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Conservatism-free Robust Stability Check of Fractional-order Interval Linear Systems^{*}

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Abstract: This paper presents a necessary and sufficient robust stability condition of fractional-order interval linear time invariant systems. The state matrix is considered a parametric interval uncertain matrix and fractional commensurate order is considered belonging to $1 \leq \alpha < 2$. Using existence condition of Hermitian $P = P^*$ for complex Lyapunov inequality, we show that a fractional-order interval linear system is robust stable if and only if there exist Hermitian matrices $P = P^*$ such that complex Lyapunov inequalities are satisfied for all vertex matrices, which is a set of selected matrices. Two numerical examples are presented to verify the validity of the proposed approach.

1. INTRODUCTION

Recently, the fractional order linear time invariant (FO-LTI) systems have attracted lots of attention in control systems society (Lurie, 1994; Podlubny, 1999b; Oustaloup et al., 1995, 1996; Raynaud and Zergainoh, 2000) even though fractional-order control problems were investigated as early as 1960's (Manabe, 1960, 1961). The fractional order calculus plays an important role in thermodynamics, mechatronics systems, chemical mixing, and biological system as well. It is recommended to refer to (Oustaloup, 1981; Axtell and Bise, 1990; Vinagre and Chen, 2002; Xue and Chen, 2002; Machado, 2002; Ortigueira and Machado, 2003) for the further engineering applications of FO-LTI systems. In the field of fractional-order control systems, there are many challenging and unsolved problems such as robust stability, input-output stability, internal stability, robust controllability, frequency domain analysis, robust observability, etc. (Rugh, 1993; Vidyasagar, 1971; Skaar et al., 1988; Matignon, 1996, 1998c,a,b; Bonnet and Partington, 2000; Matignon and d'Andréa Novel, 1996; Moze and Sabatier, 2005). In the fractional order controller, the fractional order integration or derivative of the output error is used for the current control force calculation. For the robust stability analysis of the fractional-order systems, model uncertainty, disturbance, and stochastic noises have been considered. Recently, parametric interval concept has been utilized to take account of the parameter variation in fractional-order uncertain dynamic systems (Petráš et al., 2004, 2005; Chen et al., 2005b,a; Ahn et al., 2007). Noticeably, matrix perturbation theory was used in (Chen et al., 2005a) to find the ranges of interval eigenvalues and Lyapunov inequality was used in (Ahn et al.,

2007) to reduce the conservatism in the robust stability test of interval uncertain FO-LTI systems. However, (Chen et al., 2005a; Ahn et al., 2007) do not provide exact robust stability condition; instead the methods proposed in (Chen et al., 2005a; Ahn et al., 2007) estimate the robust stability property under some restrictive conditions. This paper is an extension of (Ahn et al., 2007); specifically this paper presents a necessary and sufficient condition for the robust stability of fractional-order linear interval systems with fractional commensurate order of $1 \leq \alpha < 2$.

In the following section, we provide some backgrounds of FO linear interval systems. In Section 3, main results of the paper are presented. In Section 4, two examples are provided to validate the results. Conclusion will be given in Section 5.

2. ROBUST STABILITY OF FRACTIONAL-ORDER LINEAR INTERVAL SYSTEMS

Let us consider the FO-LTI systems governed by the following state-space form:

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t) \quad (1)$$

where $\alpha \in \mathbf{R}$, $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, and α is the fractional commensurate order. The fractional-order interval linear time invariant systems (FO-ILTI) are defined as the FO-LTI systems whose "A" matrix is interval uncertain in parameter-wise. That is, when "A" matrix is defined as $A \in A^I = [a_{ij}^I]$ where a_{ij}^I is lower and upper bounded such as $a_{ij}^I := [a_{ij}, \bar{a}_{ij}]$, we call the system (1) fractional-order interval linear time invariant systems (FO-ILTI). Note that A^I can be also defined as $[\underline{A}, \bar{A}]$

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where $\underline{A} = [\underline{a}_{ij}]$ and $\overline{A} = [\overline{a}_{ij}]$. We call $A \in A^I$ interval matrix; \underline{A} lower boundary matrix; and \overline{A} upper boundary matrix. Moreover, we define vertex matrices of A^I such as $A^v = \{A = [a_{ij}] : \forall a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}\}$. Thus, the FO-ILTI system have parametric interval uncertainties in elements of A matrix. The robust stability problem of $0 < \alpha < 1$ was studied in (Chen et al., 2005a); so this paper focuses on the robust stability of $1 < \alpha < 2$, which was also studied in (Ahn et al., 2007) with some restrictions. Let us use Caputo definition for fractional derivative of order α of any function $f(t)$, which allows utilization of initial values of classical integer-order derivatives with known physical interpretations (Caputo, 1967; Podlubny, 1999a):

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau) d\tau}{(t - \tau)^{\alpha+1-n}}, \quad (2)$$

where n is an integer satisfying $n - 1 < \alpha \leq n$. As commented in (Moze and Sabatier, 2005), with $1 \leq \alpha < 2$, when A matrix is deterministic without uncertainty, the stability condition for $\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)$ is clearly

$$\min_i |\arg(\lambda_i(A))| > \alpha\pi/2, \quad i = 1, 2, \dots, N. \quad (3)$$

Thus, the robust stability condition of FO-ILTI systems is derived as follows:

$$\min_i |\arg(\lambda_i(A))| > \alpha\pi/2, \quad i = 1, 2, \dots, N; \quad \forall A \in A^I. \quad (4)$$

For more detailed introduction to the robust stability of FO-ILTI systems, the interested readers are referred to (Ahn et al., 2007).

3. MAIN RESULTS

Based on (Molinari, 1975), it is easily proved that the FO-ILTI system is robust stable if and only if there exist positive definite Hermitian matrices $P = P^* > 0$ and $Q = Q^* > 0$ such that the following equality holds:

$$\beta PA + \beta^* A^T P = -Q, \quad \forall A \in A^I \quad (5)$$

where $\beta = \eta + j\zeta$, and η and ζ are defined from $\tan(\pi/2 - \theta) = \eta/\zeta$ with $\theta = \frac{(\alpha-1)\pi}{\pi}$ (see Fig. 1 of (Ahn et al., 2007)). In (Ahn et al., 2007), a sufficient condition, which considers $P = I$, was developed. The condition given in (5) is equivalent to $\beta PA + \beta^* A^T P < 0$, $P = P^* > 0$, $\forall A \in A^I$, which means that eigenvalues of $\beta PA + \beta^* A^T P$ are negative. Therefore, we know that equality (5) holds if and only if the maximum eigenvalue of $\beta PA + \beta^* A^T P$ is negative (i.e., $\bar{\lambda}(\beta PA + \beta^* A^T P) < 0$). Let us summarize the above argument in the following lemma:

Lemma 1. Interval fractional order LTI system is robust stable if and only if there exists a positive definite Hermitian matrix $P = P^*$ such that $\bar{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^I$.

However it is impossible to check the condition of the above lemma because there are infinite number of matrices A such that $A \in A^I$. In what follows, we present that a set of finite matrices can be used for checking the condition of Lemma 1.

Let us first notice that since $\beta PA + \beta^* A^T P$ is a Hermitian matrix for any $A \in A^I$, the maximum eigenvalue is calculated as

$$\bar{\lambda} = \max_{A \in A^I} \left(\max_{\|x\|=1} x^*(\beta PA + \beta^* A^T P)x \right) \quad (6)$$

where x is a length n column vector, $x = [x_1, x_2, \dots, x_n]^T = [u_1 + jv_1, u_2 + jv_2, \dots, u_n + jv_n]^T$. Here note that since the vector x can be normalized, we can enforce $v_1 = 0$. Let us expand (6) like (10). In (10), $\text{Re}(\cdot)$ means the real part of (\cdot) ; $\text{Im}(\cdot)$ means the imaginary part of (\cdot) ; $u = [u_1, u_2, \dots, u_n]^T$ and $v = [v_1, v_2, \dots, v_n]^T$. If we denote $P = C + jD$, then we can rewrite the right-hand side of (10) like below:

$$2\eta u^T C A u + 2\eta v^T D A u - 2\eta u^T D A v + 2\eta v^T C A v - 2\zeta u^T D A u - 2\zeta v^T C A v + 2\zeta v^T C A u - 2\zeta v^T D A v \quad (11)$$

Using $(CA)_{ij} = \sum_{k=1}^n c_{ik} a_{kj}$ and $(DA)_{ij} = \sum_{k=1}^n d_{ik} a_{kj}$, we rewrite (11) like (12) (note that (12) is on the next page).

Now defining

$$\alpha(k) = \eta u_1 c_{1k} u_1 - \zeta u_1 d_{1k} u_1 + \sum_{i=2}^n (\eta u_i c_{ik} u_1 - \zeta u_i d_{ik} u_1) + \sum_{i=2}^n (\eta v_i d_{ik} u_1 + \zeta v_i c_{ik} u_1) \quad (13)$$

$$\begin{aligned} \beta(k, j) &= u_1 (\eta c_{1k} - \zeta d_{1k}) u_j \\ &+ \sum_{i=2}^n u_i (\eta c_{ik} - \zeta d_{ik}) u_j + \sum_{i=2}^n v_i (\eta c_{ik} - \zeta d_{ik}) v_j \\ &- u_1 (\eta d_{1k} + \zeta c_{1k}) v_j - \sum_{i=2}^n u_i (\eta d_{ik} + \zeta c_{ik}) v_j \\ &+ \sum_{i=2}^n v_i (\eta d_{ik} + \zeta c_{ik}) u_j, \end{aligned} \quad (14)$$

we can simplify the right-hand side of (12) as

$$x^*(\beta PA + \beta^* A^T P)x = 2 \sum_{k=1}^n \alpha(k) a_{k1} + 2 \sum_{k=1}^n \sum_{j=2}^n \beta(k, j) a_{kj} \quad (15)$$

It is required to maximize the right-hand side of (15) to find the maximum eigenvalue ($\bar{\lambda}$) of $\beta PA + \beta^* A^T P$ considering all possible interval uncertainties in $a_{ij} \in a_{ij}^I$. Here, we observe that $\bar{\lambda}$ depends on the signs of $\alpha(k)$ and $\beta(k, j)$. That is, if $\alpha(k) \geq 0$, then $\bar{\lambda}$ occurs at $\overline{a_{k1}}$; otherwise $\bar{\lambda}$ occurs at $\underline{a_{k1}}$. In the same way, if $\beta(k, j) \geq 0$, then $\bar{\lambda}$ occurs at $\overline{a_{kj}}$; otherwise $\bar{\lambda}$ occurs at $\underline{a_{kj}}$. We summarize this observation in the following lemma:

Lemma 2. For a positive definite Hermitian $P = P^*$, the maximum of the quadratic form $x^*(\beta PA + \beta^* A^T P)x$ given in (6) occurs as one of the vertex matrices of $A \in A^I$.

Proof. We need to maximize the following summation:

$$\sum_{k=1}^n \alpha(k) a_{k1} + \sum_{k=1}^n \sum_{j=2}^n \beta(k, j) a_{kj}$$

$$x^*(\beta PA + \beta^* A^T P)x = [u^T - jv^T][(\eta + j\zeta)PA + (\eta - j\zeta)A^T P][u + jv] \quad (7)$$

$$\begin{aligned} &= \eta u^T PAu + j\eta u^T PAv - j\eta v^T PAu + \eta v^T PAu + \eta u^T A^T Pu + j\eta u^T A^T Pv - j\eta v^T A^T Pu \\ &\quad + \eta v^T A^T Pv + j\zeta u^T PAu - \zeta u^T PAv + \zeta v^T PAu + j\zeta v^T PAu - j\zeta u^T A^T Pu + \zeta u^T A^T Pv \\ &\quad - \zeta v^T A^T Pu - j\zeta v^T A^T Pv \end{aligned} \quad (8)$$

$$\begin{aligned} &= \eta u^T PAu + j\eta u^T PAv - j\eta v^T PAu + \eta v^T PAu + \eta(u^T PAu)^* + j\eta(v^T PAu)^* - j\eta(u^T PAu)^* \\ &\quad + \eta(v^T PAu)^* + j\zeta u^T PAu - \zeta u^T PAv + \zeta v^T PAu + j\zeta v^T PAu - j\zeta(u^T PAu)^* + \zeta(v^T PAu)^* \\ &\quad - \zeta(u^T PAu)^* - j\zeta(v^T PAu)^* \end{aligned} \quad (9)$$

$$\begin{aligned} &= 2\eta \text{Re}[u^T PAu] + 2\eta \text{Im}[v^T PAu] - 2\eta \text{Im}[u^T PAv] + 2\eta \text{Re}[v^T PAu] - 2\zeta \text{Im}[u^T PAu] \\ &\quad - 2\zeta \text{Re}[u^T PAu] + 2\zeta \text{Re}[v^T PAu] - 2\zeta \text{Im}[v^T PAu] \end{aligned} \quad (10)$$

$$\begin{aligned} x^*(\beta PA + \beta^* A^T P)x &= 2 \sum_{j=1}^n \sum_{i=1}^n u_i \left(\sum_{k=1}^n \eta c_{ik} a_{kj} - \zeta d_{ik} a_{kj} \right) u_j + 2 \sum_{j=2}^n \sum_{i=2}^n v_i \left(\sum_{k=1}^n \eta c_{ik} a_{kj} - \zeta d_{ik} a_{kj} \right) v_j \\ &\quad - 2 \sum_{j=2}^n \sum_{i=1}^n u_i \left(\sum_{k=1}^n \eta d_{ik} a_{kj} + \zeta c_{ik} a_{kj} \right) v_j + 2 \sum_{j=1}^n \sum_{i=2}^n v_i \left(\sum_{k=1}^n \eta d_{ik} a_{kj} + \zeta c_{ik} a_{kj} \right) u_j \\ &= 2u_1 \left(\sum_{k=1}^n \eta c_{1k} a_{k1} - \zeta d_{1k} a_{k1} \right) u_1 + 2 \sum_{i=2}^n u_i \left(\sum_{k=1}^n \eta c_{ik} a_{k1} - \zeta d_{ik} a_{k1} \right) u_1 \\ &\quad + 2 \sum_{j=2}^n u_1 \left(\sum_{k=1}^n \eta c_{1k} a_{kj} - \zeta d_{1k} a_{kj} \right) u_j + 2 \sum_{j=2}^n \sum_{i=2}^n u_i \left(\sum_{k=1}^n \eta c_{ik} a_{kj} - \zeta d_{ik} a_{kj} \right) u_j \\ &\quad + 2 \sum_{j=2}^n \sum_{i=2}^n v_i \left(\sum_{k=1}^n \eta c_{ik} a_{kj} - \zeta d_{ik} a_{kj} \right) v_j - 2 \sum_{j=2}^n u_1 \left(\sum_{k=1}^n \eta d_{1k} a_{kj} + \zeta c_{1k} a_{kj} \right) v_j \\ &\quad - 2 \sum_{j=2}^n \sum_{i=2}^n u_i \left(\sum_{k=1}^n \eta d_{ik} a_{kj} + \zeta c_{ik} a_{kj} \right) v_j + 2 \sum_{i=2}^n v_i \left(\sum_{k=1}^n \eta d_{ik} a_{k1} + \zeta c_{ik} a_{k1} \right) u_1 \\ &\quad + 2 \sum_{j=2}^n \sum_{i=2}^n v_i \left(\sum_{k=1}^n \eta d_{ik} a_{kj} + \zeta c_{ik} a_{kj} \right) u_j \end{aligned} \quad (12)$$

considering all $x = [u + jv]$, which satisfies $\|x\| = 1$, and all $a_{ij} \in a_{ij}^I = [\underline{a}_{ij}, \overline{a}_{ij}]$. Noticing that $\alpha(k)$ and $\beta(k, j)$ depend on $x = [u + jv]$, let us select a particular x^\dagger , $\|x^\dagger\| = 1$, which determines $\alpha(1) = \alpha(1)^\dagger, \dots, \alpha(n) = \alpha(n)^\dagger$ and $\beta(1, 1) = \beta(1, 1)^\dagger, \dots, \beta(n, n) = \beta(n, n)^\dagger$. Then, for the particular x^\dagger we obtain:

$$\begin{aligned} &\max_{a_{ij} \in a_{ij}^I} \left(\sum_{k=1}^n \alpha(k) a_{k1} + \sum_{k=1}^n \sum_{j=2}^n \beta(k, j) a_{kj} \right) \\ &= \sum_{k=1}^n \alpha(k) \overrightarrow{a_{k1}(S_{\alpha(k)})} + \sum_{k=1}^n \sum_{j=2}^n \beta(k, j) \overrightarrow{a_{kj}(S_{\beta(k, j)})} \end{aligned} \quad (16)$$

where

$$\begin{aligned} \overrightarrow{a_{k1}(S_{\alpha(k)})} &= \begin{cases} \overline{a_{k1}}, & \text{if } \alpha(k) \geq 0; \\ \underline{a_{k1}}, & \text{if } \alpha(k) < 0; \end{cases} \\ \overrightarrow{a_{kj}(S_{\beta(k, j)})} &= \begin{cases} \overline{a_{kj}}, & \text{if } \beta(k, j) \geq 0; \\ \underline{a_{kj}}, & \text{if } \beta(k, j) < 0. \end{cases} \end{aligned} \quad (17)$$

Therefore, since at any arbitrary selection x , the maximum of $\sum_{k=1}^n \alpha(k) a_{k1} + \sum_{k=1}^n \sum_{j=2}^n \beta(k, j) a_{kj}$ occurs at one of vertex matrices of $A \in A^I$, the maximum of $x^*(\beta PA + \beta^* A^T P)x$ occurs at one of vertex matrices of $A \in A^I$.

Now based on Lemma 1 and Lemma 2, we state the following theorem:

Theorem 3. Interval fractional order LTI system is robust stable if and only if there exists a positive definite Hermitian matrix $P = P^*$ such that $\bar{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$.

Proof. (Sufficient) Based on Lemma 2, since $\bar{\lambda}(\beta PA + \beta^* A^T P)$ occurs at one of vertex matrices of A^I , $\lambda_i(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^I$ if $\lambda_i(\beta PA + \beta^* A^T P) \leq \bar{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$. Therefore if there exists a positive definite Hermitian matrix $P = P^*$ such that $\bar{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$, then interval FO-LTI system is robust stable by Lemma 1.

(Necessary) Since there should exist $P = P^*$ such that $\bar{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^I$, it is necessary to ensure the existence of $P = P^*$ such that $\bar{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$ because $A^v \subseteq A^I$.

The superiority of Theorem 3 over Lemma 1 is highlighted in the following remark.

Remark 4. Lemma 1 states that we need to check infinity number of matrices $A \in A^I$ to verify the existence of $P = P^*$ such that (5) holds. However Theorem 3 shows

that a set of selected finite vertex matrices can be enough for checking the existence of $P = P^*$. Therefore a selected finite vertex matrices can be used for checking the robust stability of FO-LTI interval systems.

4. ILLUSTRATIVE EXAMPLES

4.1 Example-1

Consider the following fractional-order linear interval system, which was studied in (Ahn et al., 2007):

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax, \quad A \in A^I \quad (18)$$

where $\alpha = 1.5$, which makes $\beta = \eta + j\zeta = 1 + j$, and $A \in A^I = [\underline{A}, \overline{A}]$ with

$$\underline{A} = \begin{pmatrix} -1.8 & 0.4 & 0.8 \\ -1.2 & -3.6 & 0.8 \\ -0.6 & -1.8 & -3.0 \end{pmatrix}; \quad \overline{A} = \begin{pmatrix} -1.2 & 0.6 & 1.2 \\ -0.8 & -2.4 & 1.2 \\ -0.4 & -1.2 & -2.0 \end{pmatrix}$$

From Theorem 3, we need to check all vertex matrices and for individual vertex matrix $A \in A^v$, there should exist $P = P^* > 0$ such that $\beta PA + \beta^* A^T P < 0$. The existence of $P = P^* > 0$ can be checked by LMI formulation. However, the system considered in this paper is complex system; thus the standard LMI approach should be reformulated based on the following fact¹:

Fact 5. A complex Hermitian H is $H < 0$ if and only if $\begin{pmatrix} \text{Re}(H) & \text{Im}(H) \\ -\text{Im}(H) & \text{Re}(H) \end{pmatrix} < 0$.

Therefore, if there exists $P = P^* > 0$ such that the following holds

$$PB + B^* P < 0 \quad (19)$$

where $B = \begin{pmatrix} \text{Re}(A) & \text{Im}(A) \\ -\text{Im}(A) & \text{Re}(A) \end{pmatrix}$, $\forall A \in A^v$, then we can conclude that the FO-interval LTI system is robust stable. The above condition can be easily checked using MATLAB LMI commands `setlmis`, `lmivar`, `lmiterm`, `getlmis`, `feasp`, `dec2mat`. Using the algorithm given in Fig. 1, we find that there exists $P = P^*$ such that inequality (19) hold for all $A \in A^v$. For example, when $A = \underline{A}$, we obtain the following symmetric matrix:

$$\begin{pmatrix} 0.6224 & 0.0264 & 0.0439 & 0.0000 & 0.0900 & 0.1144 \\ 0.0264 & 0.3861 & -0.0525 & -0.0900 & 0.0000 & 0.1573 \\ 0.0439 & -0.0525 & 0.3978 & -0.1144 & -0.1573 & 0.0000 \\ 0.0000 & -0.0900 & -0.1144 & 0.6224 & 0.0264 & 0.0439 \\ 0.0900 & 0.0000 & -0.1573 & 0.0264 & 0.3861 & -0.0525 \\ 0.1144 & 0.1573 & 0.0000 & 0.0439 & -0.0525 & 0.3978 \end{pmatrix}$$

whose eigenvalues are 0.1868, 0.1868, 0.5128, 0.5128, 0.7068, 0.7068, and when $A = \overline{A}$, we obtain the following symmetric matrix:

$$\begin{pmatrix} 0.8575 & 0.1313 & 0.1613 & -0.0000 & 0.1332 & 0.3652 \\ 0.1313 & 0.7062 & -0.0051 & -0.1332 & 0.0000 & 0.5039 \\ 0.1613 & -0.0051 & 1.0618 & -0.3652 & -0.5039 & -0.0000 \\ -0.0000 & -0.1332 & -0.3652 & 0.8575 & 0.1313 & 0.1613 \\ 0.1332 & 0.0000 & -0.5039 & 0.1313 & 0.7062 & -0.0051 \\ 0.3652 & 0.5039 & -0.0000 & 0.1613 & -0.0051 & 1.0618 \end{pmatrix}$$

whose eigenvalues are 0.2437, 0.2437, 0.7653, 0.7653, 1.6165, 1.6165.

¹ See MATLAB LMI toolbox

4.2 Example-2

Suppose we are given

$$\underline{A} = \begin{pmatrix} -1.8 & 0.4 & 0.8 \\ -1.2 & -3.6 & 0.8 \\ -0.6 & -1.8 & -3.0 \end{pmatrix}; \quad \overline{A} = \begin{pmatrix} 1.2 & 0.6 & 1.2 \\ -0.8 & -2.4 & 1.2 \\ -0.4 & -1.2 & -2.0 \end{pmatrix}$$

Using the same algorithm given in Fig. 1, however we find that there does not exist positive definite matrix P when

$$A = \begin{pmatrix} 1.2 & 0.4 & 0.8 \\ -1.2 & -3.6 & 0.8 \\ -0.6 & -1.8 & -3.0 \end{pmatrix} \in A^v$$

Therefore, the system is not robustly stable.

5. CONCLUSIONS

This paper presented an exact robust stability condition of fractional-order interval linear systems with no conservatism. The motivation of this paper is to remove conservatism of our existing result (Ahn et al., 2007). Using existence condition of Hermitian matrix $P = P^*$ for a complex Lyapunov inequality, we showed that a fractional-order interval linear system is robustly stable if and only if there exist Hermitian matrices $P = P^*$ such that complex Lyapunov inequalities are satisfied for all vertex matrices. The existence of $P = P^* > 0$ was checked by LMI formulation. However, the LMI systems considered in this paper were complex systems; thus the standard LMI approach was reformulated. Two numerical examples were presented to verify the validity of the proposed approach.

REFERENCES

- Hyo-Sung Ahn, YangQuan Chen, and Igor Podlubny. Robust stability test of a class of linear time-invariant interval fractional-order system using Lyapunov inequality. *Appl. Math. & Computation*, 187, 2007.
- M. Axtell and E. M. Bise. Fractional calculus applications in control systems. In *Proceedings of the IEEE 1990 Nat. Aerospace and Electronics Conf.*, pages 563–566, New York, USA, 1990.
- C. Bonnet and J. R. Partington. Coprime factorizations and stability of fractional differential systems. *Systems & Control Letters*, 41, 2000.
- M. Caputo. Linear models of dissipation whose q is almost frequency independent-II. *Geophys. J. R. Astronom. Soc.*, 13:529–539, 1967.
- YangQuan Chen, Hyo-Sung Ahn, and Igor Pudlubny. Robust stability check of fractional order linear time invariant systems with interval uncertainties. In *Proc. of the IEEE Conference on Mechatronics and Automation*, pages 210–215, Niagara Falls, Canada, July 2005a.
- YangQuan Chen, Hyo-Sung Ahn, and Dingyü Xue. Robust controllability of interval fractional order linear time invariant systems. In *Proceedings of ASME 2005 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference*, pages 1–9, Long Beach, CA., Sep. 24–28 2005b.
- Boris J. Lurie. Three-parameter tunable tilt-integral-derivative (TID) controller. *US Patent US5371670*, 1994.
- J. A. Tenreiro Machado. Special Issue on Fractional Calculus and Applications. *Nonlinear Dynamics (Guest Editor)*, 29:1–385, March 2002.

```

clear all

zerobin = dec2bin(0)
beta = 1+j;
Al=[-1.8, 0.4, 0.8; -1.2, -3.6, 0.8; -0.6, -1.8, -3.0];
Au=[-1.2, 0.6, 1.2; -0.8, -2.4, 1.2; -0.4, -1.2, -2.0];
Ar = (Au - Al) ;

for ii=0:1:(2^9-1)
    setlmis([])
    tt=dec2bin(ii,9)
    pp=0;
    for jj=1:1:3
        for kk=1:1:3
            pp=pp+1;
            if tt(pp)==zerobin
                Added(jj,kk) = 0;
            else
                Added(jj,kk) = Ar(jj,kk);
            end
        end
    end
end

AAA= Al+Added;
AA = beta*AAA ;

A = [real(AA), imag(AA) ; -imag(AA), real(AA)]

X = lmivar(1, [6 1]) ;
lmiterm([1 1 1 X], 1, A);
lmiterm([1 1 1 X], A', 1);
lmiterm([1 1 1 0], 0);
lmiterm([-2 1 1 X], 1, 1);
lmis = getlmis;
[tmin, xfeas] = feasp(lmis);

X = dec2mat(lmis,xfeas,X) ;
if min(eig(X))<0
    disp('Not stable')
    break
end

clear X
end

```

Fig. 1. LMI formulation for robust stability test of fractional-order interval linear time invariant systems (FO-ILTI).

- S. Manabe. The non-integer integral and its application to control systems. *JIEE (Japanese Institute of Electrical Engineers) Journal*, 80(860):589–597, 1960.
- S. Manabe. The non-integer integral and its application to control systems. *ETJ of Japan*, 6(3-4):83–87, 1961.
- D. Matignon. Stability result on fractional differential equations with applications to control processing. In *IMACS-SMC Proceedings*, pages 963–968, Lille, France, July 1996.
- D. Matignon. Stability properties for generalized fractional differential systems. In *ESAIM: Proceedings, vol. 5*, pages 145–158. URL: <http://www.emath.fr/Maths/Proc/Vol.5/>, Dec. 1998a.
- D. Matignon. Generalized fractional differential and difference equations: stability properties and modelling issues. In *Proc. of the Mathematical Theory of Networks and Systems symposium (MTNS'98)*, pages 503–506, Padova, Italy, July 1998b.
- D. Matignon. Stability properties for generalized fractional differential systems. In D. Matignon and G. Montseny, editors, *Proceedings of the Colloquium FDS'98: Fractional Differential Systems: Models, Methods and Applications*, number 5, pages 145–158, Paris, December 1998c.

- D. Matignon and B. d'Andréa Novel. Some results on controllability and observability of finite-dimensional fractional differential systems. In *Proc. of the Computational Engineering in Systems and Application Multi-conference*, volume 2, pages 952–956, Lille, France, July 1996. IMACS, IEEE-SMC.
- B. P. Molinari. The stabilizing solution of the discrete algebraic Riccati equation. *IEEE Trans. Automatic Control*, 3:393–399, 1975.
- Mathieu Moze and Jocelyn Sabatier. LMI tools for stability analysis of fractional systems. In *Proceedings of ASME 2005 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference*, pages 1–9, Long Beach, CA., Sep. 24–28 2005.
- Manuel Duarte Ortigueira and J. A. Tenreiro Machado. Special Issue on Fractional Signal Processing and Applications. *Signal Processing (Guest Editors)*, 83(11): 2285–2480, Nov. 2003.
- A. Oustaloup. Fractional order sinusoidal oscillators: Optimization and their use in highly linear FM modulators. *IEEE Transactions on Circuits and Systems*, 28(10):1007–1009, 1981.
- A. Oustaloup, B. Mathieu, and P. Lanusse. The CRONE control of resonant plants: application to a flexible transmission. *European Journal of Control*, 1(2), 1995.
- A. Oustaloup, X. Moreau, and M. Nouillant. The CRONE suspension. *Control Engineering Practice*, 4(8):1101–1108, 1996.
- Ivo Petráš, YangQuan Chen, and Blas M. Vinagre. *Robust stability test for interval fractional order linear systems*, volume 208–210, chapter 6.5: Vincent D. Blondel and Alexander Megretski (Eds.), *Unsolved problems in the mathematics of systems and control*. Princeton University Press, July 2004.
- Ivo Petráš, YangQuan Chen, Blas M. Vinagre, and Igor Podlubny. Stability of linear time invariant systems with interval fractional orders and interval coefficients. In *Proc. of the International Conference on Computational Cybernetics (ICCC04)*, pages 1–4, Viena Technical University, Viena, Austria., 8/30–9/1 2005.
- I. Podlubny. *Fractional Differential Equations*. Mathematics in Science and Engineering. Academic Press, 1999a.
- Igor Podlubny. Fractional-order systems and $PI^\lambda D^\mu$ -controllers. *IEEE Trans. Automatic Control*, 44(1):208–214, 1999b.
- H.F. Raynaud and A. Zergainoh. State-space representation for fractional order controllers. *Automatica*, 36: 1017–1021, 2000.
- W. J. Rugh. *Linear systems theory*. Prentice-Hall, Englewood Cliffs, NJ., 1993.
- S. B. Skaar, A. N. Michel, and R. K. Miller. Stability of viscoelastic control systems. *IEEE Transactions on Automatic Control*, 33(4): 348–357, April 1988.
- Mathukumalli Vidyasagar. A characterization of e^{At} and a constructive proof of the controllability criterion. *IEEE Trans. Automatic Control*, 60(2):370–371, 1971.
- Blas M. Vinagre and YangQuan Chen. Lecture notes on fractional calculus applications in automatic control and robotics. In Blas M. Vinagre and YangQuan Chen, editors, *The 41st IEEE CDC2002 Tutorial Workshop # 2*, pages 1–310. [Online] http://mechatronics.ece.usu.edu/foc/cdc02.tw2_1n.pdf, Las Vegas, Nevada, USA, 2002.
- Dingyü Xue and YangQuan Chen. A comparative introduction of four fractional order controllers. In *Proc. of The 4th IEEE World Congress on Intelligent Control and Automation (WCICA02)*, pages 3228–3235, Shanghai, China, June 2002. IEEE.

Invited Session Proposal for IFAC World Congress 2008: “Fractional differentiation and its applications”

Organizers: Luc Dugard* and Jocelyn Sabatier**

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Description and Scope: In the last two decades, fractional (or non integer) differentiation has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics signal and image processing and notably control theory. In this last field, some important considerations such as modeling, curve fitting, system identification, stability, controllability, observability and robustness are now linked to long-range dependence phenomena. The growing research and development on fractional calculus in the areas of mathematics, physics and engineering is characterized by the number of sessions now organized in international conferences, by special issues reserved by journal editors, and of articles in conferences and journals. In the field of automatic control, fractional differentiation remains a not well known tool. The scope of the session is thus to show the interest of fractional differentiation to the automatic control community. To reach this goal, it is chosen to present the state of the art and some recent developments on fractional systems in the field of system analysis and modelling, observation and control, both on theoretical and application aspects.

This session is a double session constituted of a keynote lecture and ten papers.

List of Participants

[1-2]

Title: “An overview of the CRONE approach in system analysis, modelling and identification, observation and control”

Authors: A. Oustaloup (*IMS-LAPS, Université Bordeaux 1 France; email: alain.oustaloup@laps.ims-bordeaux.fr*)

Abstract (or Brief Description): The goal of the paper is to present the fundamental definitions connected to fractional differentiation and to present an overview of the CRONE approach in the field of system analysis, modelling and identification, observation and control. Industrial applications of fractional differentiation are also described in this paper. Some recent developments are also presented.

[3]

Title: “Fractional variational principles and fractional quantization”

Authors: D. Baleanu (*Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University- 06530, Ankara, Turkey; email: dimitrubaleanu@yahoo.co.uk*)

Abstract (or Brief Description): A fractional derivative is nothing more than an operator which generalizes the ordinary derivative. As one would expect, there are several ways to set up a fractional derivative, and it is not unusual to see nowadays various different definitions. Variational calculus and fractional calculus have played a significant role in various areas of applied sciences. Recently, the fractional variational principles gained importance in studying the fractional mechanics and various versions are investigated. The fractional Lagrangians are constructed from the classical Lagrangians by replacing the classical derivatives with one chosen fractional derivatives and the fractional Euler-Lagrange equations are obtained as a result of a fractional variational procedure.

In this paper we propose a new method of finding the Euler-Lagrange equations by using the fractional generalization of the Faa di Bruno formula. The 1+1 field formalism was used in order to obtain the corresponding fractional Euler-Lagrange and Hamilton equations. The classical results are reobtained when $\alpha=1$.

[4]

Title: “Structural properties of linear discrete-time fractional-order systems”

Authors: M. Bettayeb (*Electrical and Computer Engineering Department, University of Sharjah, UAE; email: maamar@sharjah.ac.ae*) S. Djennoune (*Automatic Control Department, Mouloud Mammeri University of Tizi-Ouzou, Algeria ; email: s_djennoune@yahoo.fr*), S. Guermah (*Automatic Control Department, Mouloud Mammeri University of Tizi-Ouzou, Algeria*)

Abstract (or Brief Description): In this communication, some results on the analysis of the reachability, controllability and observability of linear discrete-time fractional order systems are given. Mathematical conditions for checking the controllability and the observability of such systems are developed. Furthermore, the concepts of the controllability realization index, the observability realization index and the structure realization index are introduced. The new concepts will be useful for future investigations of control theory of discrete-time fractional order systems.

[5]

Title: “Half-order modeling of synchronous machines - Application to stability study”

Authors: O. Enacheanu, D. Riu, N. Retières (*G2Elab/ENSIEG – Domaine universitaire – BP 46 – 38402 Saint-Martin d’Hères cedex - France; email: Octavian.Enacheanu, Delphine.Riu, Nicolas.Retiere }@g2elab.inpg.fr*)

Abstract (or Brief Description): For several years, a very close attention has been paid to the study of the electrical supply networks in order to improve their dimensioning, their quality, their safety and their performances. Indeed, the electrical supply networks become increasingly complex, with the appearance of many and new sources of various nature and the increase power electronics devices which are used as interfaces of renewable sources or devices.

To bring solutions to problems involved in the structure of the new electrical supply networks is a difficult task. Thus, for example, fine modeling on a broad operating range of the various components is delicate and generally leads to the simulation of very high order systems, i.e. comprising a very great number of parameters. One thus prefers to direct oneself towards a reduced of electrical supply networks, valid modeling on a more restricted operating range.

In the last years, work on the modeling of electric machines concerned the use of the fractional order derivation. They thus led to the development of fine, precise and compact models, and made it possible to highlight the closed links which exist between the fractional modeling of order and the fractal theory.

These problems do not relate to only the modeling of the components of the electrical supply networks. One must hold of it account also in the study of stability of the networks where a precise model is necessary.

In this article, we were interested in the study of stability of an embedded electrical supply network or an unspecified electric system, in which the synchronous machines are modelled by fractional order systems. The first objective of this work is to extend the field of application of the fractional order systems since until now, they had been useful only at ends of modeling. We thus compared using the methods of traditional characterization of stability (criterion of Nyquist, modal approaches) the results obtained from integer and fractional order models of a synchronous machine.

[6]

Title: “Synthesis of Havriliak-Negami functions for time-domain system identification”

Authors: L. Sommacal, Pierre Melchior, Rachid Malti, and Alain Oustaloup (IMS - UMR 5218 CNRS (Université Bordeaux 1, ENSEIRB, ENSCP), Département LAPS (Automatique, Productique et Signal) 351 cours de la Libération - F33405 TALENCE Cedex - FRANCE; email: pierre.melchior@laps.ims-bordeaux.fr),

Abstract (or Brief Description): Fractional differentiation models have proven their usefulness in representing high dimensional systems with only few parameters. Generally, two elementary fractional functions are used in time-domain identification: Cole-Cole and Davidson-Cole. A third elementary function, named Havriliak-Negami, generalizes both previous ones and is particularly dedicated to dielectric systems. The use of this function is however not very popular in time-domain identification because it has no simple analytical impulse response. The only synthesis method of Havriliak-Negami elementary functions proposed in the literature is based on diffusive representation which imposes restrictive conditions on fractional orders. A new synthesis method, with no such restrictions, is developed in this paper. For that purpose Havriliak-Negami function is first split into a Davidson-Cole and a complementary function. Both functions are then synthesized in a limited frequency band using poles and zeros recursive distribution developed by Oustaloup (1995).

[7]

Title: “Approximation of high order integer systems by fractional order systems”

Authors: R. Mansouri (*Electrical Engineering Department, Mouloud Mammeri, University of Tizi-Ouzou, Algeria; email: mromero@bec.uned.es*) M. Bettayeb (*Electrical and Computer Engineering Department, University of Sharjah, UAE; email: maamar@sharjah.ac.ae*) S. Djennoune (*Automatic Control Department, Mouloud Mammeri University of Tizi-Ouzou, Algeria ; email: s_djennoune@yahoo.fr*)

Abstract (or Brief Description): Fractional order systems are used in this paper to reduce the dimension of high order integer systems. It is indeed noticed that the fractional order system s^α can be approximated by an integer system with N parameters (N depending on the approximation frequency band). An inverse analysis is used in this paper to obtain approximation of high order integer systems. The method consists in substituting a set of poles and zeros or a set of poles and residues which exhibits a particular distribution by a fractional order model.

[8]

Title: “On bounded real lemma for fractional systems”

Authors: M. Moze, J. Sabatier and A. Oustaloup (*IMS-LAPS, Université Bordeaux I France; email: Jocelyn.sabatier@laps.ims-bordeaux.fr*)

Abstract (or Brief Description): This paper proposes an extension of the bounded real lemma for fractional systems. The LMI relation proposed permits as for the integer case, the computation of the H_∞ norm of the system but doesn't ensure its stability. The proposed LMI is used to compute the H_∞ norm of a sensitivity function of the CRONE suspension.

[9]

Title: “Conservatism-free Robust Stability Check of Fractional-order Interval Linear Systems”

Authors: Hyo-Sung Ahn (*Department of Mechatronics, Gwangju Institute of Science and Technology (GIST), 1 Oryong-dong, Buk-gu, Gwangju 500-712, Korea; email: hyosung@gist.ac.kr*) YangQuan Chen (*Center for Self-Organizing and Intelligent Systems (CSOIS), Dept. of Electrical and Computer Engineering, 4160 Old Main Hill, Utah State University, Logan, UT 84322-4160, USA; email: yqchen@ece.usu.edu*)

Abstract (or Brief Description): This paper presents a necessary and sufficient robust stability condition of fractional order interval linear time invariant systems. The state matrix is considered a parametric interval uncertain matrix and fractional commensurate order is considered belonging to $1 < \alpha < 2$. Using existence condition of Hermitian $P = P^*$ for complex Lyapunov inequality, we show that a fractional-order interval linear system is robust stable if and only if there exist Hermitian matrices $P = P^*$ such that complex Lyapunov inequalities are satisfied for all vertex matrices, which is a set of selected matrices. Two numerical examples are presented to verify the validity of the proposed approach.

[10]

Title: “Computation of Stability Margins for Uncertain Fractional-order Systems using Interval Constraint Propagation”

Authors: S. V. Paluri Nataraj (*Systems & Control Engg., IIT Bombay, India; email: nataraj@sc.iitb.ac.in*), K. Rambabu (*Systems & Control Engg., IIT Bombay, India; email: ram@sc.iitb.ac.in*)

Abstract (or Brief Description): The present paper proposes an algorithm for finding the stability margins and cross over frequencies for an uncertain fractional-order system using interval constraint propagation technique. It is first shown that the problem of finding the stability margins can be formulated as a interval constraint satisfaction problem and then solved using branch and prune algorithm. The algorithm guarantees that the stability margins and the cross over frequencies are computed to prescribed accuracy and that these values are reliable in the face of all kinds of computational errors. The other advantage of the method is that the stability margins and crossover frequencies can be computed without the need of frequency response plots. Two examples of uncertain fractional order systems are taken from the literature and their stability margins and cross over frequencies are computed using the proposed algorithm

[11]

Title: “On the Fractional PID Control of a Laboratory Servo System”

Authors: R. S. Barbosa, J. A. Tenreiro Machado, Isabel S. Jesus (*Institute of Engineering of Porto, Rua Dr. António Bernardino de Almeida, 4200-072 Porto, Portugal; mail: {rsb,jtm,isj}@isep.ipp.pt*)

Abstract (or Brief Description): In this paper, several types of fractional-order PID controllers are investigated for the control of a laboratory modular servo system. The fractional controller is more flexible and gives the possibility of adjusting more carefully the dynamical properties of a control system. The servo system is controlled by using a real-

time digital control system based on MATLAB/Simulink. Results are compared with those obtained from classical PID controllers. Simulation and experimental responses are presented and analyzed, showing the effectiveness of the proposed fractional-order algorithms.

[12]

Title: “GPC Control of a Fractional–Order Plant: Improving Stability and Robustness”

Authors: Miguel Romero (*Departamento de Sistemas de Comunicación y Control, UNED, Madrid, Spain; email: mromero@bec.uned.es*) Blas M. Vinagre (*Departamento de Ingeniería Eléctrica, Electrónica y Automática, Universidad de Extremadura, Badajoz, Spain; email: bvinagre@unex.es*) Ángel P. de Madrid (*Departamento de Sistemas de Comunicación y Control, UNED, Madrid, Spai ; email: angel@scc.uned.es*)

Abstract (or Brief Description): This work deals with the use of Generalized Predictive Control (GPC) with fractional order plants. Low integer–order discrete approximations will be used as models to design the controllers. The stability and robustness of the closed loop system will be studied with the Nyquist criterion. Three techniques will be proposed to enhance robustness: the improvement of the model response at low frequencies, the use of the prefilter $T(z^{-1})$, and a new recommendation to chose two of the parameters (the control horizon N_u and the error weighting sequence λ) of the GPC controller