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Linear Dependency and Independency of Interval Vectors: Theory and Its Applications to Robust Controllability Test

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Abstract

To the best knowledge of authors, none of existing literatures is available for the linear independency test of interval vectors. In this note, we present an effective way for checking the linear independency of interval vectors and its possible application to the robust control problem, for example in this note, for solve the robust controllability test and robust observability test of uncertain interval system.

Index Terms

Linear independency, Interval vectors, Robust controllability, Uncertain systems.

I. INTRODUCTION

In robust control, the model uncertainty has been effectively and popularly handled by "interval" concept. Great amount of literatures are available under the name of "interval" for example, interval algebra [1], [2], interval polynomial [3], [4], Schur stability of interval matrices [5], [6], Hurwitz stability of interval matrices [7], [8], [9], interval polynomial matrices [10], eigenvalues of interval matrices [11], [12], [13], and robust control with parameter uncertainty [14], [15]. However, to the best knowledge of authors, nobody has presented any property about the interval vectors although the interval vector concepts have been introduced in [1], [2]. In this note, our ultimate aim is to check the linear independency of interval vectors. We will show that sufficient linear independency condition of interval vectors can be effectively used in checking the robust controllability of the uncertain linear time invariant (LTI) system.

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The note consists of as follows: In Section II, we provide sufficient linear dependency and independency conditions of the interval vectors. In Section III, the developed linear independency condition of interval vectors is used to check the sufficient controllability condition of the uncertain LTI system. Conclusions are given in Section IV.

II. LINEAR DEPENDENCY AND INDEPENDENCY OF INTERVAL VECTORS

Throughout the note, we need the following basic definitions. Our discussions are limited to the real system.

Definition 2.1: A real interval scalar x^I is defined as: $x^I := [\underline{x}, \overline{x}]$, where $\underline{x}, \overline{x} \in \mathcal{R}$ and for all $x \in x^I$, there exists a corresponding λ such that $x = \lambda\overline{x} + (1-\lambda)\underline{x}$ with $0 \leq \lambda \leq 1$, $\lambda \in \mathcal{R}$. The n-dimensional real column interval vector $\mathbf{x}^I$ is defined as: $\mathbf{x}^I := (x_1^I, \ldots, x_n^I)^T$ and the $n \times m$ dimensional real interval matrix is defined from the interval vectors as:

$$X^I := \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \ldots, \mathbf{x}_m^I\right)$$

The interval vector and interval matrix can be written as: $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $X^I = \left[\underline{X}, \overline{X}\right]$. Or, they can be written as: $\mathbf{x}^I = [\mathbf{x}_0 - \Delta\mathbf{x}, \mathbf{x}_0 + \Delta\mathbf{x}]$ and $X^I = [X_0 - \Delta X, X_0 + \Delta X]$, where $\mathbf{x}_0 = \frac{\overline{\mathbf{x}}+\underline{\mathbf{x}}}{2}$, $X_0 = \frac{\overline{X}+\underline{X}}{2}$, $\Delta\mathbf{x} = \frac{\overline{\mathbf{x}}-\underline{\mathbf{x}}}{2}$, and $\Delta X = \frac{\overline{X}-\underline{X}}{2}$.

Based on [1], [2], the following interval arithmetics are used in this note.

Definition 2.2: The intersection of two real interval scalars x^I and y^I is defined as: $x^I \cap y^I := \left\{z \mid z \in x^I \text{ and } z \in y^I\right\}$. The union of two real interval scalars x^I and y^I is defined as: $x^I \cup y^I := \left\{z \mid z \in x^I \text{ or } z \in y^I\right\}$.

Definition 2.3: The addition of two real interval scalars x^I and y^I is defined and calculated as: $x^I \oplus y^I = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right]$, the substraction is $x^I \ominus y^I = \left[\underline{x} - \overline{y}, \overline{x} - \underline{y}\right]$, and the multiplication is

$$x^I \otimes y^I = \left[\min\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}, \max\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}\right]$$

The division should be carefully defined as [2]:

$$\begin{aligned}
\frac{1}{x^I} &= \emptyset \text{ if } x^I = [0,0] \\
&= \infty \text{ if } x^I = (0, 0^+] \\
&= -\infty \text{ if } x^I = [0^-, 0) \\
&= \left[\frac{1}{\overline{x}}, \frac{1}{\underline{x}}\right] \text{ if } x^I > 0 \\
&= \left[\frac{1}{\underline{x}}, \frac{1}{\overline{x}}\right] \text{ if } x^I < 0 \\
&= [-\infty, \infty] \text{ if } \underline{x} < 0 \text{ and } \overline{x} > 0
\end{aligned} \tag{1}$$

Then, the division of two interval scalars is simply defined and calculated as: $x^I \oslash y^I = x^I \otimes \frac{1}{y^I}$.

Definition 2.4: The *ratio* $\mathbf{r}_{xy}$ between two interval vectors is defined and calculated as:

$$\mathbf{r}_{xy} := \mathbf{x}^I \setminus \mathbf{y}^I = \left(x_1^I \oslash y_1^I, \dots, x_n^I \oslash y_n^I\right)$$

The addition, substraction, dot-product, and cross-product of two interval vectors and interval matrices can be defined based on above scalar arithmetics.

The interval arithmetics of a real interval scalar by itself should be distinguished from the arithmetics of two different scalar intervals. For the LTI system [1], we use the following definitions:

Definition 2.5: If x^I does not time dependent (i.e., time invariant), the addition of a real interval scalar x^I is defined and calculated as: $x^I \oplus x^I = [\underline{x} + \underline{x}, \overline{x} + \overline{x}]$, the substraction is $x^I \ominus x^I = 0$, and the multiplication is $x^I \otimes x^I = [\alpha^2, \beta^2]$, where $\alpha = \min\{|\underline{x}|, |\overline{x}|\};\ \beta = \max\{|\underline{x}|, |\overline{x}|\}$. The division is defined as: $x^I \oslash x^I = 1$ if $x^I \neq [0, 0]$.

Remark 2.1: Definition 2.5 is important, let us consider the following simple interval system:

$$\begin{aligned} x_{k+1} &= a^I x_k + bu \\ y_k &= c x_k \end{aligned} \tag{2}$$

where a^I is an interval. For example, in robust control problem, in calculating the impulse response of the system, we need to multiply a^I. If system (2) is time-invariant, then the bounds of $a^I \otimes a^I \otimes \cdots \otimes a^I$ should be calculated by Definition 2.5, not from Definition 2.3. The result calculated by Definition 2.5 will be less conservative than the result calculated by Definition 2.3.

In linear algebra, the following linear (in)dependency condition of the linear vectors is popularly used.

Definition 2.6: Without interval, when n different vectors are given as: $\mathbf{x}_1, \dots, \mathbf{x}_n$, they are called in *linearly independent* iff there exist only trivial solutions ($a_1 = a_2 = \cdots = a_n = 0$) such that $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = 0$. Otherwise, they are *linearly dependent*. If they are linearly independent, any $\mathbf{x}_i$ cannot be produced by any combinations of other vectors.

Now, with the basic definitions given above, we define the linear (in)dependency of interval vectors.

Definition 2.7: With interval, let us suppose we have n different interval column vectors given as: $\mathbf{x}_1^I, \dots, \mathbf{x}_n^I$. They are called in *linearly independent* iff there exist only trivial solutions ($a_1 = a_2 = \cdots = a_n = 0$) such that $a_1\mathbf{x}_1^I + a_2\mathbf{x}_2^I + \cdots + a_n\mathbf{x}_n^I = 0$. Otherwise, we say that the interval vectors are in *linearly dependent*.

Before considering the general case, let us first consider the linear independency of two interval vectors. Supposing that two interval vectors are given as: $\mathbf{x}_1^I$ and $\mathbf{x}_2^I$, and based on Definition 2.7, two interval vectors are linearly

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[1] For linear time varying case, we have to use Definition 2.3.

independent iff there exist only trivial solutions $a_1 = a_2 = 0$ such that

$$a_1 \mathbf{x}_1^I + a_2 \mathbf{x}_2^I = 0. \tag{3}$$

Here, notice that it is not easy to get solutions for (3) directly. However, if we use "ratio" concept, we can check the linear independency property easily, which is expressed in the following theorem:

Theorem 2.1: Two n dimensional LTI interval vectors $\mathbf{x}^I, \mathbf{y}^I$ with $0 \notin x_1^I \cap x_2^I \cap \ldots \cap x_n^I$, $0 \notin y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, are linearly independent iff, from the ratio $\mathbf{r}_{xy}$ of $\mathbf{x}^I, \mathbf{y}^I$, the following equality holds:

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = \emptyset, \tag{4}$$

where $(\mathbf{r}_{xy})_i$ can be defined as $\frac{x_i^I}{y_i^I}$.

Proof: Sufficient: From $a_1 \mathbf{x}_1^I + a_2 \mathbf{x}_2^I = 0$, we have

$$a_1 \left[x_1^I, x_2^I, \ldots, x_n^I \right]^T = -a_2 \left[y_1^I, y_2^I, \ldots, y_n^I \right]^T \tag{5}$$

From Definition 2.4 and Definition 2.5, and using the commutative and associative property of interval scalars, the ratio of each elements are

$$\begin{aligned} x_i^I \oslash y_i^I &= (\mathbf{r}_{xy})_i \\ \Leftrightarrow x_i^I \otimes \frac{1}{y_i^I} &= (\mathbf{r}_{xy})_i \\ \Leftrightarrow x_i^I \otimes \frac{1}{y_i^I} \otimes y_i^I &= (\mathbf{r}_{xy})_i \otimes y_i^I \\ \Leftrightarrow x_i^I &= (\mathbf{r}_{xy})_i \otimes y_i^I \end{aligned} \tag{6}$$

By inserting (6) to the left-hand side of (5), the followings are true:

$$\begin{aligned} a_1 \left[(\mathbf{r}_{xy})_1 \otimes y_1^I, (\mathbf{r}_{xy})_2 \otimes y_2^I, \ldots, (\mathbf{r}_{xy})_n \otimes y_n^I \right]^T &= -a_2 \left[y_1^I, y_2^I, \ldots, y_n^I \right]^T \\ \Leftrightarrow a_1 \left[(\mathbf{r}_{xy})_1, (\mathbf{r}_{xy})_2, \ldots, (\mathbf{r}_{xy})_n \right]^T &= -a_2 \left[1, 1, \ldots, 1 \right]^T \\ \Leftrightarrow \left[(\mathbf{r}_{xy})_1, (\mathbf{r}_{xy})_2, \ldots, (\mathbf{r}_{xy})_n \right]^T &= -\frac{a_2}{a_1} \left[1, 1, \ldots, 1 \right]^T \end{aligned} \tag{7}$$

Here, from (7), we have

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = -\frac{a_2}{a_1}, \tag{8}$$

so, $\frac{a_2}{a_1} = \emptyset$; thus, by definition, only $a_1 = 0$ is the solution, henceforth, since $0 \notin x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ and $0 \notin y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, we have $a_2 = 0$.

Necessity: Let us suppose that

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n \neq \emptyset,$$

then we can have $a_2 = 0$ and $a_1 \neq 0$, or $a_2 \neq 0$ and $a_1 \neq 0$. Thus, by definition, this is not linearly independent. ■

Let us further think the case $0 \in x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ or $0 \in y_1^I \cap y_2^I \cap \ldots \cap y_n^I$.

Theorem 2.2: If $0 \in x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ or $0 \in y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, two interval vectors are then linearly dependent.

Proof: With any a_1 and $a_2 = 0$, or with $a_1 = 0$ and any a_2, the following equality can be true:

$$a_1\mathbf{x}_1^I + a_2\mathbf{x}_2^I = 0.$$

So, By Definition 2.7, the proof is completed. ■

Although above theorems are effective for checking the linear (in)dependency of two interval vectors, it is difficult to extend above theorems to more than 3 interval vectors. Let us suppose that we have three different interval vectors, which are given as: $\mathbf{x}^I, \mathbf{y}^I, \mathbf{z}^I$ and we want to check the linear (in)dependency of them. The first task is to check the linear dependency between two interval vectors. This task can be performed from proceeding results, but we also have to check the linear combination case. That is, we have to check if there exist trivial solutions $a_1 = a_2 = a_3 = 0$ such that

$$a_1\mathbf{x}^I + a_2\mathbf{y}^I + a_3\mathbf{z}^I = 0.$$

It looks quite tough to solve this simple equation, furthermore our ultimate goal is to find the general case such as:

$$a_1\mathbf{x}_1^I + a_2\mathbf{x}_2^I + \cdots + a_3\mathbf{x}_n^I = 0.$$

So, apparently, it is almost impossible to check the linear (in)dependency of the interval vectors [2]. In the sequel, we suggest one simple but very effective sufficient condition for checking the linear (in)dependency of interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ where an $\mathbf{x}_i^I$ is an interval vector in $\mathcal{R}^m$. For the accurate description of our idea, we separately consider three different cases.

$$\text{Case}-1: m > n. \quad \text{Case}-2: m = n. \quad \text{Case}-3: m < n$$

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[2] As far as authors are concerned, nobody has suggested this kind of questions and there is no existing solution.

We only investigate Case-1. In fact, Case-2 and Case-3 can be investigated using the analysis of Case-1. For convenience, the following concepts are necessary. In the following $m \times n,\ m > n$, matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ m_{31} & m_{32} & \dots & m_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m1} & m_{m2} & \dots & m_{mn} \end{bmatrix}$$

let us select whole possible $n \times n$ sub-matrices, which are expressed as:

$$S^1 = \begin{bmatrix} s^1_{11} & s^1_{12} & \dots & s^1_{1n} \\ s^1_{21} & s^1_{22} & \dots & s^1_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s^1_{n1} & s^1_{n2} & \dots & s^1_{nn} \end{bmatrix},\ S^2 = \begin{bmatrix} s^2_{11} & s^2_{12} & \dots & s^2_{1n} \\ s^2_{21} & s^2_{22} & \dots & s^2_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s^2_{n1} & s^2_{n2} & \dots & s^2_{nn} \end{bmatrix},\ \cdots,\ S^k = \begin{bmatrix} s^k_{11} & s^k_{12} & \dots & s^k_{1n} \\ s^k_{21} & s^k_{22} & \dots & s^k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s^k_{n1} & s^k_{n2} & \dots & s^k_{nn} \end{bmatrix}.$$

Then, it is easy to notice that the total number of possible sub-matrices S^i is calculated by the binomial coefficient calculation formula: $k = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$. Sub-matrices S^i are composed of n different row vectors of M. The index of n different row vectors of S^i is represented by a set such as: $s^i = \{\text{index of row vectors of } M\},\ i = 1, \dots, k$. For example, from the matrix

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}$$

we can find total 4 different sub-matrices and they are

$$S^1 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},\ S^2 = \begin{bmatrix} a & b & c \\ d & e & f \\ j & k & l \end{bmatrix},\ S^3 = \begin{bmatrix} a & b & c \\ g & h & i \\ j & k & l \end{bmatrix},\ S^4 = \begin{bmatrix} d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}.$$

So, $s^1 = \{1, 2, 3\}$, $s^2 = \{1, 2, 4\}$, $s^3 = \{1, 3, 4\}$, and $s^4 = \{2, 3, 4\}$. For the accurate translation of our idea, we make definition as:

Definition 2.8: In this note, we call sub-matrices $S_M = \{S^i,\ i = 1, \dots, k\}$ as *square set* and S^i as *sub-square matrices*, and $s_M = \{s^i,\ i = 1, \dots, k\}$ is called *index set* and s^i is called *index*.

Now, we consider the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$. Let us write these interval vectors in an interval matrix form such as:

$$X^I = \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I\right) \tag{9}$$

Then, X^I is an $m \times n$ interval matrix, so based on Definition 2.8, the corresponding square set of X^I can be found as: $S_X = \{S^i,\ i = 1, \ldots, k\}$ where $k = \begin{pmatrix} m \\ n \end{pmatrix}$, and the corresponding index set of X^I can be found as $s_X = \{s^i,\ i = 1, \ldots, k\}$. Here, we further define the center square matrices S_{X_c} and calculate them as:

$$S_{X_c} = \left\{ S_0^i = \frac{\underline{S^i} + \overline{S^i}}{2},\ i = 1, \ldots, k \right\},$$

and define the radius square matrices ΔS_X and calculate them as:

$$\Delta S_X = \left\{ \Delta S^i = \frac{\overline{S^i} - \underline{S^i}}{2},\ i = 1, \ldots, k \right\}$$

For our main result, notating the absolute value of a matrix A by $\|A\| = (|a_{ij}|)$, the following lemma can be adopted from [16].

Lemma 2.1: For interval matrix X^I, let its center matrix X_0 be nonsingular and the spectral radius $\rho\left(\|(X_0)^{-1}\|\,\Delta X\right) < 1$, then X^I is nonsingular.

Now, for the linear independency test of the interval vector set, we suggest the following theorem:

Theorem 2.3: For $S^I \in S_X$, if there exists at least one corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$ such that S_0 is nonsingular and $\rho\left(\|(S_0)^{-1}\|\,\Delta S\right) < 1$, then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: Let us consider $X^I = \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I\right)$, which is an $m \times n,\ m > n$, interval matrices composed of the interval vectors. It is a fact that the column vectors are linearly independent if (and only if in the point of "rank") the rank of X^I is n. Also from the fact that the row rank is equal to the column rank, so if S^I has rank n, then the column rank of X^I is also n. Therefore, if any one of $S^I \in S_X$ has row rank n, then X^I has n column rank. So, by Lemma 2.1, for S_0 and ΔS corresponding to S^I, if S_0 is nonsingular and $\rho\left(\|(S_0)^{-1}\|\,\Delta S\right) < 1$, then X^I has full column rank, because the nonsingular condition is equivalent to the full rank condition. Thus, since the full column rank indicates the linear independency, the proof is completed. ■

Remark 2.2: Theorem 2.3 checks the linear independency of the interval vector set using finite interval matrices set. The key idea of Theorem 2.3 is to investigate the linear independency of the interval vectors on the form of interval matrix. Using the fact that the row rank is equal to column rank and the full rank condition is equivalent to the linear independency condition, Theorem 2.3 easily checks the linear independency of the interval vectors.

However, although Theorem 2.3 is represented in a simple form, the result could be conservative in checking the condition $\rho\left(\left\|(S_0)^{-1}\right\|\Delta S\right) < 1$, because $\left\|(S_0)^{-1}\right\|$ is used. To reduce the conservatism, the following result can be obtained based on Theorem 2.3.

Corollary 2.1: For at least one $S^I \in S_X$ and for its corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$, if there exists a matrix R such that

$$\rho\left(\|I - RS_0\| + \|R\|\,\Delta S\right) < 1,$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: The proof can be completed by the proof of Theorem 2.3 and theorem 3.1 of [16]. ■

Using the proof of Theorem 2.3 and using the results of [16], we also can find the sufficient condition for linear dependency of the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$. Let us use the following lemma for this purpose.

Lemma 2.2: For interval matrix X^I, there exist a matrix R and a natural number p such that, in element-wisely,

$$\left(I + \|I - X_0 R\|\right)_p \leq \left(\Delta X\,\|R\|\right)_p$$

where $p \in \{1, \ldots, n\}$ and $(\cdot)_p$ represents p^{th} column, then interval matrix X^I is singular.

Proof: See theorem 3.3 of [16]. ■

Corollary 2.2: For all $S^I \in S_X$ and for all its corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$, if there exist a matrix R and a natural number p such that, in element-wisely,

$$\left(I + \|I - S_0 R\|\right)_p \leq \left(\Delta S\,\|R\|\right)_p,$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly dependent.

Proof: Theorem 2.3 shows that the interval vectors are linearly independent if there exists at least one S^I such that the conditions of Theorem 2.3 hold. So, to eliminate the case of Theorem 2.3, we have to check all $S^I \in S_X$ for the linearly dependent test. Hence, by checking all S^I and based on the proof of Theorem 2.3 and Lemma 2.2, the proof of Corollary 2.2 can be completed. ■

Above results use the inverse of S_0, but, as commented in [16], this approach may be ineffective in the calculation of S_0^{-1}. Without using the inverse, we can derive the sufficient conditions for checking the linear dependency or independency. Based on theorem 4.1 of [16], the following result can be derived.

Corollary 2.3: For any $S^I \in S_X$, if there exist at least one corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$ such that

$$\lambda_{max}\left(\Delta S^T \Delta S\right) < \lambda_{min}\left(S_0^T S_0\right),$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: By theorem 3.3 of [16] and due to the same reason as Theorem 2.3, the proof is straightforward. ■

The sufficient condition for the linear dependency can also be obtained using eigenvalues as:

Corollary 2.4: For all $S^I \in S_X$, if there exist corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$ such that

$$\lambda_{max}\left(S_0^T S_0\right) \leq \lambda_{min}\left(\Delta S^T \Delta S\right),$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly dependent.

Proof: By theorem 3.3 of [16] and due to the same reasons as Theorem 2.3 and Corollary 2.2, the proof is straightforward. ■

The above results are all sufficient conditions, so the conservatism is indispensable, but if the size of X^I is small, and a few elements are intervals, then the following direct checking method will be less conservative.

Corollary 2.5: If there exists at least one $S^I \in S_X$ such that

$$\det\left(S^I\right) \neq 0$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: Since S^I is full rank if $\det\left(S^I\right) \neq 0$ and due to the same as Theorem 2.3, the proof is immediate. ■

Next, let us consider Case-3 which is $m < n$ with interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ and $\mathbf{x}_i^I \in \mathcal{R}^m$. This problem is dual to Case-1, because, in this case, we can also define "square set" and "index set" as done in Case-1. Then using the same procedure as performed in Theorem 2.3 and in above corollaries, the linear independency and dependency can be checked. However, in Case-3, it is recommended checking the linear dependency of two column vectors using Theorem 2.1 first. This approach will save the computational amount. In Case-2, since X^I is an interval matrix, without making square set and index set, directly Case-1 results can be utilized.

A the expense of the huge computational amount and an iteration loop, we can further derive the necessary and sufficient condition for all theorems and corollaries above. As an example, we only provide the necessary and sufficient condition of Theorem 2.3. The following is the result:

Theorem 2.4: The interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent if and only if, for any $S^I \in S_X$, there is a finite set of subinterval matrices of S^I $\left(\text{i.e., } \left[\underline{(S^I)_i}, \overline{(S^I)_i}\right] \subseteq \left[\underline{S}, \overline{S}\right] = S^I,\ i = 1, \ldots, l\right)$ such that

$$\left[\underline{S}, \overline{S}\right] = \bigcup_{i=1}^{i=l} \left[\underline{(S^I)_i}, \overline{(S^I)_i}\right]$$

and $(S^I)_i$ holds the conditions of Theorem 2.3.

Proof: From [8], [17], [18], the proof can be performed easily. ■

In this section, we suggested “linear dependency” and ”linear independency” problem of “interval vectors”. Even though it looks as an NP hard problem, we solved these problem by forming interval matrices. Our key idea is straightforward and simple, hence the sufficient conditions are also very simple. Notice that in interval vector, in addition to the linear dependency and independency problems discussed in this note, there exist many interesting issues such as “interval vector norm”, “null space of interval matrices”, “interval multi-input control problem”, and etc. Authors observe that the linear independency and dependency problem of interval vectors can be attacked in other mathematical frameworks. These works will be further studied in our future efforts. In next section, we will show that the linear (in)dependency property of interval vectors can be effectively used in checking the robust controllability and observability of the uncertain interval LTI system or the robust un-controllability and un-observability of the uncertain interval LTI system.

III. ROBUST CONTROLLABILITY TEST OF INTERVAL SYSTEM USING INTERVAL VECTORS

The robust controllability problem of uncertain linear system has been steadily studied in [18], [17] and therein references. Most notably, the methods suggested in [18], [17] provide algebraically elegant derivations. However, unfortunately, their methods, in instinct, cannot avoid the conservatism; hence regardless the algebraic simplification, their significance could be limited. In this section, we provide an alternative method developed based on interval vectors, which is very simple but much less conservative. The following LTI uncertain system is considered:

$$\dot{x} = Ax + Bu \tag{10}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^r$, $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times r}$, rank$(B) = r$, and $A \in A^I = [\underline{A}, \overline{A}]$ and $B \in B^I = [\underline{B}, \overline{B}]$. We call the interval uncertain system (10) is controllable if rank$(C^I) = n$, where

$$C^I = [B^I, A^I \otimes B^I, A^I \otimes A^I \otimes B^I, \ldots, \underbrace{A^I \otimes \cdots \otimes A^I}_{n-r} \otimes B^I]$$

which is $n \times (n-r+1)\cdot r$ interval matrix. For convenience, $m \equiv (n-r+1)\cdot r$. In fact, the main source of conservatism of [18], [17] is due to the fact that they used C^I without any modification for the controllability test. We explain this in more detail in the sequel.

First let us consider the case without interval such as:

$$\dot{x} = A_0 x + B_0 u \tag{11}$$

and corresponding the controllability matrix like

$$C_0 = [B_0, A_0 B_0, (A_0)^2 B_0, \ldots, (A_0)^{n-r} B_0].$$

If the system is controllable, then always rank$(C_0) = n$. To distinguish the interval case from the without interval case, let us suppose that the rank of following sub-matrix of C_0

$$C_0' = [B_0, A_0B_0, (A_0)^2B_0, \dots, (A_0)^{n-r-q}B_0]$$

where $q \geq 1$, is n (i.e., rank$(C_0') = n$). Then, without interval, it is always true that rank(C_0') = rank$(C_0) = n$. Now, let us include interval. In this case, we have to check the rank of C^I, but since C^I is $n \times m$ interval matrices, it is not easy to find the rank of C^I. Thus, in [18], [17], inevitably, they tried to find some inequality conditions in matrix norm to guarantee the sufficient conditions of LTI interval system (see Eq. (3.9) in [17] and Eq. (10) in [17]). Using these inequalities, they found the upper boundaries for sufficient condition, but in this upper boundary calculation, the formula is so conservative (see the derivation of Theorem 1 of [18] and Eq. (3.6) of [17]). So, even there is ignorable interval uncertainty in $(C^I)'$, which is defined as:

$$(C^I)' = [B^I, A^I \otimes B^I, A^I \otimes A^I \otimes B^I, \dots, \underbrace{A^I \otimes \cdots \otimes A^I}_{n-r-q} \otimes B^I]$$

the overall upper bounds are calculated from the maximum interval uncertainty of C^I. So, the controllability checking methods of [18], [17] instinctively are conservative. Hence, their approach must investigate the sufficient conditions based on C^I using maximum interval uncertain element of the interval A^I. This is the main reason why their methods are so conservative.

However, if we can check the rank of C^I directly, the result will be much less conservative, which can be done by checking the linear independency property of the interval vectors. Based on the results of Section II, easily we have the formula for controllability check of the uncertain LTI system:

Theorem 3.1: If the controllability matrix C^I satisfies the linear independency conditions of Theorem 2.3, then the uncertain interval system is controllable.

Proof: Since the interval system is controllable if its controllability matrix has rank n and the full rank condition is equivalent to the linear independency condition, the proof is immediate. ■

Corollary 3.1: If the controllability matrix C^I satisfies the linear independency conditions of Corollary 3.1, then the uncertain interval system is controllable.

Next, let us compare the test results using the three examples given in [18].

Example 1:

$$A \in A^I = \begin{pmatrix} 1 \pm 0.05 & 0 & 0 \\ 0 & 1 \pm 0.04 & 1 \pm 0.03 \\ 0 & -2 \pm 0.08 & 4 \pm 0.4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The controllability matrix C is calculated from the interval arithmetics as:

$$C \in C^I = \begin{pmatrix} 1 & 0 & 1 \pm 0.05 & 0 \\ 0 & 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 0 & 4 \pm 0.4 \end{pmatrix}$$

So, we have four sub-square matrices:

$$S^1 \in \begin{pmatrix} 1 & 0 & 1 \pm 0.05 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; S^2 \in \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 4 \pm 0.4 \end{pmatrix};$$

$$S^3 \in \begin{pmatrix} 1 & 1 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 0 & 4 \pm 0.4 \end{pmatrix}; S^4 \in \begin{pmatrix} 0 & 1 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 1 & 0 & 4 \pm 0.4 \end{pmatrix}$$

Then, from S^2, we have

$$S_0^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}; \quad \Delta S^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.03 \\ 0 & 0 & 0.4 \end{pmatrix}$$

Therefore, since S_0^2 is nonsingular and $\rho \left(\| (S_0^2)^{-1} \| \, \Delta S^2 \right) = 0.03 < 1$, easily we confirm that the interval system is controllable. However, in [18], they conclude that their method cannot check the controllability directly, which is due to the conservatism of their method. Clearly, our method is much less conservative. In [18], the following sign variant problem was given:

Example 2:

$$A \in A^I = \begin{pmatrix} 0 \pm 0.05 & 0 & 0 \\ 0 & 1 \pm 0.04 & 1 \pm 0.03 \\ 0 & 0 \pm 0.08 & 0 \pm 0.4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The controllability matrix C is calculated as:

$$C \in C^I = \begin{pmatrix} 1 & 0 & 0 \pm 0.05 & 0 \\ 0 & 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 0 & 0 \pm 0.4 \end{pmatrix}$$

So, we have four sub-square matrices:

$$S^1 \in \begin{pmatrix} 1 & 0 & 0 \pm 0.05 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; S^2 \in \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 0 \pm 0.4 \end{pmatrix};$$

$$S^3 \in \begin{pmatrix} 1 & 0 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 0 & 0 \pm 0.4 \end{pmatrix}; S^4 \in \begin{pmatrix} 0 & 0 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 1 & 0 & 0 \pm 0.4 \end{pmatrix}$$

From S^2, S_0^2 is nonsingular and $\rho\left(\|(S_0^2)^{-1}\| \, \Delta S^2\right) = 0.03 < 1$. So, regardless the sign variation, easily we find that the interval system is controllable. However, in [18], they used controller K to guarantee the controllability, but as resulted from our method, the system is already controllable. So, their approach requires the extra work, which is not necessary in our method. The following example includes the interval in B.

Example 3:

$$A \in A^I = \begin{pmatrix} 1 \pm 0.02 & 0 & 0 \\ 0 & 1 \pm 0.02 & 1 \pm 0.02 \\ 0 & -2 \pm 0.05 & 4 \pm 0.09 \end{pmatrix} \quad B \in B^I = \begin{pmatrix} 1 \pm 0.025 & 0 \\ 0 & 0 \\ 0 & 1 \pm 0.02 \end{pmatrix}$$

The controllability matrix C is calculated as:

$$C \in C^I = \begin{pmatrix} 1 \pm 0.025 & 0 & 1 \pm 0.0455 & 0 \\ 0 & 0 & 0 & 1 \pm 0.0404 \\ 0 & 1 \pm 0.02 & 0 & 4 \pm 0.1718 \end{pmatrix}$$

So, we have four sub-square matrices:

$$S^1 \in \begin{pmatrix} 1 \pm 0.025 & 0 & 1 \pm 0.0455 \\ 0 & 0 & 0 \\ 0 & 1 \pm 0.02 & 0 \end{pmatrix}; S^2 \in \begin{pmatrix} 1 \pm 0.025 & 0 & 0 \\ 0 & 0 & 1 \pm 0.0404 \\ 0 & 1 \pm 0.02 & 4 \pm 0.1718 \end{pmatrix};$$

$$S^3 \in \begin{pmatrix} 1 \pm 0.025 & 1 \pm 0.0455 & 0 \\ 0 & 0 & 1 \pm 0.0404 \\ 0 & 0 & 4 \pm 0.1718 \end{pmatrix}; S^4 \in \begin{pmatrix} 0 & 1 \pm 0.0455 & 0 \\ 0 & 0 & 1 \pm 0.0404 \\ 1 \pm 0.02 & 0 & 4 \pm 0.1718 \end{pmatrix}$$

Since from S^2, S_0^2 is nonsingular and $\rho\left(\|(S_0^2)^{-1}\| \, \Delta S^2\right) = 0.04 < 1$, the system is controllable. From these examples, it is clear our method is much simple and much less conservative than the existing method in checking the robust controllability of the uncertain LTI system. The robust observability is dual to the robust controllability problem and can be verified based on our method easily.

IV. CONCLUSIONS

In this note, we suggested the concept of "linear dependency" and "linear independency" of interval vectors and for the possible application, we applied our result to the robust controllability test of the uncertain interval LTI system. With much less conservatism, we could verify the controllability of the existing examples easily. This is just one possible application of the independency property of interval vectors. In our future efforts, we will consider more applications of interval vectors in checking the stability of the uncertain system.

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