fractional order system with input of white noise. The degree of LRD in time series is analyzed by estimating their Hurst parameter. Fractional Fourier transform based Hurst parameter es-
timator will be proved by this thesis to have a better performance than the other existing estimators for many time series including fractional Gaussian noise, biocorrosion processes data, and Great Salt Lake water-surface-elevation data.

2.1 Autoregressive Fractional Integrated Moving Average

A typical fractional system is called autoregressive fractional integrated moving average (ARFIMA) or fractional autoregressive integrated moving average (FARIMA) [9]. It is a generalization of autoregressive moving average (ARMA) [39] model.

2.1.1 Moving Average Model (MA)

The notation $MA(q)$ means a moving average model with $q$ terms. An $MA(q)$ model can be written as

$$y_t = x_t + \theta_1 x_{t-1} + \ldots + \theta_q x_{t-q}$$

for some coefficients $\theta_1, \ldots, \theta_q$. A moving average model is essentially a finite impulse response (FIR) filter.

2.1.2 Autoregressive Model (AR)

The notation $AR(p)$ means an autoregressive model with $p$ terms. An $AR(p)$ model can be written as

$$y_t = \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p}$$

for some coefficients $\phi_1, \ldots, \phi_p$. An autoregressive model is essentially an infinite impulse response (IIR) filter.
2.1.3 Autoregressive Moving Average Model (ARMA)

The notation $ARMA(p, q)$ means a model with $p$ autoregressive terms and $q$ moving average terms. This model subsumes the AR and MA models,

$$y_t = \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + x_t + \theta_1 x_{t-1} + \ldots + \theta_q x_{t-q}. \quad (2.3)$$

2.1.4 Autoregressive Integrated Moving Average (ARIMA)

An ARIMA model $ARIMA(p, d, q)$ is a generalization of ARMA model. The $p$, $d$, and $q$ are integers greater than or equal to zero and refer to the order of the autoregressive, integrated, and moving average parts of the model, respectively.

According to (2.3), an $ARMA(p, q)$ model can be written as

$$(1 - \sum_{i=1}^{p} \phi_i L^i)X_t = (1 + \sum_{i=1}^{q} \theta_i L^i)\varepsilon_t, \quad (2.4)$$

where $L$ is the lag operator such that $LX_t = X_{t-1}$, $X_t$ is a given time series and $\varepsilon_t$ are error terms. The error terms $\varepsilon_t$ are assumed to be normally distributed with zero mean such as Gaussian white noise.

Then, the ARIMA model is generalized by adding a differencing parameter $d$ to (2.4).

$$(1 - \sum_{i=1}^{p} \phi_i L^i)(1 - L)^dX_t = (1 + \sum_{i=1}^{q} \theta_i L^i)\varepsilon_t, \quad (2.5)$$

$$(1 - L)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-L)^k$$

where $d$ is a positive integer.

A well-known special case, $ARIMA(0, 1, 0)$ model, is given by:

$$X_t = X_{t-1} + \varepsilon, \quad (2.6)$$

which is simply a random walk when $\varepsilon$ is a white noise.
2.1.5 Autoregressive Fractional Integrated Moving Average (ARFIMA)

The autoregressive fractional integrated moving average model is generalized by permitting the degree of differencing to take fractional values. The fractional differencing operator is defined as an infinite binomial series expansion in powers of the lag operator $L$. Fractionally differenced processes may exhibit long-term memory (long-range dependence) or antipersistence (short term memory) [9].

The usefulness of this model has been proved by numerous studies. For instance, Diebold and ReRudebush applied ARFIMA to real GNP data [41]; Baillie et al. found long memory in time series of inflation [42]; Carto and Rothamn used an ARFIMA(0, $d$, 1) model for annual bond yields [43]; Bertacca et al. applied an ARFIMA based analysis in sea SAR imagery [44]; Liu et al. used ARFIMA for network traffic modeling [45].

An ARFIMA ($p$, $d$, $q$) process may be differenced a finite integral number until $d$ lies in the interval $[-\frac{1}{2}, \frac{1}{2}]$, and will then be stationary and invertible [9]. This range is the most useful set of $d$.

1) When $d = -\frac{1}{2}$, the ARFIMA ($p$, $-\frac{1}{2}$, $q$) process is stationary but not invertible.

2) When $-\frac{1}{2} < d < 0$, the ARFIMA ($p$, $d$, $q$) process has a short memory, and decay monotonically and hyperbolically to zero.

3) When $d = 0$, the ARFIMA ($p$, 0, $q$) process can be white noise.

4) When 0 < $d$ < $\frac{1}{2}$, the ARFIMA ($p$, $d$, $q$) process is a stationary process with long memory, and is very useful in modelling long-range dependence (LRD). The autocorrelation of a LRD time series decays slowly as a power law function.

5) When $d = \frac{1}{2}$, the spectral density of the process is

$$s(\xi) = \frac{\theta(e^{i\xi})\theta(e^{-i\xi})}{\phi(e^{i\xi})\phi(e^{-i\xi})} \{ (1 - e^{i\xi})(1 - e^{-i\xi}) \}^{-d}$$

$$\sim \left( \frac{1 - \theta_1 - \ldots - \theta_q}{1 - \phi_1 - \phi_p} \right)^2 \left( 2\sin \frac{1}{2}\xi \right)^{-2d} \sim \frac{C}{\xi^{2.7}}$$

as $\xi \to 0$. Thus the ARFIMA ($p$, $\frac{1}{2}$, $q$) process is a discrete-time “1/f noise.”
2.2 1/f Noise

We may define a fractional stochastic process as the output of a fractional dynamic system [46]. The input could be simply white noise. Fractional dynamic system can be described by fractional differential equations [47]. The integrators, differentiators, and constant multipliers are the simplest of these systems. There are two definitions for fractional integral and derivative [3, 48, 49]. The first one is Gr"{u}newald-Letnikov definition.

\[ D_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha}{j} \right) f(t - jh), \quad (2.8) \]

where \( \left( \frac{\alpha}{j} \right) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - j + 1)} \), \( \Gamma(z) = \int_0^\infty e^{-u}u^{z-1}du \) for all \( z \in \mathbb{R} \), and \( D_t^\alpha \) is a FOC operator. A generalization of fractional derivative and integral operator is \( aD_t^\alpha \).

\[
aD_t^\alpha = \begin{cases} 
\frac{d^\alpha}{dt^\alpha} & \mathbb{R}(\alpha) > 0, \\
1 & \mathbb{R}(\alpha) = 0, \\
\int_a^t (d\tau)^{-\alpha} & \mathbb{R}(\alpha) < 0.
\end{cases} \quad (2.9)
\]

The second one is Riemann-Liouville definition. For \( a, \alpha \in \mathbb{R} \),

\[
aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t - \tau)^{\alpha-n+1}}d\tau, \quad (n - 1 < \alpha < n), \quad (2.10)
\]

where \( a \) and \( t \) are the limits of \( aD_t^\alpha f(t) \).

The Laplace transform of \( aD_t^\alpha f(t) \) from the Gr"{u}newald-Letnikov definition is as follows:

\[
\int_0^\infty e^{-st}aD_t^\alpha f(t)dt = s^\alpha F(s); \quad (2.11)
\]

from the Riemann-Liouville definition:

\[
\int_0^\infty e^{-st}aD_t^\alpha f(t)dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^kD_t^\alpha f(t)|_{t=0}; \quad (2.12)
\]

for \( n - 1 < \alpha \leq n \), where \( F(s) = \mathcal{L}\{f(t)\} \) is the normal Laplace transformation.
Define fractional linear time-invariant (FLTI) systems described by a differential equation with the following general form:

\[
\sum_{n=0}^{N} a_n D^{\alpha_n} y(t) = \sum_{m=0}^{M} b_m D^{\alpha_m} x(t), \tag{2.13}
\]

where \( D \) is the derivation operator and the \( \alpha \) is the order of differentiation. According to (2.11), applying Laplace transform to (2.13), the following transfer function can be easily obtained, provided that \( Re(s) > 0 \) or \( Re(s) < 0 \).

\[
H(s) = \frac{\sum_{m=0}^{M} b_m s^{\alpha_m}}{\sum_{n=0}^{N} a_n s^{\alpha_n}} \tag{2.14}
\]

Among various fractional random signals, the \( 1/f \) noise is of special importance. The \( 1/f \) noise is a random process defined in terms of the shape of its power spectral density \( S(f) \). The power or square of some variable associated with the random process, measured in a narrow bandwidth, is roughly proportional to reciprocal frequency:

\[
S(f) = \frac{\text{constant}}{|f|^{\gamma}}. \tag{2.15}
\]

The typical spectrum of \( 1/f \) noise is shown in fig. 2.1.

Models of \( 1/f \) noise were developed by Bernamont [50] in 1937 for vacuum tubes and by Mcwhorter [51] in 1955 for semiconductors. Keshner [10] referred several examples of \( 1/f \) noise. From his considerations, we may conclude that those signals seem to belong to one of two types: those with spectrum of the form \( 1/f \) for every \( f \) and the others with that form only for \( f \) above a given value. This means that the former can be considered as the output of a fractional integrator, while the second may be the output of a low-pass fractional system with a pole near the origin. For the first case, the system can be defined by the transfer function \( H(s) = 1/s^\beta \) and according to Riemann-Liouville integral, its impulse response \( h(t) = t^{\beta-1}/\Gamma(\beta) \) with \( 0 < \beta < 1 \). Let the input be a continuous-time stationary white
noise $w(t)$ with variance $\sigma^2$. The output can be written in convolution form as follows:

$$y_\beta(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{t} w(\tau)(t-\tau)^{\beta-1} d\tau. \quad (2.16)$$

The autocorrelation function of the output $y_\beta(t)$ is then,

$$R(\tau) = \sigma^2 \frac{|\tau|^{2\beta-1}}{2\Gamma(2\beta) \cos \beta \pi}. \quad (2.17)$$

There is an important property shown in (2.17). The autocorrelation of the $1/f$ noise decays slowly as a power law function, which indicates the long-range dependence [15].

Fractional Gaussian noise is a kind of $1/f$ noise. It is long-range dependent with power law behavior in all frequencies [52]. Alternatively, the FGN process can be defined as a process $X_i, i = 1, 2...$ satisfying the condition for any timescales $k$ and $l$ [52]

$$(Z_i^{(k)} - k\mu) \overset{d}{=} \left( \frac{k}{l} \right)^H (Z_j^{(l)} - l\mu), \quad (2.18)$$

Fig. 2.1: $1/f$ noise spectrum.
where $Z_i$ is denoting the aggregated process of $X_i$ on the timescale $k$, $Z_j$ is denoting the aggregated process of $X_i$ on the timescale $l$, $\mu$ is the mean value of $X_i$, the symbol $\overset{d}{=} \equiv$ stands for equality in distribution, and $H$ is a positive constant ($0 < H < 1$) known as the Hurst parameter. Hurst parameter will be discussed in detail in the next section.

It is shown [52] that, the autocovariance function $r(\tau)$ of FGN $X_i$ is given by

\[
r(\tau) = \frac{1}{2}[(\tau + 1)^{2H} + (\tau - 1)^{2H}] - \tau^{2H}, \tau > 0.
\]

(2.19)

Apart from small $\tau$, this function is very well approximated by

\[
r_\tau = H(2H - 1)\tau^{2H-2},
\]

(2.20)

which shows that autocorrelation is a power function of lag $\tau$.

Considering a continuous time process $Y(t)$ with autocorrelation $\text{Cov}(Y(t), Y(t+\tau)) = c\tau^{2H-2}$, $c = H(2H - 1)$, we can also obtain (2.19) by making $Y(t)$ discrete using time intervals of any length $\delta$ and taking $X_i$ as the average of $Y(t)$ in the interval $[i\delta, (i+1)\delta]$. This enables an approximate calculation of the power spectrum of the process as

\[
s^{(k)}(\xi) \approx 4 \int_0^\infty c\tau^{2H-2} \cos(2\pi\tau\xi)d\tau \approx c'\xi^{1-2H},
\]

(2.21)

which is a power law of the frequency $\xi$.

Several FGN generating methods has been proposed. Some widely used ones are listed below.

- **A Fast Fractional Gaussian Noise Generator [53]**

The fast fractional gaussian noise generator is based on an algorithm to compute an approximation to discrete fractional Gaussian noise. The algorithm computes fast fractional Gaussian noise (FFGN) [53] as a sum of a low frequency term and a high frequency term.
The high frequency term is a Markov-Gauss process and the low frequency term is a weighted sum of Markov-Gauss processes. Then, the FFGN is given by

\[ X_i = X_i^{(L)} + X_i^{(H)}, \]  

(2.22)

where \( X_i^{(L)} \) and \( X_i^{(H)} \) denote low and high frequency terms, respectively. Writing \( M_i^{(n)} \) for the \( n \)th Markov-Gauss process with variance 1 and weight factor \( W_{2n}^2 \), we define

\[ X_i^{(L)} = \sum_{n=0}^{N} W_{2n}^2 M_i^{(n)}, \]  

(2.23)

\[ W_n = \frac{H(2H-1)}{\Gamma(3-2H)}(B^{1-H} - B^{H-1})B^{2n(H-1)}. \]  

(2.24)

The high frequency term \( X_i^{(H)} \) is used to compensate the deficiency due to the approximation leading to (2.24). Finally, there remains only the choice of \( B \) and \( N \). Mandelbrot found 2,3 or 4 convenient for \( B \) [53], and Chi et al. (1973) recommend about 15-20 for \( N \) [54].

**A Multiple Timescale Fluctuation Approach [52]**

It is shown that the autocorrelation function of the FGN process at the basic timescale could be approximated by the weighted sum of exponential functions of the time lag [52]. This observation can lead to an algorithm to generate FGN. The best (in terms of mean square error) approximation of (2.19) is given by the following equation

\[ r = 1.52(H - 0.5)^{1.32}. \]  

(2.25)

It should be mentioned that this algorithm is based on the same principle with the FFGN algorithm [53].
• A Disaggregation Approach [40, 52]

The disaggregation approach is a mathematical induction technique which is made possible by the expressions of the statistics of the aggregated FGN process [40]. The first step is sufficient to describe the method application. Assume that the generation is completed at the timescale \( k \leq n \) and the time series will be generated at the next timescale \( k/2 \). In the generation step, the higher-level amount \( Z_i^{(k)}(1 < i < n/k) \) is disaggregated into two lower-level amounts \( Z_{2i-1}^{(k/2)} \) and \( Z_{2i}^{(k/2)} \) such that

\[
Z_{2i-1}^{(k/2)} + Z_{2i}^{(k/2)} = Z_i^{(k)}. \tag{2.26}
\]

Therefore, (2.26) can be used to generate \( Z_{2i-1}^{(k/2)} \) and then obtain \( Z_{2i}^{(k/2)} \). At this generation step, both the values of previous lower-level time steps, i.e., \( Z_{2i}^{(k/2)} \) and the values of next higher-level time steps, i.e., \( Z_{2i+1}^{(k)} \) have become available. To simplify the method, the correlations of \( Z_{2i-1}^{(k/2)} \) with only one higher-level time step behind and one ahead are considered. Thus, \( Z_{2i-1}^{(k/2)} \) can be generated from the linear relationship

\[
Z_{2i-1}^{(k/2)} = a_2 Z_{2i-3}^{(k/2)} + a_1 Z_{2i-2}^{(k/2)} + b_0 Z_i^{(k)} + b_1 Z_{i+1}^{(k)} + V, \tag{2.27}
\]

where \( a_2, a_1, b_0, \) and \( b_1 \) are parameters to be estimated and \( V \) is innovation whose variance has to be estimated. The unknown parameters can be estimated according to the correlations in the form of \( \text{Cor}[Z_{2i+1}^{(k/2)}, Z_{2i+1+j}^{(k/2)}] = r_j \), where \( r_j \) is given by (2.19).

• A Symmetric Moving Average Approach [52, 55]

The symmetric moving average (SMA) approach could be used to generate stochastic process with any kind of autocorrelation structure [55]. For an input white noise \( W_i \), the SMA filter takes the weighted average of a number of \( W_i \) to produce the output, for example

\[
X_i = \sum_{j=-q}^{q} a_{|j|} W_{i+j} = a_q W_{i-q} + ... + a_1 W_{i-1} + a_0 W_i + a_1 V_{i+1} + ... + a_q V_{i+q}, \tag{2.28}
\]
where $a_j$ are the weights symmetric about a center ($a_0$) that corresponds to the variable $W_i$ and $q$ theoretically is infinity but in practice can be restricted to a finite number. The autocovariance implied by (2.28) is

$$r_j = \sum_{i=-q}^{q-j} a_{ij}a_{i+j}, \quad j = 0, 1, 2, \ldots . \tag{2.29}$$

It is also shown [55] that the discrete Fourier transform $s_a(\xi)$ of the $a_j$ sequence is related to the power spectrum of the process $s_r(\xi)$ by

$$s_a(\xi) = \sqrt{2s_r(\xi)}. \tag{2.30}$$

We can use (2.19) to approximate the inverse Fourier transform of $s_a(\xi)$ to get $a_j$:

$$a_j \approx \sqrt{\frac{(2 - 2H)r_0}{3 - 2H}}[(j + 1)^{H+0.5} + (j - 1)^{H+0.5} - 2j^{H+0.5}], \quad j > 0. \tag{2.31}$$

In the end, the generation scheme (2.28) with coefficients $a_j$ can lead to the SMA approach.

### 2.3 Hurst Parameter Estimation

The Hurst parameter $H$ characterizes the degree of LRD. A process is said to have long range dependence when $0.5 < H < 1$. A Hurst parameter satisfying $0 < H < 0.5$ means that the process is negatively correlated, or anti-persistent. The middle value $H = 0.5$ is a special case because it defines two distinct regions in the interval $(0, 1)$ and may suggest white noise.

Hurst (1951) formulated mathematically his discovery in terms of the so-called rescaled range, which is a storage-related feature of a time series [17]. Several types of models such as fractional Gaussian noise models, fast fractional Gaussian noise models, broken line models [56], ARFIMA/FARIMA models, and symmetric moving average models have been proposed to reproduce the Hurst phenomenon when generating synthetic time series.
When the Hurst parameter is well defined and can be reproduced, how to correctly measure $H$ from a given time series becomes a problem. As mentioned in Chapter 1, many methods have been proposed for Hurst parameter estimation. We will discuss them in more details in this section.

### 2.3.1 Time-domain Estimators

- **R/S Analysis [19]**

  Let $R(n)$ be the range of the data aggregated over blocks of length $n$ and $S^2(n)$ be the sample variance of the data aggregated at the same scale. For long-range dependent time series the ratio $R/S(n)$ follows

  $\mathbb{E}[R/S(n)] \sim C_H n^H,$ \hspace{1cm} (2.32)

  where $C_H$ is a positive, finite constant independent of $n$. Thus, a log-log plot of $R/S(n)$ versus $n$ should have a constant slope as $n$ becomes large. An important thing should be considered is choosing the values of $n$. For small $n$ short-range dependence dominate and the readings may not be valid. A large $n$ results in few samples and the value of $R/S(n)$ will also not be accurate.

- **Aggregated Variance Method [20]**

  The aggregated variance method divides the original time series $X = \{X_i, i \geq 1\}$ into blocks of size $m$ and average within each block, that is, consider the aggregated series

  \[
  X^{(m)}(k) = \frac{1}{m} \sum_{i=(k-1)m+1}^{km} X(i) \quad k = 1, 2, ... \hspace{1cm} (2.33)
  \]

  for successive values of $m$. Then, divide the data, $X_1, ..., X_N$, into $N/m$ blocks of size $m$, and calculate its sample variance

  \[
  \overline{\text{Var}}X^{(m)} = \frac{1}{N/m} \sum_{k=1}^{N/m} (X^{(m)}(k))^2 - \left( \frac{1}{N/m} \sum_{k=1}^{N/m} X^{(m)}(k) \right)^2. \hspace{1cm} (2.34)
  \]
Since for FGN and ARFIMA, \( \text{Var}(X^{(m)}) \sim \sigma^2 m^\beta \) as \( m \to \infty \) where \( \sigma \) is the scale parameter and \( \beta = 2H - 2 < 0 \), the sample variance \( \hat{\text{Var}}(X^{(m)}) \) should be asymptotically proportional to \( m^{2H-2} \) for large \( N/m \) and \( m \), and the resulting points should form a straight line with slope \( \beta = 2H - 2, -1 \leq \beta < 0 \).

- **Absolute Value Method [21]**

  For absolute value method, the data is divided in the same way as (2.33) to form aggregated series. Instead of computing the sample variance, the sum of the absolute values of the aggregated series is computed,

  \[
  \frac{1}{N/m} \sum_{k=1}^{N/m} |X^{(m)}(k)|. \tag{2.35}
  \]

  Then, the logarithm of this statistic is plotted versus the logarithm of \( m \). For long-range dependent time series with parameter \( H \), the result should be a line with slope \( H - 1 \).

- **Variance of Residuals Method [22]**

  Variance of residuals method involves several steps [57]. First, divide the time series into blocks of size \( m \). Then, within each of the blocks, calculate the partial sums of the series. Third, fit a least-squares line to the partial sums within each block and compute the sample variance of the residuals. In the end, repeat this procedure for each of the blocks, and average the resulting sample variances. We should get a straight line with a slope of \( 2H \) if the result is plotted on a log-log plot versus \( m \).

2.3.2 Frequency-domain Estimators

- **Local Whittle Estimator [23]**

  Local Whittle estimator is a maximum likelihood estimator which assumes a functional form for the spectral density of the time series and seeks to minimize parameters based upon this assumption. In order to use local Whittle estimator, one must specify the functional
form expected, typically either FGN or FARIMA. Errors may occur if the user fails to specify the model. Local Whittle is a semi-parametric estimation method.

- **Periodogram Method** [24]

  The periodogram is defined by

  \[
  I(\xi) = \frac{1}{2\pi N} \left| \sum_{j=1}^{N} X_j e^{ij\xi} \right|^2,
  \]

  where \( \xi \) is the frequency and \( i = \sqrt{-1} \). For a series with finite variance, \( I(\xi) \) is an estimate of the spectral density of the series. A log-log plot of \( I(\xi) \) should have a slope of \( 1 - 2H \) close to the origin.

- **Wavelet-Based Method**

  Wavelet analysis has been used with success to measure the Hurst parameter in recent years [11, 16, 18, 23, 25, 31, 32, 46, 58–60]. Therefore, the wavelet-based estimator is the best choice for comparison in this thesis. Wavelets can be thought of as akin to Fourier series but using waveforms other than sine waves. The Hurst exponent is calculated from the wavelet spectral density by fitting a linear regression line through a set of \( x_j, y_j \) points, where \( x_j \) is the octave and \( y_j \) is the logarithm of the normalized power. The slope of this regression line is proportional to the estimate for the Hurst exponent. A 95% confidence interval can be given.

- **FrFT-Based Estimator**

  We proposed to use a fractional Fourier transform (FrFT) based estimator [28, 61]. It uses the spectrum calculated by FrFT for estimation. The details will be discussed in Chapter 3. The FrFT-based local estimator has proven to have better performances than other existing methods in validation experiments of this thesis.
2.4 Fractional Fourier Transform (FrFT)

Fractional Fourier transform may be considered as a fractional power of the classic Fourier transform [62]. The first idea of fractional power of the Fourier operator appears in 1929 [63]. Like the complex exponentials are the basis functions in Fourier analysis, the chirps (signals sweeping all frequencies in a certain frequency interval) are the basis in fractional Fourier analysis. Since FrFT can provide a richer picture in time-frequency analysis [12] and it is nothing more than a variation of the standard Fourier transform [64], it can improve the performance of some signal processing applications.

2.4.1 Formulation and Derivation

Let us start with interpretation of a variable along a rotated axis system. Like the time and frequency variables in the time-frequency plane, let $x$ be the variable along the $x$-axis and $\xi$ is the variable along the $\xi$-axis (fig. 2.2). Let $x_a$ and $\xi_a$ represent the rotated variables $x$ and $\xi$, respectively. We have

$$
\begin{bmatrix}
    x_a \\
    \xi_a
\end{bmatrix} =
\begin{bmatrix}
    \cos \alpha_F & \sin \alpha_F \\
    -\sin \alpha_F & \cos \alpha_F
\end{bmatrix}
\begin{bmatrix}
    x \\
    \xi
\end{bmatrix},
$$

(2.37)

which $\alpha_F = a\pi/2$ is the rotating angle. It is always assumed that $\xi_a = x_{a+1}$. Therefore, $x_a$ and $\xi_a$ are always orthogonal.
If \( f(x) \) is a “time” signal of the variable \( x \), it lives on the horizontal axis. Its Fourier transform (FT) \( \mathcal{F}(f(x)) = F(\xi) \) is a function of the frequency \( \xi \), and hence it lives on the vertical axis. Fourier transform changes signal \( f(x) \) in time domain \( x \) to \( F(\xi) \) in frequency domain \( \xi \), which corresponds to a counterclockwise rotation over an angle \( \pi/2 \) in the \((x, \xi)\) plane. Since applying FT twice to \( f(x) \) results in

\[
(\mathcal{F}^2 f)(x) = (\mathcal{F}(\mathcal{F} f))(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x} d\xi = f(-x),
\]

(2.38)

\( \mathcal{F}^2 \) is called the parity operator. Thus, the time axis rotated over an angle \( \pi \) for \( \mathcal{F}^2 \).

Similarly, it follows that

\[
(\mathcal{F}^3 f)(\xi) = (\mathcal{F}(\mathcal{F}^2 f))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x) e^{-i\xi x} dx = F(-\xi),
\]

(2.39)

which corresponds to a rotation of the representation axis over \( 3\pi/2 \). If we apply one more \( \mathcal{F} \), we should get another rotation over \( \pi/2 \), which brings us back to the original time axis. Therefore,

\[
\mathcal{F}^4(f) = f \quad \text{or} \quad \mathcal{F}^4 = I.
\]

(2.40)

In conclusion, the FT operator corresponds to a rotation of the axis over an angle \( \pi/2 \) in the time-frequency plane.

There are six definitions of FrFT [62]. The most intuitive way to define FrFT is by generalizing the rotation concept of classical FT. Like FT corresponds to a rotation in the time-frequency plane over an angle \( \alpha_F = \pi/2 \), the FrFT will correspond to a rotation over an arbitrary angle \( \alpha_F = a\pi/2 \) with \( a \in \mathbb{R} \).

For a more formal definition, FrFT can be defined through eigenfunctions. We can also start with defining the classical FT. According to (2.40), the eigenvalues are all in the set \( \{1, -i, -1, i\} \) and thus FT has only four different eigenspaces. A possible choice for the eigenfunctions of the operator \( \mathcal{F} \) that is generally agreed upon is given by the set of
normalized Hermite-Gauss functions [65]:

$$\phi_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x), \quad (2.41)$$

where $H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}$, $D = d/dx$ is a Hermite polynomial of degree $n$. Therefore, there is an eigenvalue $\lambda_n$ such that

$$\mathcal{F}\phi_n = e^{-in\pi/2} \phi_n. \quad (2.42)$$

Thus, the eigenvalue for $\phi_n$ is given by $\lambda_n = e^{-in\pi/2} = \lambda^n$ with $\lambda = -i = e^{-i\pi/2}$ representing a rotation over an angle $\pi/2$.

Similarly, FrFT can be defined for a rotating angle $\alpha_F = a\pi/2$ by

$$\mathcal{F}^a\phi_n = e^{-ina\pi/2} \phi_n = \lambda_n^a \phi_n = \lambda_n^a \phi_n. \quad (2.43)$$

Since any function $f$ in the eigenspaces can be expanded in terms of these eigenfunctions $f = \sum_{n=0}^{\infty} a_n \phi_n$ with

$$a_n = \frac{1}{\sqrt{2^n n! \pi \sqrt{2}}} \int_{-\infty}^{\infty} H_n(x) e^{-x^2/2} f(x) dx, \quad (2.44)$$

we can get

$$f_a := \mathcal{F}^a f = \mathcal{F}^a \left[ \sum_{n=0}^{\infty} a_n \phi_n \right] = \sum_{n=0}^{\infty} a_n \mathcal{F}^a \phi_n = \sum_{n=0}^{\infty} a_n e^{-ina\pi/2} \phi_n. \quad (2.45)$$
For computational purposes, (2.45) should be changed to an integral representation by replacing the $a_n$ in the series by their integral expression in (2.44):

\[
 f_a(\xi) = \sum_{n=0}^{\infty} \left[ \int_{-\infty}^{\infty} \phi_n(x) f(x) dx \right] e^{-ina\pi/2} \phi_n(\xi) \\
 = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-ina\pi/2}H_n(\xi)H_n(x)}{2^n n! \sqrt{\pi}} e^{-(x^2+\xi^2)/2} f(x) dx \\
 = \frac{1}{\sqrt{\pi} \sqrt{1 - e^{-2i\alpha_F}}} \int_{-\infty}^{\infty} \exp \left\{ \frac{2ix\xi e^{-i\alpha_F} - e^{-2i\alpha_F}(\xi^2 + x^2)}{1 - e^{-2i\alpha_F}} \right\} \exp \left\{ -\frac{\xi^2 + x^2}{2} \right\},
\]

(2.46)

where in the last step Mehler’s formula [66] has been used.

\[
 \sum_{n=0}^{\infty} \frac{e^{-ina\pi/2}H_n(\xi)H_n(x)}{2^n n! \sqrt{\pi}} = \frac{\exp \left\{ \frac{2ix\xi e^{-i\alpha_F} - e^{-2i\alpha_F}(\xi^2 + x^2)}{1 - e^{-2i\alpha_F}} \right\}}{\sqrt{\pi} (1 - e^{-2i\alpha_F})}.
\]

(2.47)

With the following expressions,

\[
 \frac{2x\xi e^{-i\alpha_F}}{1 - e^{-2i\alpha_F}} = -ix\xi \csc \alpha_F,
\]

(2.48)

\[
 \frac{1}{\sqrt{\pi} \sqrt{1 - e^{-2i\alpha_F}}} = \frac{e^{-i(\bar{\alpha}_F - \alpha_F)}}{\sqrt{2\pi} |\sin \alpha_F|},
\]

(2.49)

\[
 \frac{e^{-2i\alpha_F}}{1 - e^{-2i\alpha_F}} + \frac{1}{2} = -\frac{i}{2} \cot \alpha_F,
\]

(2.50)

where $\alpha_F = \text{sgn}(\sin \alpha_F)$, we can obtain a more tractable integral representation of $F$ as (2.51).

\[
 f_a(\xi) := (F^a f)(\xi) = \frac{e^{-i(\bar{\alpha}_F - \alpha_F)}}{\sqrt{2\pi} |\sin \alpha_F|} \int_{-\infty}^{\infty} \exp \left\{ -i \frac{x\xi}{\sin \alpha_F} + \frac{i}{2} x^2 \cot \alpha_F \right\} f(x) dx,
\]

(2.51)
where $0 < |\alpha_F| < \pi$.

It is shown that defining via (2.51), the FrFT exists in the same conditions as in which the FT exists [62]. Therefore, we have proved the definition of FrFT in linear integral form as follows.

The $a$th order of fractional Fourier transform of $f(x)$ is

$$f_a(\xi) = \int_{-\infty}^{\infty} K_a(\xi, x) f(x) dx,$$

(2.52)

where $K_a(\xi, x) = A_{\alpha_F} \exp[i\pi(\cot \alpha_F \xi^2 - 2 \csc \alpha_F \xi x + \cot \alpha_F x^2)], \ A_{\alpha_F} = \sqrt{1 - i \cot \alpha_F}, \ \alpha_F = a\pi/2.$

### 2.4.2 Properties of the Kernel Function $K_a(\xi, x)$

By $F^b = F^{b-a}F^a$, we have

$$f_a(\xi) := (F^a f)(\xi) = \frac{e^{-\frac{i}{2} (2\alpha - \alpha_F) \xi^2 \cot \alpha_F}}{\sqrt{2\pi |\sin \alpha_F|}} \int_{-\infty}^{\infty} \exp\left\{ - i \frac{x \xi}{\sin \alpha_F} + i \frac{x^2}{2} \cot \alpha_F \right\} f(x) dx.$$  

(2.53)

Using the expressions in (2.53) and the interpretation of the FrFT as a rotation, it is directly verified that the kernel $K_a$ has the following properties [65].

*If $K_a(\xi, x)$ is the kernel of the FrFT as in (2.52), then*

1. $K_a(\xi, x) = K_a(x, \xi)$;
2. $K_{-a}(\xi, x) = K_a(\xi, x)$;
3. $K_a(-\xi, x) = K_a(\xi, -x)$;
4. $\int_{-\infty}^{\infty} K_a(\xi, t) K_b(t, x) dt = K_{a+b}(\xi, x)$;
5. $\int_{-\infty}^{\infty} K_a(t, \xi) K_b(t, x) dt = \delta(\xi - x)$. 
2.4.3 Convolution of FrFT

Since FrFT is a generalization of FT, it has similar properties as FT. Therefore we only mention the convolution property here. FT transforms a convolution in time domain into a product in frequency domain. This property remains true when it concerns the convolution of two functions in the domain of the FrFT \( F^a \): if \( g_a(x_a) = f_a(x_a) * h_a(x_a) \), then its FT becomes \( g_{a+1}(\xi_a) = f_{a+1}(\xi_a)h_{a+1}(\xi_a) \)\(^{[65]} \). Thus,

\[
F\{F[f_a(x_a) * h_a(x_a)]\} = F\{f_{a+1}(x_{a+1})h_{a+1}(x_{a+1})\} = f_{a+2}(x_{a+2}) * h_{a+2}(x_{a+2}). \tag{2.54}
\]

2.4.4 FrFT of Some Common Functions

The fractional Fourier transform of some common functions can be derived by the use of (2.52).

- **FrFT of Delta Function \([62]\)**

\[
FrFT_{\alpha_F}\{\delta(\xi)\} = \sqrt{1 - i \cot \alpha_F \exp(i\pi \xi^2 \cot \alpha_F)} \tag{2.55}
\]

\[
FrFT_{\alpha_F}\{\delta(\xi - \tau)\} = \sqrt{1 - i \cot \alpha_F \exp[i\pi(\xi^2 \cot \alpha_F - 2\xi \tau \csc \alpha_F + \tau^2 \cot \alpha_F)]} \tag{2.56}
\]

Here, \( \alpha_F \) is rotating angle of FrFT. As can be seen from (2.55) and (2.56), a delta function is transformed into a linear chirp function by the FrFT.

- **FrFT of a Sinusoid \([62]\)**

\[
FrFT_{\alpha_F}\{\exp(i2\pi \tau \xi)\} = \sqrt{1 + i \tan \alpha_F \exp[-i\pi(\xi^2 \tan \alpha_F - 2\xi \tau \sec \alpha_F + \tau^2 \tan \alpha_F)]} \tag{2.57}
\]

Notice that similar to the delta function, the FrFT of a sinusoid is also a linear chirp function.
• FrFT of a Constant \[62\]

\[
FrFT_{\alpha_F}\{C\} = C\sqrt{1 + i \tan \alpha_F} \exp(-i\pi \xi^2 \tan \alpha_F)
\] (2.58)

Since the DC term can be considered as a sinusoid with zero frequency, the result is a special case of the previous FrFT pair with \(\tau = 0\).

• FrFT of a Chirp Function \[62\]

\[
FrFT_{\alpha_F}\{\exp(i\pi \chi \xi^2)\} = \sqrt{\frac{1 + i \tan \alpha_F}{1 + \chi \tan \alpha_F}} \exp\left[i\pi \xi^2 \frac{\chi - \tan \alpha_F}{1 + \chi \tan \alpha_F}\right]
\] (2.59)

The FrFT of a linear chirp function is also a linear chirp function with a special type of sweeping rate.

• FrFT of a Gaussian Function \[62\]

\[
FrFT_{\alpha_F}\{\exp(-\pi \xi^2)\} = \exp(-\pi \xi^2)
\] (2.60)

It is interesting to notice that the FrFT of a Gaussian function is still a Gaussian function. In fact, the result is independent of the transform angle \(\alpha_F\).

Figure 2.3 is an example of fractional Fourier transform of a sine function. The orders of FrFT vary from 0.0 to 2.0.

For a symmetric delta function shown in fig. 2.4, the FrFT spectra with orders in (0,2) are plotted in fig. 2.5.
Fig. 2.3: Fractional Fourier transform of a sine function.

Fig. 2.4: A sample of symmetric delta function.
Fig. 2.5: Fractional Fourier transform of the symmetric delta function.
Chapter 3
Fractional Fourier Transform-Based Hurst Parameter Estimation

The fractional Fourier transform is a generalization of Fourier transform and has a strong relationship with the wavelet transform [60] which is very suitable for analyzing LRD. Therefore, in long-range dependence applications, it is possible to achieve an improved performance using FrFT. Since FrFT has been shown to have a computational complexity proportional to the wavelet transform, the performance improvements may come without additional cost.

3.1 Derivation of the Main Result

Consider a stationary stochastic process \( Y = f(u) \). \( Y \) is said to be long-range dependent if it can be modeled by a power-law decay of the autocorrelation:

\[
    r_Y(\tau) = E[f(u)f(u - \tau)] \sim c_Y|\tau|^{-\gamma}, \quad \tau \to \infty, \quad 0 < \gamma < 1, \tag{3.1}
\]

where \( \sim \) means the ratio of the left and the right hand sides converges to 1. Imposing the condition (3.1) on the spectral density \( s_Y \) of \( Y \), as \( \xi \to 0 \) we get

\[
    s_Y(\xi) \sim c_s|\xi|^{-\alpha}, \quad 0 < \alpha < 1, \tag{3.2}
\]

where \( c_s > 0 \) and \( s_Y(\xi) = (2\pi)^{-\frac{1}{2}} \sum_{\tau \in \mathbb{Z}} e^{i\xi \tau} r_Y(\tau) \). With smoothness assumptions, \( \alpha \) and \( \gamma \) have the following relationship

\[
    \alpha = 1 - \gamma. \tag{3.3}
\]
Since the Hurst parameter is related with $\alpha$ by $H = (1 + \alpha)/2$ and $\alpha$ could be estimated by the spectral density $s_Y$ according to (3.2), the Hurst parameter $H$ could then be estimated according to $s_Y$.

According to sec. 2.4, the $a$th order of fractional Fourier transform of a signal $f(u)$ is

$$f_a(\xi) = \int_{-\infty}^{\infty} K_a(\xi, u) f(u) du,$$  \hspace{1cm} (3.4)

where $K_a(\xi, u) = A_{\alpha_F} \exp[i\pi(\cot \alpha_F \xi^2 - 2 \csc \alpha_F \xi u + \cot \alpha_F u^2)], \ A_{\alpha_F} = \sqrt{1 - i \cot \alpha_F},\ \alpha_F = a\pi/2$.

Making the change of variable by $j = \xi \sec \alpha_F$ and denoting the left-hand side of (3.4) by $g(j) = f_a(j/\sec \alpha_F)$, we can obtain the following result:

$$g(j) = C(\alpha_F) e^{-i\pi j^2 \sin^2 \alpha_F} \int_{-\infty}^{\infty} \exp \left[ i\pi \left( \frac{j - u}{\tan \frac{1}{2} \alpha_F} \right)^2 \right] f(u) du,$$  \hspace{1cm} (3.5)

where $C(\alpha_F) = \sqrt{1 - i \cot \alpha_F} \exp(i\pi j^2)$ is a constant that depends on $\alpha_F$ only.

Note that (3.5) has certain similar characteristics of a wavelet transform. The continuous wavelet transform [59] is defined as

$$WT_f(j, k) = \int f(u) \psi_{j,k}^*(u) du,$$  \hspace{1cm} (3.6)

$$\psi_{j,k}(u) = \frac{1}{|k|^{\frac{1}{2}}} \psi \left( \frac{u - j}{k} \right),$$

where $j, k, \psi(t), \psi_{j,k}(t)$ show scale, scan time, mother wave and wavelet, respectively. Let the mother wave $\psi(t) = \exp(i\pi t^2)$, the wavelet transform in (3.6) can be changed to

$$WT_f(j, k) = \frac{1}{|k|^{\frac{1}{2}}} \int \exp(i\pi(u - j)^2/k^2) f(u) du.$$  \hspace{1cm} (3.7)
Clearly, (3.7) has the same form as FrFT in (3.5) except the scaling factor. Let \( k = \tan^{\frac{1}{2}} \alpha_F \), (3.7) becomes

\[
WT_f(j, k) = \frac{1}{|\tan \alpha_F|^{\frac{1}{4}}} \int \exp \left[ i\pi \left( \frac{j - u}{\tan \frac{1}{2} \alpha_F} \right)^2 \right] f(u) du
= \exp[i\pi j^2 \sin^2 \alpha_F] \frac{C(\alpha_F)|\tan\alpha_F|^\frac{1}{4}}{C(\alpha_F)|\tan\alpha_F|^\frac{1}{4}} g(j),
\]

(3.8)

which establishes the relationship between FrFT and wavelet transform.

For wavelet \( \psi_{j,k}(t) \), \( f(u) \) has the expansion [16]

\[
f(u) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} d_{j,k}(f) \psi_{j,k}(u),
\]

(3.9)

where \( d_{j,k}(f) \) are wavelet coefficients of the function \( f(u) \). Let \( \varepsilon_j \) denote the mean energy of the wavelet coefficients at the scale \( j \), that is

\[
\varepsilon_j = E[d_{j,k}^2(f)].
\]

(3.10)

Given a finite sample \( Y = f(k), k = 1, 2, ..., N_j \), a triangular array of approximate wavelet coefficients can be obtained by Mallat’s algorithm [23]. Thus, one can estimate \( \log_2(\varepsilon_j) \) by using the sample energy of these coefficients:

\[
\log_2 \left( \frac{1}{N_j} \sum_{k=1}^{N_j} WT_f^2(j, k) \right) \approx \log_2(\varepsilon_j) = \log_2(E[d_{j,k}^2(f)]).
\]

(3.11)

According to (3.8) and (3.11), the log-scale FrFT spectrum could be derived as

\[
G(j) = \log_2(E[g^2(j)]) = \log_2 \left( \frac{C(\alpha_F)|\tan\alpha_F|^\frac{1}{4}}{\exp[i\pi j^2 \sin^2 \alpha_F]} \right) \frac{1}{2} E[d_{j,k}^2(f)].
\]

(3.12)

Indeed,

\[
s_Y(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi \tau} r_Y(\tau) d\tau.
\]

(3.13)
By using the Parseval identity and a change of variables, it can be shown that

\[
Ed_{j,k}^2(f) = \int_R \psi_{j,k}(t) \int_R \psi_{j,k}(s) r_Y(t - s) ds dt \\
= \int_R |\hat{\psi}_{j,k}(\xi)|^2 s_Y(\xi) d\xi \\
= 2^j \int_R |\hat{\psi}(2^j \xi)|^2 s_Y(\xi) d\xi \\
= \int_R |\hat{\psi}(\eta)|^2 s_Y(\eta/2^j) d\eta,
\]

(3.14)

where \(\hat{\psi}(\xi) = \sqrt{2\pi} \int_R e^{i\xi t} \psi(t) dt\) denotes the Fourier transform of the function \(\psi\). The expression in (3.14) relates the mean energy \(\varepsilon_j\) of the wavelet coefficient \(d_{j,k}(Y)\) to the spectral density of the stationary signal \(Y(t)\). For large scales \(j\), the function \(s_Y(\eta/2^j), \eta \in R\) can be viewed as a zoomed version of the spectral density \(s_Y(\eta)\) around the zero frequencies. Therefore, the integral in the right hand side of (3.14) picks out the spectral behavior of \(Y\).

Substituting (3.2) into (3.14) yields, as \(j \to \infty\)

\[
Ed_{j,k}^2(Y) \sim c_f \int_R |\hat{\psi}(\eta)|^2 |\eta/2^j|^{-\alpha} d\eta = c_f C 2^j \alpha,
\]

(3.15)

where \(C = C(\psi, \alpha) = \int_R |\hat{\psi}(\eta)|^2 |\eta|^{-\alpha} d\eta\).

Note that in (3.12), \(\frac{C(\alpha_F) |\tan \alpha_F|^{1/4}}{\exp(|\pi j^2 \sin^2 \alpha_F|)} = \sqrt{1 - \cot \alpha_F \exp(i\pi j^2 \cos^2 \alpha_F)} |\tan \alpha_F|^{1/4}\), and it only depends on \(\alpha_F\). Therefore, according to \(H = (1 + \alpha)/2\) and the relationship between log-scale FrFT spectrum and \(Ed_{j,k}^2\) in (3.12), the Hurst parameter can be estimated using the fact that

\[
G(j) \sim (2H - 1)j + const.
\]

(3.16)

By using a linear regression of the log-scale FrFT spectrum \(G(j)\) on the scale \(j\), where \(1 < j_1 < j_2 < J\), with \(J\) the total number of scales, the Hurst parameter \(H\) of \(Y\) can be
estimated as

\[
\hat{H}_{[j_1,j_2]} = \frac{1}{2} \sum_{j=j_1}^{j_2} \omega_j G(j) + \frac{1}{2}, \tag{3.17}
\]

where the weights \(\omega_j\)'s are such that \(\sum_{j=j_1}^{j_2} \omega_j = 0\) and \(\sum_{j=j_1}^{j_2} j \omega_j = 1\).

### 3.2 Local Analysis

Using the above established FrFT based Hurst parameter estimation method, \(\hat{H}_{[j,j+1]}\) can be given by

\[
\hat{H}_{[j,j+1]} = \frac{G(j + 1) - G(j)}{2} + 0.5, \quad j = 1, 2, \ldots, J - 1. \tag{3.18}
\]

Inside (3.18), \(G(j + 1) - G(j)\) is the local slope of the FrFT spectrum. From (3.16), the \(\hat{H}_{[j,j+1]}\) should be very close to \(H\) at large scale \(j\). Therefore, the \(\hat{H}_{[j_1,j_2]}\) in (3.17) can be expressed as

\[
\hat{H}_{[j_1,j_2]} = \frac{1}{2} \sum_{j=j_1}^{j_2} v_j \hat{H}_{[j,j+1]}, \tag{3.19}
\]

where \(v_j = \omega_{j+1} + \ldots + \omega_{j_2}\).

Besides, the stochastic process \(Y = f(k)\) \((k = 1, \ldots, N)\) can be divided into some non-overlapping windows for local analysis. Assume that the window size is \(w\) \((w \leq N)\), \(Y_r\) \((r = 1, \ldots, N/w)\) is the time series corresponding to the \(r\)th window and the size of the last window is \(N - w(N/w - 1) \geq w\). Then the FrFT spectrum can be computed within each window to obtain a matrix \(G\), the \((j, r)\)th element of which is defined as

\[
G_j(r) = \log_2 \left( \frac{1}{N_j} \sum_{k=1}^{N_j} g_j^2(Y_r) \right), \tag{3.20}
\]

where \(g_j(Y_r)\) is the FrFT coefficients of time series \(Y_r\).
3.3 Effect of Nonstationarity

The main reason of utilizing local analysis is that most of the Hurst parameter estimation methods are based on the assumption that the time series is stationary [17]. Nonstationarity will result in poor performance or even fail the estimator [7]. Stove et al. (2004) have shown that nonstationarity effects such as abrupt shifts in the mean would yield a steep wavelet spectrum and overestimate the Hurst parameter [16]. The two plots below are wavelet spectrum and FrFT spectrum of the electrochemical noise calculated on scale \( j \).

The testing electrochemical noise is the corrosion potential of the stainless steel electrode with bacteria attached in the artificial saliva called Jenkin’s Solution [33]. The electrode is put in the solution for 24 hours at room temperature. The sampling period is 0.5 second. Details of the biocorrosion experiments will be discussed in Chapter 5.

The Hurst parameter of the biocorrosion data, estimated over the range of scales \( j \), is about 0.89. Figure 3.1 presents the local wavelet spectrum of the biocorrosion signal in each window as described in (3.20). Figure 3.2 presents its local fractional Fourier spectrum in each window. As shown in the plots, both wavelet spectra and FrFT spectra are consistent...
with long range dependence. However, fig. 3.1 shows that at the large scales, there are some obvious variabilities in the wavelet spectra. When \(6 < j < 7\), there are also a few variabilities for FrFT spectra in fig. 3.2. These variabilities indicate the existence of nonstationarity [16] and may affect the global analysis of long range dependence for Hurst parameter. Nevertheless, compared to wavelet spectrum, FrFT spectrum is more robust to nonstationarity. Therefore, FrFT is more suitable for Hurst parameter estimation of the time series with some parts nonstationary if local analysis is used.

In the next chapter, we will present the detailed validation test for the proposed FrFT based Hurst parameter estimator using the benchmark fractional Gaussian noise.
Chapter 4

Fractional Gaussian Noise Analysis

4.1 White Noise

Before implementing the FrFT-based estimator, the right fractional order of FrFT should be considered. It is because the mean value of the local Hurst parameter estimations will be taken as the estimated Hurst parameter, while different fractional orders of FrFT result in different variances of local Hurst parameter estimations.

Figure 4.1 shows the variances of the Hurst parameter estimations with the order of FrFT $\alpha$ varied from 0.1 to 1.0. We use different orders of FrFT to estimate the Hurst parameter of the FGN. Then, the statistic variance of each estimation is calculated correspondingly. In the end, the “order-variance” plot is plotted. The white noise is generated

![Graph showing the variances of Hurst parameter estimations with different orders of FrFT for white noise.](image)

Fig. 4.1: The variances of Hurst parameter estimations with different orders of FrFT for white noise.
by Matlab using sample size 1,000,000. As shown in fig. 4.1, the minimum variance appears at \( a = 0.6 \). Therefore, 0.6 will be chosen as the order of FrFT for estimating the Hurst parameter of the white noise. The results are compared with wavelet based local analysis.

The Hurst parameter of a random white noise should be 0.5 [29]. In our experiment, the estimation by the FrFT-based estimator is 0.50977, while the estimation by wavelet-based estimator is 0.47122. The results show that the FrFT-based estimator is more accurate.

4.2 Fractional Gaussian Noise

In order to test the accuracy of the Hurst parameter estimation and prevent the bias from the FGN generating algorithm, we use two methods to generate FGN for comparison.

4.2.1 Fractional Integrator Method

Before applying the fractional operator, the simple integer order system shown in fig. 4.2 is implemented for an overview. The transfer function of the system is \( H(s) = \frac{1}{s+c} \) with the impulse response \( h(t) = e^{-ct}u(t) \) with \( u(t) \) the unit step function. Let the input be a continuous-time white noise \( \omega(t) \) with variance \( \sigma^2 \). We have the output signal \( y(t) \) as follows:

\[
y(t) = \int_{-\infty}^{t} \omega(\tau)e^{-c(t-\tau)}d\tau. \tag{4.1}
\]

The autocorrelation of the output \( y(t) \) is

\[
R(\tau) = \sigma^2 h(\tau) * h(-\tau) = \sigma^2 \int_{-\infty}^{\infty} e^{c\tau}u(\tau)e^{-c(t-\tau)}u(t - \tau)d\tau = \frac{\sigma^2}{2c} e^{-c|\tau|}. \tag{4.2}
\]

Since the outputs of a conventional integer order system do not have long-range dependence, their autocorrelation should decay very fast as an exponential law distribution shown in fig. 4.3. In this picture, the solid (blue) line is the autocorrelation of the output of the first
order system, while the dashed (red) line is the fitting exponential law function according to (4.2). The $\hat{c}$ used for fitting the autocorrelation of the output is almost the same as the $c$ used for generating $y(t)$ and obtaining $R(\tau)$.

As explained in Chapter 2, the FGN can be considered as the output of a fractional integrator. The system can be defined by the transfer function $H(s) = s^\nu = \frac{1}{s^\beta}, \nu = -\beta$ with impulse response $h(t) = \frac{t^{-\nu-1}}{\Gamma(-\nu)}, \nu < 0$. $\beta$ is a real number. The system block diagram is shown in fig. 4.4.
Let $\omega(t)$ be a continuous-time white noise with variance $\sigma^2$, the $\nu$th order fractional noise $y_{\nu}(t)$ can be expressed as

$$y_{\nu}(t) = \frac{1}{\Gamma(-\nu)} \int_{-\infty}^{t} \omega(\tau)(t-\tau)^{-\nu-1} d\tau,$$

(4.3)

where $y_0(t) = \omega(t)$. Its autocorrelation function [7],

$$R(\tau) = \sigma^2 \frac{|\tau|^{-2\nu-1}}{2\Gamma(-2\nu)\cos(\nu\pi)},$$

(4.4)

To obtain this function, we compute the following convolution.

$$R(\tau) = \sigma^2 h(\tau) * h(-\tau) = \sigma^2 \frac{1}{\Gamma^2(-\nu)} \int_{0}^{\infty} t^{-\nu-1}(t+\tau)^{-\nu-1} u(t+\tau) dt = \sigma^2 \frac{|\tau|^{-2\nu-1}B(1+2\nu,-\nu)}{\Gamma^2(-\nu)},$$

(4.5)

where $B(x,y)$ is the beta function. As $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we obtain (4.4). Also, we can get its power spectrum

$$S(\xi) = \sigma^2 |\xi|^{2\nu},$$

(4.6)

which has the same form as (3.2). Therefore we have $H = \frac{1+\alpha}{2} = \frac{1+(-2\nu)}{2} = \frac{1}{2} - \nu.$

Fig. 4.4: FGN by fractional integrator.
Because $0.5 < H < 1$ indicates long-range dependence, correspondingly $-0.5 < \nu < 0$. The autocorrelation of the output FGN is shown in fig. 4.5. The solid (blue) line is its computed autocorrelation and the dashed (red) one is the power law fitting function based on (4.4). The order $\nu$ of both plots are very close, $\nu = -0.3$. As we can see, its autocorrelation decays slowly as a power law function, which proves the existence of long-range dependence.

In our LRD experiment, the sample size of the input white noise is 100,000. The spectrum of a sample FGN is illustrated in fig. 2.1. With the proper order of FrFT which results in minimum variance of corresponding local estimations, the Hurst parameters of the output fractional Gaussian noises will be estimated according to the FrFT-based Hurst parameter estimator. One hundred FGN with their Hurst parameters in the range of 0.01 to 1.00 with the interval 0.01 will be generated for testing. Besides, local wavelet-based method and aggregated variance method will be implemented for comparison.

In fig. 4.6, the $x$-axis is the value of Hurst parameters of FGN, the $y$-axis is the estimated value of the Hurst parameters. There is an ideal estimation plot of $y = x$ for reference. As we can see, the estimation by FrFT-based estimator is the closest to the
ideal estimation line except some Hurst parameters very close to 1.0. For $0.5 < H < 1$ which indicates long-range dependence, our FrFT-based estimator gives out satisfactory estimations. Besides, as shown in this plot the estimation methods utilizing local analysis including FrFT-based method and wavelet-based method have better performance than that of the aggregated variance method which utilizes global analysis. For $0.01 < H < 0.5$, the results of both three estimation methods are becoming greater than the real value. These results may suggest some bias of the FGN generating method as well as the Hurst parameter estimation methods. Therefore, in the next section, we use symmetric moving average filter to generate the paths of FGN for examination.

4.2.2 Symmetric Moving Average Filter Method

In this section, the paths of fractional Gaussian noise are generated by using a truncated symmetric moving average filter. The filter coefficients are computed via IFFT of the square root of the FFT of the covariances of the FGN. The moving average is also computed by
Fig. 4.7: Comparison of three Hurst parameter estimation methods for FGN produced by symmetric moving average filter with 100 Hurst parameters from 0.01 to 1.00.

using the FFT algorithm.

Similar as the experiment in sec. 4.2.1, 100 $H$ in the range of 0.01 to 1.00 with the interval 0.01 will be tested for Hurst parameter estimation of FGN generated by moving average filter. The sample size of FGN is 100,000, with generating filter length 200,000. Window size is 5000 for all the local estimator including FrFT-based and wavelet-based. Proper order of FrFT has been tested for the minimum variance of the Hurst parameter estimations. The results of three estimation methods are compared in fig. 4.7.

As we can see, the FrFT-based estimations are the closest to the ideal line. When the Hurst parameter of FGN approaches to 1.0, the wavelet-based estimations become greater than the actual value and it almost reaches 1.5 at the values of $H$ very close to 1.0. On the other hand, the fluctuations in the results of aggregated variance method suggest that its estimated values vary more than the other two. Besides, when the real Hurst parameters approach to 1.0, the estimations based on aggregated variance method become much smaller than 1.0. In conclusion, the FrFT-based estimator outperforms the other two in estimating
Hurst parameters of FGN produced by symmetric moving average filter. One thing that needs to be mentioned is, for $0.01 < H < 0.50$, especially for $0.01 < H < 0.10$, all three methods have estimations smaller than the real values, which is opposite to the results in fig. 4.6. Therefore, we can tell that the misleading estimations may be caused by the generating methods of FGN.

In conclusion, the results in both sec. 4.2.1 and sec. 4.2.2 show that for long-range dependent fractional Gaussian noises, the FrFT-based estimation method performs very well in analyzing their Hurst parameters and gives out better estimations than the other two methods.
Chapter 5

Applications to Bioelectrochemical Corrosion Processes

According to the research in recent years, when the biocorrosion signals possess long-range dependence, the conventional methods fail to precisely characterize the differences between them [33, 34]. This is the motivation of implementing FOSP techniques. In this thesis, the proposed FrFT-based Hurst parameter estimator will be introduced to characterize the LRD of the biocorrosion signals.

In the field of biosensor technology, electrochemistry deals basically with behaviors of implant devices in terms of corrosion. Today, medical implant devices in the body cover a range of materials and applications. Safety is the most important property of any biomaterial used as a bioimplant. Biocompatibility of a metallic implant is closely associated with the interaction of the implant with the surrounding environment. Metal release from the implant into the surrounding tissue may occur as a consequence of various mechanisms like 1) mechanical nature, i.e., due to wear phenomena; 2) electrochemical nature, i.e., corrosion processes; 3) due to unstable metallic state [67]. The implantation of a metal object into the body inevitably leads to some degree of local tissue response. Depending on the material utilized, the metallic implant may induce a reaction in cells distant from the surgery site. These reactions may be simply moderate or transient, but in more severe cases, serious tissue damage with permanent morphological and structural changes can occur [68].

Corrosion is an important interfacial process of materials. Corrosion can be defined as a chemical or electrochemical reaction between a material, usually a metal, and its environment that produces a deterioration of the metal and its properties [34]. Different types of microbes are used in microbiologically influenced corrosion (MIC) such as sulphur-reducing bacteria, sulphur-oxidizing bacteria, organic acid-producing bacteria, iron-oxidizing bacteria, etc. Various metals which are affected by MIC are stainless steel, titanium, gold, etc.
Fig. 5.1: An example of electrochemical noise measurement. ECN data is obtained from Prof. A. Zhou’s lab with permission.

We used stainless steel in our experiments. Stainless steel was first identified as a suitable material for orthopaedic implants. Today, it is still one of the most frequently used biomaterials for implants because of its suitable mechanical properties and excellent clinical application. Also, stainless steel has very low corrosion resistance and has low production cost [69]. Electrochemical noise (ECN) techniques [70] are used for extracting the biocorrosion signals in the experiment in Chapter 5. It gets the information from fluctuations either in potential or current that is observed on corroding electrodes.

Figure 5.1 shows an example of electrochemical noises (ECN) obtained from a stainless steel electrode that was exposed to the Tomasi’s artificial saliva solution for 30 minutes. This ECN data typically consists of three sets of measurements, the corrosion potential of the working electrode (WE), the counter electrode (CE), and the coupling current between WE and CE. In this thesis, the potential of the WE was used for signal processing. It is possible that the random fluctuations in fig. 5.1 may possess nonstationarity in some parts when the time series becomes very long.
Table 5.1: VMP2 software and parameters setup.

<table>
<thead>
<tr>
<th>Software</th>
<th>EC-Lab for windows v9.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrode connection</td>
<td>Standard</td>
</tr>
<tr>
<td>Electrode surface area</td>
<td>0.1765 cm²</td>
</tr>
<tr>
<td>tR1 (h:m:s)</td>
<td>0:00:10.0000</td>
</tr>
<tr>
<td>dtR1 (s)</td>
<td>0.5000</td>
</tr>
<tr>
<td>ti (h:m:s)</td>
<td>0:30:0.0000</td>
</tr>
<tr>
<td>I Range</td>
<td>Auto</td>
</tr>
<tr>
<td>Bandwidth</td>
<td>5</td>
</tr>
</tbody>
</table>

The details about the bioelectrochemical experiments including the preparation of electrodes, bacteria growth and attachment and the preparation of artificial saliva solutions are in the Appendix.

5.1 Electrochemical Noise Measurements

A versatile multi-channel potentiostat 2/Z (VMP2) is used for ECN measurements. The VMP2 is a multi-channel instrument designed to study intercalated compounds with long time experiments. It has the ability for impedance measurements and the BiStat. Table 5.1 shows the details about the software and the parameters setup.

Figure 5.2 is the schematic plot of the bioelectrochemical experiment equipments. Figure 5.3 is the real picture of our experiment.

In the schematic plot, the corrosion behaviors of the stainless steel (SS) electrode in three artificial saliva solutions were studied by using the Zero Resistance Ammeter (ZRA). In the ZRA measurement configuration, two identical electrodes (materials and size), namely, working electrodes (WE) and counter electrode (CE), are immersed in the solution of interest. The fluctuation of the potential of WE and CE versus reference electrode (RE) as well as the coupling current between WE and CE will be measured simultaneously. ZRA could be used as a current to voltage converter. It gives a voltage output proportional to the current between its two input terminals while imposing a “zero” voltage drop to the external circuit. VTT is a reference voltage converter which is connected to RE. Finally, all the current and potential signals are collected by computer after A/D converter.

In fig. 5.3, stainless steel is used as both working and counter electrode and Ag/AgCl is
Fig. 5.2: Schematic plot of bioelectrochemical experiment for ECN test.

Fig. 5.3: Bioelectrochemical experiment equipment for ECN test. The picture is taken from Prof. A. Zhou’s lab with permission.
used as reference electrode. The blue equipment under the beaker is VMP2. Zero Resistance Ammeter (ZRA) is a built-in technique in its multichannel potentiostat.

5.2 Time-Frequency Analysis

5.2.1 Time-domain

Mean is the average value of potential measurements. According to (5.1)

\[
\mathcal{E} = \frac{1}{n} \sum_{k=1}^{n} E[k],
\]

where \(E[k]\) is the potential value.

Variance is a measurement of the average AC Power in the signal. It is also referred to as noise power.

\[
V = \frac{1}{n} \sum_{k=1}^{n} (E_n[k] - \mathcal{E})^2
\]

Skewness is a nondimensional measurement of the symmetry of a distribution. A zero value means that the distribution is symmetrical about the mean. A positive value indicates there is a tail in the positive direction and a negative value implies the presence of tail in the negative direction. A time record consisting of unidirectional transient will typically be heavily skewed, and this may be useful to detect transient associated with metastable pitting.

\[
skewness = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{E_n[k] - \mathcal{E}}{\sqrt{E_n[k]^2}} \right)^3
\]

Kurtosis is a measurement of whether the data are peaked or flat relative to a normal distribution. That is, data sets with high kurtosis tend to have a distinct peak near the mean, decline rapidly, and have heavy tails. Data sets with low kurtosis tend to have a flat top near the mean rather than a sharp peak. Positive kurtosis indicates a “peaked”
Table 5.2: Four statistical components: mean, variance, skewness, and kurtosis of the ECN in three different solutions.

<table>
<thead>
<tr>
<th>Stainless steel</th>
<th>Solution A</th>
<th>Solution B</th>
<th>Solution C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>-0.2388</td>
<td>-0.3192</td>
<td>-0.3808</td>
</tr>
<tr>
<td>Variance</td>
<td>1.3896e-004</td>
<td>2.2726e-005</td>
<td>4.1023e-006</td>
</tr>
<tr>
<td>skewness</td>
<td>-1.4296</td>
<td>1.8382</td>
<td>0.4142</td>
</tr>
<tr>
<td>kurtosis</td>
<td>3.9443</td>
<td>4.7590</td>
<td>2.5538</td>
</tr>
</tbody>
</table>

distribution and negative kurtosis indicates a “flat” distribution.

\[
kurtosis = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{E_n[k] - \bar{E}}{\sqrt{E_n[k]^2}} \right)^4
\] (5.4)

Table 5.2 gives out the results of the four statistical components for the ECN of stainless steel WE electrode in three different solutions tested for 30 minutes. The mean values of the potential noises in Table 5.2 indicate that the potentials grow from solution A to C. While the variance values indicate that the noise power is decreased from solution A to C. The skewness values show that the distribution of ECN in solution A has a negative tail while that of solution B and that of solution C have positive tails. The kurtosis values show that all the potential noises are “peaked” distributions meaning they have large fluctuations.

### 5.2.2 Frequency-domain

We use Welch method in Matlab to get the power spectral density of potential noises used for Table 5.2. The plots are shown in fig. 5.4.

For comparison, fig. 5.5 is the power spectral density plot of the same ECN by fractional Fourier transform. The order of FrFT is 0.5.

Figure 5.6 shows the power spectra of the ECN by FrFT with the fractional order from 0.1 to 0.8. As we can see, the passband becomes narrower when \( a \) increases from 0.1 to 0.8. The plot also shows that the potential noise is dominated by low frequency components, since it is usually the low frequency information that is useful for noise impedance computations. A rough estimate of noise impedance can be obtained from the FrFT spectrum by comparing the magnitude.
Fig. 5.4: Power spectrum density estimate of potential noises in three artificial saliva via Welch.

Fig. 5.5: Power spectrum density estimate of potential noises in three artificial saliva via FrFT.
5.3 LRD Analysis and Discussion

In this part, the FrFT-based estimator is applied to the bioelectrochemical corrosion signals which may possess some partial nonstationarities as discussed in sec. 3.3. The testing signal is potential noise of stainless steel electrode in solution A with bacteria coated for 24 hours.

First, the fractional order $a$ of FrFT varied from 0.1 to 1.0 is considered. For each order there will be a different statistic variance of the corresponding local Hurst parameter estimations. As shown in fig. 5.7, the fractional order $a$ equal to 0.4 results in the smallest variance $9.3160e-006$, which indicates that we should choose $a = 0.4$ for the FrFT-based estimator in analyzing the long-range dependence of the following biocorrosion noises. When implementing the method, it is important to choose the right starting scale $j_1$. This is because at large scales the long-range dependent time series become self-similar, while at small scales the FrFT spectrum may not have the same slope. A graphical tool [16] will be used for choosing the range of scales. It displays deviations from the local Hurst parameter
estimations. Every column of the plot of this graphic tool corresponds to the vector of \( \hat{H}_{[j,j+1]} \) in (3.18). The blue (black) cell represents \( \hat{H}_{[j,j+1]} \) is above a 95% confidence interval of the estimate \( \hat{H}_{[j_1,j_2]} \); the red (white) cell represents \( \hat{H}_{[j,j+1]} \) is below this confidence interval; and the purple (gray) represents \( \hat{H}_{[j,j+1]} \) is within the confidence interval.

Next, we set \( j_1 = 5 \) for each window, and estimate local Hurst parameters \( \hat{H}(r) \) of the biocorrosion signal. In fig. 5.8, the top plot is the local \( H(r) \) estimate with the mean and a band of their 95% confidence intervals while the bottom plot visualizes the deviations in the local FrFT spectrum. The presence of red (white) and blue (black) indicates that the choice of scales \( j_1 \) and \( j_2 \) are not suitable and these Hurst parameter estimates may not be meaningful. Both the slope of FrFT below the average and the slope of FrFT above the average are too high when compared to the slope around the average.

According to the importance of choosing the starting scale and the results shown in fig. 5.8, the choice of \( j_1 \) should be changed. After checking the FrFT estimator with several \( j_1 \), fig. 5.9 gives a good estimation where the starting scale \( j_1 = 7 \). The bottom plot of
Fig. 5.8: Local estimations of Hurst parameter by FrFT estimator. Applied to biocorrosion potential noise of stainless steel electrode in Jenkin’s Solution for 24 hours.

Fig. 5.9 shows that there are fewer red (white) or blue (black) colors which means that the local Hurst parameter estimates in fig. 5.9 is more reliable than those in fig. 5.8. From the top plot in fig. 5.9, the mean value of Hurst parameter is 0.77291, right between 0.5 and 1, indicating that the biocorrosion data has long-range dependence.

Now, let us apply the wavelet spectrum-based Hurst parameter estimation method for a comparison. The same biocorrosion data was used. The top plot in fig. 5.10 shows the estimates of the local Hurst estimator, their mean value, and their 95% confidence intervals over the window number parameter $j$. The bottom plot is the visualization of deviations. The starting scales are the same $j_1 = 7$. As the figure shows, the wavelet estimator mainly has two disadvantages. First, the bottom plot of wavelet estimator has more red (white) and blue (black) patches, which indicates the big deviation and misleading Hurst estimates. Second, both the subplots in fig. 5.10 shows some nonstationary effects. For window number $r = 3, 8, 14, 19$, there are big fluctuations in the estimates in the top plot. The red (white) and blue (black) patches in the bottom plot of fig. 5.10 also appear around those window
numbers. On the contrary, the plots in fig. 5.9 show smoother estimation which indicates that the FrFT-based estimator is more robust to nonstationarity.

In what follows, the aggregated variance method is used to estimate the Hurst parameters of biocorrosion noises globally. The results are shown in Table 5.3.

The corrosion potential of a stainless steel electrode in three different simulated saliva solutions, namely, (A) Jenkin’s Solution, (B) Tomasi’s Solution, and (C) NaCl solution, are determined via an electrochemical noise (ECN) technique. The electrodes are put into the three solutions for three time scales, 5 minutes, 30 minutes, and 24 hours. The testing sample period is 0.5 second. As shown in Table 5.3, the estimated the Hurst parameters

Table 5.3: Hurst parameters estimation of electrochemical noises of stainless steel electrode with bacteria in three different artificial saliva via the aggregated variance method.

<table>
<thead>
<tr>
<th>Hurst parameter</th>
<th>5 min</th>
<th>30 min</th>
<th>24 hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution A</td>
<td>0.82</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td>Solution B</td>
<td>1.02</td>
<td>0.98</td>
<td>1.00</td>
</tr>
<tr>
<td>Solution C</td>
<td>0.85</td>
<td>0.94</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Fig. 5.10: Local estimations of Hurst parameter by wavelet estimator. Applied to biocorrosion potential noise of stainless steel electrode in Jenkin’s Solution for 24 hours. Vary for three different solutions in 5 minutes and 30 minutes tests. As testing time extends to 24 hours, the collected data set becomes very large. The Hurst parameter estimations of biocorrosion noises in three solutions have the same value of 1.00, which is not realistic. The aggregated variance method fails to distinguish the differences of the Hurst parameters between those biocorrosion noises of 24 hours test.

Then, the local FrFT-based estimator is applied to those biocorrosion noises of 24 hours test. The results are presented in fig. 5.11, where the Hurst parameters of three different biocorrosion signals are estimated with different window sizes. The window numbers of $x$-axis are calculated by dividing the sizes of the time series by window sizes. On the contrary of the global estimator, the results by local estimator are clearly distinguishable. One can easily tell the differences between the three artificial saliva. Note that the window number should be used with care, because the estimations of Hurst parameters may be misleading when the window number is less than 10 as shown in the figure.

In conclusion, the long-range dependence in the biocorrosion signals has been explored.
Fig. 5.11: Comparison of local Hurst parameter estimations using FrFT. Applied to biocorrosion potential noise of stainless steel electrode in three different artificial saliva for 24 hours.

The local FrFT-based estimator has been successfully implemented to characterize the biocorrosion noises that have long-range dependence and are long enough to have some parts non-stationary. The superiority of the proposed estimator has already been validated in Chapter 4. Although we do not have ground truth that our FrFT-based Hurst parameter estimator has the best performance, its superiority to some other popular estimator on the biocorrosion signals has been approved in this chapter with additional benefit of distinguishing the difference between solution A, B, and C.
Chapter 6
Applications to the Elevation Records of Great Salt Lake

6.1 A Short Introduction

The United States Geological Survey (USGS) has been collecting water-surface-elevation data from Great Salt Lake since 1875 [71]. Great Salt Lake is divided into a north and a south part by a rock-fill causeway. The U.S. Geological Survey (USGS) operates gages that collect water-surface elevation data on the south part of the lake at the Boat Harbor gage, and on the north of the lake at the Saline gage [72]. In this thesis, we use the south levels of the GSL for experiment.

6.2 LRD Analysis and Discussion

The levels of the south Great Salt Lake are measured twice a month including the 1st day and 15th day of the month. The observed one-dimensional 1880 - 2002 GSL south-level time series is shown in fig. 6.1. The figure shows dramatic rise and falls of GSL levels at different times throughout the measurement period. We can tell the flood in 1982 - 1986 from the peak of the plot.

Let \( I_t \) represent the levels of Great Salt Lake. We define \( r_I \) as the logarithmic difference of the adjacent measurements.

\[
\begin{align*}
    r_I &= \log(I_t) - \log(I_{t-1}) = \Delta \log(I_t) \\
\end{align*}
\]

Then, we define \( v_I \) as the logarithmic standard deviation of squared \( r_I \).

\[
    v_I = \frac{1}{2} \log((r_I)^2)
\]

The idea of (6.2) comes from the volatility of financial data [73,74]. While the data set seems
to be approximately uncorrelated, it is commonly accepted that they are not independent because the volatility clustering [75]. We use logarithm of squared $r_I$ rather than squared $r_I$ themselves because $r_I$ is highly skewed while log transformation reduces this skewness and makes unconditional distributions closer to the normal distribution. Thus, in the experiments we use $v_I$ as the initialized GSL data for processing.

The power spectral density (PSD) of $v_I$ is plotted in fig. 6.2. As shown in the picture, the PSD of the initialized GSL data is decaying like power-law distribution in the frequency domain. This is the same as the PSD of $1/f$ noise and may suggest long-range dependence. Therefore, we can test the data with some Hurst parameter estimation methods mentioned in Chapter 2 to see if the estimated results are between 0.5 and 1.0. We have the following experimental results. Some of the Hurst parameter estimators are obtained from using the SELFIS tool in [76].

- **R/S Analysis** [76]

  This method uses the rescaled range statistic. The R/S statistic is the range of partial
Fig. 6.2: Power spectral density of the volatility of GSL water-surface elevation.

Fig. 6.3: Hurst parameter estimation of Great Salt Lake data using R/S analysis.
sums of deviations of a time-series from its mean, rescaled by its standard deviation. A log-log plot of the R/S statistic versus the number of points of the aggregated series should be a straight line with the slope being an estimation of the Hurst exponent. As shown in fig. 6.3, the estimator can not fit the data with a line for estimation. Thus, the result \( H = 0.172 \) may not be valid.

- **Aggregated Variance Method** [76]

This method plots log scale of the sample variance versus the block size of each aggregation level. It can be observed from fig. 6.4 that the estimator has fitted a straight line from the results. If the series is long-range dependent then the slope \( \beta \) of the fitted straight line is greater than \(-1\). The estimation of \( H \) is given by \( H = 1 + \frac{\beta}{2} \). In this case \( H = 0.602 \).

- **Absolute Value Method** [76]

By inspection from fig. 6.5, it is obvious that the log-log plot of the aggregated GSL water level versus the absolute first moment of the aggregated GSL water level is a straight line. The slope of the fitted line should equal to \( H - 1 \). Therefore, the Hurst parameter estimated by absolute value method is 0.632.
Fig. 6.5: Hurst parameter estimation of Great Salt Lake data using absolute value method.

Fig. 6.6: Hurst parameter estimation of Great Salt Lake data using variance of residuals method.
Fig. 6.7: Hurst parameter estimation of Great Salt Lake data using periodogram method.

- **Variance of Residuals Method** [76]

  As discussed in Chapter 2, the method uses the least-squares method to fit a line to the partial sum of each block \( m \). In fig. 6.6, a log-log plot of the aggregation level versus the average of the variance of the residuals after the fitting for each level is a straight line with slope of \( H/2 \). \( H = 0.630 \).

- **Periodogram Method** [76]

  In fig. 6.7, the logarithm of the spectral density of the GSL water level versus the logarithm of the frequencies has been plotted. The slope in the figure provides an estimate of \( H = 0.547 \).

- **Wavelet-Based Method**

  Similar as Chapter 5, the mean value of the Hurst parameter estimations by local wavelet-based method is 0.79367 with a 95% confidence interval as shown in fig. 6.8.

- **FrFT-Based Estimator**

  As shown in fig. 6.9, the order of FrFT-based estimator equals to 0.2 results in the minimum variance for estimating Hurst parameter of GSL data.

  Setting 0.2 as the order of FrFT-based estimator, we get the Hurst parameter estimation in fig. 6.10. It can be observed from the picture that the mean value of the estimated local
The local estimates of $H(r)$. Scales: $(j_1, j_2) = (2, 5)$, $w = 400$

Changes in local wavelet spectra. Deviations from linearity.

Estimated $H(r)$
Mean = 0.79367

Fig. 6.8: Hurst parameter estimation of GSL data by wavelet-based estimator.

Fig. 6.9: The variances of Hurst parameter estimations with different orders of FrFT for Great Salt Lake data.
Hurst parameters is 0.63909.

For Great Salt Lake data, all the valid Hurst parameter estimation methods give out results between 0.5 and 1.0. Each estimator looks at a different property of a given time-series. Thus, applying all those estimators to the GSL water-surface-elevation data provides with a more complete overall picture of its possible self-similar nature. From the experimental results, we can conclude that the volatility of the GSL water-surface-elevation data possesses long-range dependence and it is possible to fit the fractional order system model to it and do forecasting.

6.3 ARFIMA Modeling and Forecasting

As discussed in sec. 2.3, ARFIMA model may give a better fit for long-range dependent time series. In this section, we will apply ARFIMA model to fit the volatility of GSL water-surface-elevation data. The Exact Maximum Likelihood (EML) method is used for
estimations of parameters of ARFIMA model. Let $y$ be the sample time series. The log-
likelihood of the estimation is simply
\[
\log L(d, \phi, \theta, \beta, \sigma^2_\varepsilon) = -\frac{T}{2} \log(2\pi) - \frac{1}{2} \log |\Sigma| - \frac{1}{2} z' \Sigma^{-1} z, \tag{6.3}
\]
where $z = y - \mu$ with $\mu$ the mean value of $y$; $\beta$ is the regression parameter in $z = \beta x$ where $x$ is a vector with all ones; $\sigma^2_\varepsilon$ is the variance of $\varepsilon_t$ in (2.5); $T$ is the size of sample series; and $\Sigma$ is a Toeplitz matrix of autocovariances of $y$. By writing $\Sigma = R\sigma^2_\varepsilon$, (6.3) becomes
\[
\log L(d, \phi, \theta, \beta, \sigma^2_\varepsilon) \propto -\frac{1}{2} \log |R| - \frac{T}{2} \log \sigma^2_\varepsilon - \frac{1}{2\sigma^2_\varepsilon} z' R^{-1} z. \tag{6.4}
\]
Then, by differentiating both sides with respect to $\sigma^2_\varepsilon$, we have
\[
\hat{\sigma}^2_\varepsilon = T^{-1} z' R^{-1} z. \tag{6.5}
\]
Finally, the concentrated likelihood is like
\[
l_c(d, \phi, \theta, \beta) = -\frac{T}{2} \log(2\pi) - \frac{T}{2} - \frac{1}{2} \log |R| - \frac{T}{2} \log(T^{-1} z' R^{-1} z). \tag{6.6}
\]

The length of the available GSL water-surface-elevation data is 2771. We use the
volatility of the first 2748 (1880.1 $\sim$ 2001.4) data samples as experimental time series for
fitting ARFIMA model. Then, the identified model will be used for forecasting the GSL
water-surface-elevation data of the next year, which are 24 measurements. The forecasted
data will be compared with the real data (data samples from 2748 to 2771) for the model
validation.

Using EML method, we get estimated $d = 0.3232$, $\phi(1) = 0.732$, $\phi(2) = -0.204$,
$\theta(1) = -0.242$ and $\theta(2) = -0.131$ for the ARFIMA model. Then, the forecasts of the
next 24 measurements are compared with the actual ones in fig. 6.11. As we can see,
the forecasted values successfully approach to the actual values of the volatilities of GSL
water-surface-elevation data. The mean error of the forecasts is -0.0157, and the variance
Fig. 6.11: GSL volatility and a year (24) out-of-sample ARFIMA forecasts (all previous data used).

is 0.0094.

If we use less data samples for fitting the ARFIMA model, e.g., 1099 data samples from 1650 (1955.6) to 2748 (2001.4), we get the forecasts shown in fig. 6.12. As we can see, at the starting time scales the forecasts are close to actual values, while the forecasts become misleading as the time scales become large. The mean error of the forecasts is -0.0439 and the variance is 0.0165. Both of them are greater than the forecasts in fig. 6.11. Therefore, more previous data samples result in better forecasting performance.

By using the forecasted volatility values with (6.1) and (6.2), the actual water-surface-elevation forecasts can easily be obtained.

In conclusion, the LRD in the volatility of GSL water-surface-elevation has been explored. The Hurst parameter of the GSL data has been successfully estimated by our FrFT-based estimator with comparison of most existing Hurst parameter estimation methods. Based on what we explored, ARFIMA model has been applied to the GSL data for forecasting. The quality of prediction is affected by the amount of previous data samples.
Fig. 6.12: GSL volatility and a year (24) out-of-sample ARFIMA forecasts (the 1099 previous samples used).
Chapter 7

Conclusion

7.1 Summary and Conclusions

Fractional order signal processing techniques are very useful in analyzing real-world data and are developing incredibly fast. In this thesis, we have reviewed various existing FOSP techniques and found out their relationships with fractional calculus. The output of fractional systems can have both short- and long-range dependence. While the autocorrelation of short-range dependence corresponds to an exponential law distribution, the autocorrelation of long-range dependence corresponds to a power law distribution.

Fractional Fourier transform which is a generalization of the common Fourier transform provides a richer picture of the time-frequency analysis of time series. It is especially useful for analyzing biocorrosion signals such as electrochemical noises of biosensors.

In this thesis, fractional Fourier transform is first introduced for analyzing long-range dependence. The FrFT-based Hurst parameter estimator outperforms the other existing methods in most cases, and is more robust to the nonstationary effects. Besides, the local analysis method we implemented provides a more accurate Hurst parameter estimation. On the other hand, different orders of FrFT may result in different variances of Hurst parameter estimation. It is important to choose the right order for minimum variance before testing. Finally, choosing the right starting scales and window sizes in our estimator may help characterizing the long-range dependence.

Fractional Gaussian noises are long-range dependent processes. Testing results of their Hurst exponents gave us strong evidences of the accuracy, efficiency, and robustness of our FrFT-based estimator. There are many methods of generating FGN. The bias of the generating methods should be considered.
Also in this thesis, the FOSP techniques has been applied to some real-world data like biocorrosion noises and the Great Salt lake water-surface-elevation data. In Chapter 5, our experiments show that bioelectrochemical corrosion noises possess long-range dependence under certain experimental conditions. Their dependence structure is intricate and may have some nonstationary parts. The FrFT-based estimator is efficient enough to characterize the long-range dependence in biocorrosion noises.

In Chapter 6, the LRD in the water-surface-elevation data of Great Salt Lake is analyzed. After processing the data with the same processing method for financial data, its LRD properties are found out. The LRD properties provide the possibility of fitting fractional order models. ARFIMA model has been successfully implemented for fitting the volatility of GSL water-surface-elevation data. It has been found to provide satisfactory forecast of the water-levels in the future. The experiments also show that more previous data samples result in better forecasting.

7.2 Future Work

There is a lot remaining in the fractional order signal processing field for us to explore. Here are some suggested future research efforts.

1. ARFIMA model is the most popular model for modeling LRD time series. Since we have analyzed the long-range dependence in the Great Salt Lake data, how can we fit the ARFIMA model to forecast the water-level of GSL in the future?

2. Based on our cooperations with the researchers in BIE department of Utah State University, we would like to apply the FOSP techniques in a wider range of bio-signals including biocorrosion processes with nanoparticles coated on bioimplants.

3. Another very important application is the financial data. The analysis and forecast of the stock market time series with fractional order signal processing techniques is also interesting to us.
4. For the ARFIMA model, what if there is an extra input signal? The ARFIMAX is also one of our research interests because of the variety of time series.

5. The Kalman filter is an efficient recursive filter which estimates the state of a dynamic system from a series of incomplete and noisy measurements. It is also very important for us to explore the fractional Gaussian noise with Kalman filter.

6. Since the fractional Fourier transform generalizes Fourier transform, it is interesting to check the time-frequency distribution of stochastic processes by FrFT. Besides, the optimal filtering in FrFT domain will also be an interesting research topic in the future.
References


Appendix
Bioelectrochemical Experiments

- Preparation of Electrodes

The stainless steel material was cut in the cylindrical shape. Then this material was cleaned with the help of sand paper. A copper wire is soldered at one end of the stainless steel electrode. Then, the total surface of the electrode is coated with an insulating material (epoxy) leaving only the surface of the electrode. The coated electrode with epoxy was left for 24 hours to dry totally. The connectivity of the surface of the electrode and that of the copper wire was checked with the help of a multimeter. The stainless steel electrode prepared this way was found to be conductive. For the tests, these electrodes are prepared by polishing it with the polishing pads. Two different polishing pads were used to clean the stainless steel surface before the start of every new experiment. The electrodes were then rinsed off with distilled water and they were made ready to use. While polishing, care was taken that there are no scratches on the surface of the stainless steel electrode. The Ag/AgCl reference electrode was stored in the KCl solution with 3M concentration. The Pt wire was used as the counter electrode.

- Bacteria Growth and Attachment

The bacteria, Acidithiobacillus Ferrooxidans (ATF), were inoculated in 250 ml growth medium (No.2049 ATCC medium), cultivated on shaker for 72 hours and the temperature was kept at 30 °C. The cultivation solution is then filtered, and the filtrate is centrifuged at 11,000 rpm for 15 minutes. The precipitation after the centrifuge is suspended by 0.02M H$_2$SO$_4$ and again centrifuged at 9,000 rpm for 10 minutes. This washing via H$_2$SO$_4$ and centrifuge are repeated for 2-3 times until there is no orange color of Fe$^{3+}$ with precipitation of bacteria. The concentration of the bacteria in 0.02M H$_2$SO$_4$ was decided by spectrophotometer at wavelength of 500 nm; calculation of the cell density refers to method from Jean Lacombe Barron, which means that an optical OD$_{500}$ value of 0.2 corresponds to $1 \times 10^9$ cells/ml. With suspending in 0.02M H$_2$SO$_4$, bacteria concentration was finally adjusted to $1 \times 10^9$ cells/ml. To attach bacteria onto the electrode surface, three stainless steel
electrodes were simultaneously immersed into $H_2SO_4$ solution in which the bacteria concentration had been adjusted to $1 \times 10^9$ cells/ml. All electrodes were immersed for 72 hours, and then electrochemical test such as electrochemical noise (ECN) was run on electrodes in three artificial saliva solutions respectively.

- **Preparation of Artificial Saliva Solutions**

The three different types of simulated saliva solutions were used for the stainless steel electrode, namely, (A) Jenkin’s solution, (B) Tomasi’s solution and (C) NaCl solution. The chemical composition of the Jenkin’s solution in 250 ml of distilled water is: $NaCl$ 0.2125 mg, $KCl$ 0.3 mg, $CaCl_2 \cdot 2H_2O$ 0.0375 mg, $MgCl_2 \cdot 6H_2O$ 0.0125 mg, $K_2HPO_4$ 0.0875 mg, $KSCN$ 0.0250mg, $NaF$ 0.0025 mg, $H_2O_2$ 0.0750 mg and Sorbic acid 0.0125mg. The chemical composition of the Tomasi’s solution in 250 ml of distilled water is: $NaCl$ 0.1685 mg, $KCl$ 0.24 mg, $CaCl_2 \cdot 2H_2O$ 0.02925 mg, $MgCl_2 \cdot 6H_2O$ 0.010125 mg, $K_2HPO_4$ 0.02275 mg. The chemical composition of the NaCl solution in 250ml of distilled water is: $NaCl$ 2.5 mg.