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9/15/2004

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intelligent systems

Technical Report No. : USU-CSOIS-TR-04-13
(Accepted by IFAC World Congress 2005 at Prague, www.ifac.cz 12/15/2004)

STABILITY ANALYSIS OF ITERATIVE LEARNING CONTROL SYSTEM WITH INTERVAL UNCERTAINTY

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Abstract: This paper presents a stability analysis of the iterative learning control (ILC) problem when the plant Markov parameters are subject to interval uncertainty. Using the super-vector approach to ILC, vertex Markov matrices are employed to develop sufficient conditions for both asymptotic stability and monotonic convergence of the ILC process. It is shown that Kharitonov segments between vertex matrices are not required for checking the stability of interval super-vector ILC systems, but instead checking just the vertex Markov matrices is sufficient.

Keywords: Iterative learning control; monotonic convergence; interval uncertainty; Schur stability; vertex matrices

1. INTRODUCTION

Iterative learning control (ILC) using the super-vector analysis approach has been well established in the literature. The advantage of the super-vector notation is that the 2-dimensional problem of ILC is changed as the 1-dimensional multi-input multi-output (MIMO) problem. As shown in, for example, Moore (1993, 1998); Moore and Chen (1999), most discrete-time ILC problems can be expressed in the form

$$Y_k = HU_k$$

where k is the iteration index, $Y_k, U_k \in R^n$, where n is the trial length, and H is a lower-triangular Toeplitz matrix whose elements are the Markov parameters of the system to be controlled in the linear case. For time-varying systems and some classes of affine nonlinear systems a similar representation can be developed, with the key feature being that the matrix H is lower triangular. The super-vector approach to ILC is to design a learning gain matrix Γ so the resulting "closed-loop system" in the iteration domain, given by

$$E_{k+1} = (I - H\Gamma)E_k$$

where $E_k = Y_d - Y_k$ is the error, for some desired trajectory Y_d, is either asymptotically and/or monotonic convergent along the iteration axis in an appropriate norm topology. Such stability conditions have been analyzed in Moore and Chen (2002); Chen and Moore (2002a); and design issues have been considered in Chen and Moore (2001, 2002b).

In the ILC literature, robust design of the learning gain matrix has been considered using standard techniques such as H_∞-ILC, LQ-ILC, optimal-ILC, etc. However, though it is natural to con-

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sider interval uncertainties in the system matrix H when using the super-vector representation, to date there has been little or no research on this topic. In this paper we will study the stability of the ILC problem when the plant Markov parameters are subject to interval uncertainty.

In the robust control literature there are numerous results related to Hurwitz stability for interval matrixes, including Jiang (1987); Petkovski (1988), and Schur stability in Batra (2003); Rohn (1994). Kharitonov's theorem has also been very popular for interval matrix stability analysis, e.g., Bhattacharyya et al. (1995); Kokame and Mori (1991). However, all these works require lots of calculation and cannot be directly applied for checking the monotonic convergence of interval ILC. In this paper, an analysis method is developed for checking the convergence properties of the interval ILC problem. Similar to the Kharitonov vertex polynomial method, it will be shown that the extreme values of the interval Markov parameters provide a sufficient condition for monotonic convergence of the interval ILC.

This paper is organized as follows. Section 2 introduces some basic ILC results and describes the interval ILC problem. In Section 3 sufficient stability conditions for interval ILC are derived. A simulation example and conclusions are given in Sections 4 and 5, respectively.

2. INTERVAL ILC

Let the ILC learning gain matrix (Γ) discussed above be given as

$$\Gamma = \{\gamma_{ij}\}, i, j = 1, \cdots, n, \tag{1}$$

where the gains γ_{ij} are the elements Γ. We call the gains Arimoto-like if $\gamma_{ij} = 0, i \neq j$ and $\gamma_{ij} = \gamma, i = j$. The gains γ_{ij} are called causal ILC gains for $i > j$ and non-causal ILC gains for $i < j$. If the gains do not exhibit Toeplitz-like symmetry we call the learning algorithm time-varying. In ILC, there are two stability concepts: asymptotic stability and monotonic convergence.

In asymptotic stability, two concepts should be differentiated according to the ILC gain matrix structure. When Arimoto-like gains and purely causal gains are used, the stability condition is defined as:

$$|1 - \gamma_{ii} h_1| < 1, i = 1, \cdots, n, \tag{2}$$

where h_1 is the first non-zero Markov parameter. When non-causal gains are used, the asymptotic stability condition is defined as: $\rho(I - H\Gamma) < 1$, where ρ is the spectral radius of $(I - H\Gamma)$, and H is the Markov matrix.

Monotonic convergence is defined in appropriate norm topology as follows:

Definition 1. If $\|I - H\Gamma\|_1 < 1$, then $\|E_k\|$ is monotonically convergent to zero in l_1-norm topology.

Definition 2. If $\|I - H\Gamma\|_\infty < 1$, then $\|E_k\|$ is monotonically convergent to zero in l_∞-norm topology.

We now describe the interval ILC problem using the following definitions.

Definition 3. A scalar a, is called an *interval parameter* if it lies between two boundaries according to $a \in [\underline{a}, \overline{a}]$, where $\underline{a}$ is the minimum value of a and $\overline{a}$ is the maximum value of a.

Definition 4. An *interval matrix* (A^I) is defined as a matrix that is a member of the interval plant $\mathcal{A}^I$ given by:

$$\mathcal{A}^I = \{A^I : a^I_{ij} \in [\underline{a_{ij}}, \overline{a_{ij}}], i, j = 1, \cdots, n\},$$

where $\overline{a_{ij}}$ is the maximum extreme value of the i^{th} row and j^{th} column element of the uncertain plant, and $\underline{a_{ij}}$ is the minimum extreme value of the i^{th} row and j^{th} column element of the uncertain plant.

Definition 5. The *upper bound matrix* ($\overline{A}$) is a matrix whose elements are $\overline{a_{ij}}$. The *lower bound matrix* ($\underline{A}$) is a matrix whose elements are $\underline{a_{ij}}$. The *vertex matrices* (A^v) are defined by:

$$\mathcal{A}^v = \{A^v : a^v_{ij} \in \{\underline{a_{ij}}, \overline{a_{ij}}\}, i, j = 1, \cdots, n\}$$

Definition 6. If the Markov parameters are intervals such as: $h^I_i \in [\underline{h}_i, \overline{h}_i]$, then ILC system has interval uncertainties. The *interval Markov matrix* is denoted as H^I.

The interval ILC problem is concerned with the analysis and design of the ILC system when the system to be controlled is subjected to structured uncertainties in its Markov parameters. There are two classes of problems. First, given an *interval Markov matrix* H^I and a gain matrix Γ, what are the stability and convergence properties of the closed-loop system? Second, given an *interval Markov matrix* H^I, design Γ, so as to achieve desired stability and convergence properties of the closed-loop system. In the next section we consider the first problem.

3. STABILITY CONDITIONS OF INTERVAL ILC

We consider separately asymptotic stability and monotonic convergence.

3.1 Asymptotic Stability

For the asymptotic stability test of the interval ILC, the following lemmas are adopted from literature.

Lemma 1. [Shih et al. (1998); Han and Lee (1994)] With a given interval matrix A^I, the spectral radius of A^I is bounded by the maximum value of the spectral radii of vertex matrices A^v.

Lemma 2. [Delgado-Romero et al. (1996)] Let the interval matrix be given as $\underline{A} \leq A^I \leq \overline{A}$. If $\beta = \max\{\rho(MS_1), \rho(MS_2)\} < 1$, where $MS_1 = \overline{a_{ij}}$ if $i = j$ and $MS_1 = \max\{|\underline{a_{ij}}|, |\overline{a_{ij}}|\}$ if $i \neq j$; $MS_2 = \underline{a_{ij}}$ if $i = j$ and $MS_2 = \min\{-|\underline{a_{ij}}|, -|\overline{a_{ij}}|\}$ if $i \neq j$, then the interval matrix A^I is Schur stable.

Now, with above definitions and lemmas, we are ready to present our main results. Based on (2), the following theorem is suggested.

Theorem 1. Let the first Markov parameter h_1 be an interval parameter given by $h_1^I \in [\underline{h_1}, \overline{h_1}]$ and let Arimoto-like/causal ILC gains be used in Γ. Then the interval ILC system, $E_{k+1} = (I - H^I\Gamma)E_k$, is asymptotically stable if

$$\max\{|1 - \gamma_{ii}\underline{h_1}|, |1 - \gamma_{ii}\overline{h_1}|\} < 1, i = 1, \cdots, n, \quad (3)$$

Proof: Using the fact that H^I is a lower Toeplitz triangular matrix and Γ is a lower triangular matrix, then $I - H^I\Gamma$ is a lower triangular matrix. So, the diagonal terms of $I - H^I\Gamma$, given as $\{1 - \gamma_{ii}h_1^I\}, i = 1, \cdots, n$, are the eigenvalues of $I - H^I\Gamma$. When $i = k$, the maximum value of $|1 - \gamma_{kk}h_1^I|$ occurs at one of $h_1^v \in \{\underline{h_1}, \overline{h_1}\}$, because $|1 - \gamma_{kk}h_1^I|$ is the absolute value of $1 - \gamma_{kk}h_1^I$. Therefore, the maximum of $\{|1 - \gamma_{ii}\underline{h_1}|, |1 - \gamma_{ii}\overline{h_1}|\}$ occurs at one of $h_1^v \in \{\underline{h_1}, \overline{h_1}\}$. So, if $\max\{|1 - \gamma_{ii}\underline{h_1}|, |1 - \gamma_{ii}\overline{h_1}|\} < 1$ is satisfied, the system is asymptotically stable from (2). ■

Now consider the case of a general Γ. In $I - H^I\Gamma$, the interval matrix is H^I. So, the lower bound and the upper bound of $I - H^I\Gamma$ should be re-calculated. For convenience, let $T = H\Gamma$, calculated as:

$$t_{ij} = \sum_{k=1}^{i} h_k \gamma_{(i+1-k)j}, i, j = 1, \cdots, n$$

where t_{ij} are elements of T and $\gamma_{(i+1-k)j}$ are ILC learning gains. Similarly define $T^I = H^I\Gamma$ and also define $P = I - T$ and $P^I = I - T^I$. The lower and upper bounds of P^I, i.e., $\underline{P}$ and $\overline{P}$, can be calculated easily from the lower triangular Toeplitz matrix structure of T^I. Then, using the lower and upper bounds of P^I, it can be shown from Lemma 1 that the maximum spectral radius of $I - H^I\Gamma$ occurs at one of vertex matrices, P^v, of P^I. However, it is quite messy to check all the vertex matrices. Thus, it is suggested that Lemma 2 should be used to check asymptotic stability for the case of a general Γ.

3.2 Monotonic Convergence

To prove our next result the following lemmas are required.

Lemma 3. Let $x^I \in [\underline{x}, \overline{x}]$ be an interval parameter. Then for

$$y = |\gamma_{11}x^I + \gamma_{12}| + |\gamma_{21}x^I + \gamma_{22}|, \\ \forall \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22} \in \Re, \quad (4)$$

the $\max\{y\}$ occurs at a vertex point of x (i.e., $x^v \in \{\underline{x}, \overline{x}\}$).

Lemma 4. Let $x^I \in [\underline{x}, \overline{x}]$ be an interval parameter. Then for

$$y = |\gamma_{11}x^I + \gamma_{12}| + |\gamma_{21}x^I + \gamma_{22}| + \cdots + \\ |\gamma_{n1}x^I + \gamma_{n2}|, \forall \gamma_{i1}, \gamma_{i2} \in \Re, i = 1, \cdots, n, \quad (5)$$

the $\max\{y\}$ occurs at one of vertex points of x^v (i.e., $x^v \in \{\underline{x}, \overline{x}\}$).

The following lemma considers multiple interval parameters.

Lemma 5. Let $x^j \in [\underline{x^j}, \overline{x^j}], j = 1, \cdots, m$ be interval parameters (for convenience we omit the superscript I and v). Then for

$$y = |(\gamma_{11}^1 x^1 + \gamma_{12}^1) + \cdots + (\gamma_{11}^m x^m + \gamma_{12}^m)| + \cdots \\ + |(\gamma_{n1}^1 x^1 + \gamma_{n2}^1) + \cdots + (\gamma_{n1}^m x^m + \gamma_{n2}^m)|, \\ \forall \gamma_{i1}^j, \gamma_{i2}^j \in \Re, i = 1, \cdots, n, j = 1, \cdots, m, \quad (6)$$

the $\max\{y\}$ occurs at the vertices of x^j.

The proofs of Lemma 3, Lemma 4, and Lemma 5 are given in the Appendix. Next, using these lemmas, the following theorems can be proven.

Theorem 2. Given interval Markov parameters $h_i^I \in [\underline{h_i}, \overline{h_i}]$, the interval ILC system is monotonically convergent in the l_∞-norm topology if

$$\max\{\|I - H^v\Gamma\|_\infty\} < 1, \quad (7)$$

where H^v are vertex Markov matrices of the interval plant.

Proof: Based on Definition 2, the theorem can be proved by showing that $\max\{\|I - H^I\Gamma\|_\infty\} = \max\{\|I - H^v\Gamma\|_\infty\}$ (Note: in this proof, H^I denotes a matrix in the interval matrix set $\mathcal{H}^I = \{H^I\}$. Furthermore, for convenience, we omit I in H^I for notational simplicity). From the expansion of $I - H\Gamma$, the row vectors of $I - H\Gamma$ are expressed as:

$$(I - H\Gamma)_n = [-(h_n\gamma_{11} + h_{n-1}\gamma_{21} + \cdots + h_1\gamma_{n1}), \\ -(h_n\gamma_{12} + h_{n-1}\gamma_{22} + \cdots + h_1\gamma_{n2}), \ldots, \\ 1 - (h_n\gamma_{1n} + h_{n-1}\gamma_{2n} + \cdots + h_1\gamma_{nn})], \quad (8)$$

where $(I - H\Gamma)_i$ is the i^{th} row vector. Then, $\|I - H\Gamma\|_\infty$ is the function of h_i and γ_{ij}, because the following is true:

$$\|I - H\Gamma\|_\infty = \max\{\|(I - H\Gamma)_1\|_1, \|(I - H\Gamma)_2\|_1, \cdots, \|(I - H\Gamma)_n\|_1\}, \quad (9)$$

where $\|(I - H\Gamma)_i\|_1$ is the l_1-norm of each row vector. Thus, assuming fixed ILC gains γ_{ij}, $\|I - H\Gamma\|_\infty$ is expressed in the general form such as:

$$\begin{aligned}\|I - H\Gamma\|_\infty = & |-(h_i\gamma_{11} + \cdots + h_1\gamma_{i1})| + \cdots \\ & + |1 - (h_i\gamma_{1i} + \cdots + h_1\gamma_{ii})| + \cdots \\ & + |-(h_i\gamma_{1n} + \cdots + h_1\gamma_{in})|, \quad (10)\end{aligned}$$

where i means the i^{th} row. We see that (10) is the same form as (6) of Lemma 5. Note, in Lemma 5, x^j are intervals with $\forall\gamma_{i1}^j, \gamma_{i2}^j \in \Re, i = 1, \cdots, n, j = 1, \cdots, m$, and in (10), h_i are intervals with $\forall\gamma_{ij} \in \Re, i, j = 1, \cdots, n$. Therefore, from Lemma 5, the maximum of $\| I - H^I\Gamma\|_\infty$ occurs at one of vertex Markov matrices of the plant. ■

Theorem 3. Given interval Markov parameters $h_i^I \in [\underline{h_i}, \overline{h_i}]$, the following equality is true:

$$\max\{\|I - H^I\Gamma\|_1\} = \max\{\|I - H^v\Gamma\|_1\}, \quad (11)$$

where $\|\cdot\|_1$ is a matrix 1-norm, which is defined as: $\|A\|_1 = \max_{j=1,\cdots,n}\sum_{i=1}^n |A_{ij}|$.

Proof: The proof can be completed using the same procedure as above. ■

4. SIMULATION ILLUSTRATION

Let us consider a single-input, single-output system given as:

$$A = \begin{bmatrix} 0.72 & 0.0 & 0.0 \\ 1.0 & -1.04 & -0.81 \\ 0.0 & 0.81 & 0.0 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

and $C = [1.0 - 0.98 - 1.09]$, which has first and second Markov parameters given as $h_1 = CB = 1$ and $h_2 = CAB = -0.266$. It is assumed that there are interval uncertainties in h_1 and h_2 given as

$$h_1^I \in [0.9, 1.1]; \quad \text{and } h_2^I \in [-0.366, -0.166].$$

We consider two ILC learning gain matrices. For "Case-1" we suppose Arimoto-like ILC (only diagonal terms). "Case-2" uses the inverse of the nominal (without interval) Markov matrix. Thus for both cases Theorem 1 applies. From Theorem 1, $\max\{|1 - \gamma_{ii}\underline{h_1}|, |1 - \gamma_{ii}\overline{h_1}|\} = 0.1$. When the vertex matrix checking method of Lemma 2 is used, β is calculated as 0.1. So, Lemma 2 has the same result as Theorem 1. Thus, clearly, the system is asymptotically stable for both gains. However, from the four vertex points of $\|I - H^v\Gamma\|$ based on Theorem 2, the maximum ∞-norm of Case-1 is bigger than 1, while the maximum ∞-norm of the Case-2 is less than 1 (see Fig. 1). So, Case-1 might not always be be monotonic convergent for every plant in the interval system, while Case-2 is not only asymptotically stable but also monotonic convergent in l_∞-norm topology. Fig. 2 shows the ILC performance result using the sinusoidal reference signal. The figures shows the maximum, minimum, and average absolute errors at each iteration trial. The upper figure is the result of Case-1, and the bottom figure is the result of Case-2. In the Case-1 test, we used two different γ_1: the dot-dashed lines are results with $\gamma_1 = \frac{1}{h_1}$, and the solid lines are results with $\gamma_1 = \frac{1}{\overline{h_1}}$. The reason why we use $\gamma_1 = \frac{1}{\overline{h_1}}$ is that $\overline{h_1}$ is more robust than $h_1 = 1.0$. However, both cases show that the signals are not monotonically converging even if the solid lines are more robust than the dot-dashed lines.

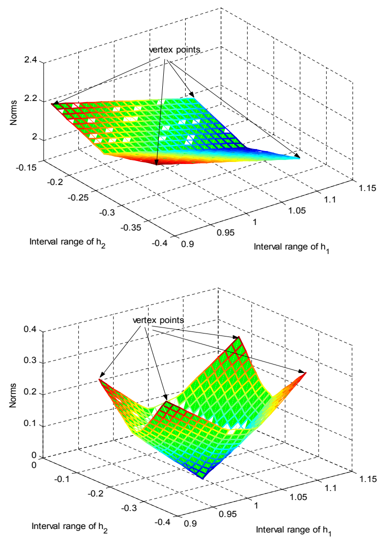

Fig. 1. Upper: norms for Case-1; bottom: norms for Case-2.

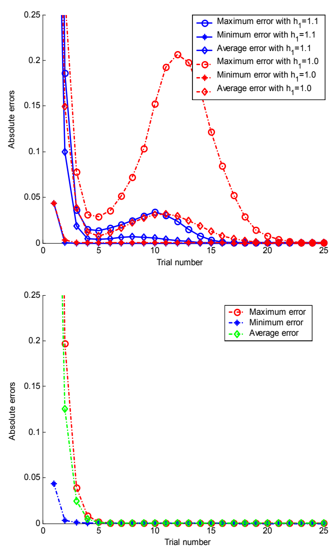

Fig. 2. ILC performance results.

5. CONCLUSION

We have presented a stability analysis of the ILC problem when the plant Markov parameters are subject to interval uncertainty. It was shown that checking just the vertex Markov matrices of an interval plant is enough to determine the asymptotic stability and the monotonic convergence properties of the interval ILC system. This is a powerful result from a computational perspective. In future research we will consider the design problem: given an *interval Markov matrix* H^I, find Γ so as to achieve desired stability and convergence properties of the closed-loop system.

6. APPENDIX

In the following proofs, superscripts I and v are omitted for simplicity.

Proof: (Lemma 3): The upper figure of Fig. 3 shows the line drawings of $|\gamma_{11}x+\gamma_{12}|$ and $|\gamma_{21}x+\gamma_{22}|$. Let us check the three different regions $R_1 \in [\underline{x}, \underline{x_0}]$, $R_2 \in [\underline{x_0}, \overline{x_0}]$, and $R_3 \in [\overline{x_0}, \overline{x}]$. In region R_1, $\max\{y\}$ occurs at $\underline{x}$, because $y_1+y_2 > y_3+y_4$. Also, in region R_3, $\max\{y\}$ occurs at $\overline{x}$, because $y_8+y_9 > y_6+y_7$. Now consider R_2. In region R_2, y is the just summation of two linear straight lines (i.e, the line connecting from y_6 to y_4 and the line connecting from y_3 to y_7) like $y = \gamma_{11}x + \gamma_{12} + \gamma_{21}x + \gamma_{22} = (\gamma_{11} + \gamma_{21})x + \gamma_{12} + \gamma_{22}$, which is represented by line l_1. So, in region R_2, the value of y linearly increases or linearly decreases. So, $\max\{y\}$ in R_2 occurs at $x \in \{\underline{x_0}, \overline{x_0}\}$. Finally, from the upper figure of Fig. 3, since the following relationship is true:

$$\max\{y_1 + y_2, y_8 + y_9\} > \max\{y_3 + y_4, y_6 + y_7\},$$

the proof of Lemma 3 is completed. ∎

Proof: (Lemma 4): From the upper figure of Fig. 3, consider R_2 again. In region R_2, the value of y (summation of two lines) is l_1. Here, let us change the upper figure to the bottom figure. In the bottom figure, $y = |\gamma_{11}x+\gamma_{12}|+|\gamma_{21}x+\gamma_{22}|$ is represented by lines l_1, l_2, and l_3. To prove Lemma 4, draw a supplementary line (the dashed line in the figure) from point y_{10} to point y_6. Then, the line connecting y_{10} and y_6; and the line connecting y_6 and y_{11} can be represented by a line such as $y = |\gamma_1^1 x + \gamma_2^1| + \triangle y_1$ with $\gamma_1^1, \gamma_2^1 \in \Re$, and $\triangle y_1 \in \Re^+$. Note that this approach does not change the result, because the triangular area included by points y_3, y_6, and y_{10} does not add any value to vertex point values (i.e., y_{10} and y_{11}). Whereas, if the maximum value still occurs at a vertex point after the triangular area is added, it is certain that the maximum value always occurs at a vertex point. Now, let us check the following:

$$y = |\gamma_{11}x + \gamma_{12}| + |\gamma_{21}x + \gamma_{22}| + |\gamma_{31}x + \gamma_{32}|,$$

which is rewritten as: $y = |\gamma_1^1 x + \gamma_2^1| + \triangle y_1 + |\gamma_{31}x + \gamma_{32}|$. Here, ignore $\triangle y_1$, because $\triangle y_1$ is a constant value at all x (i.e., for all $x \in [\underline{x}, \overline{x}]$). Then, from Lemma 3, the maximum value of y occurs at a vertex of x (i.e., $x \in \{\underline{x}, \overline{x}\}$). In this way, by induction, the maximum value of $y = |\gamma_{11}x+\gamma_{12}|+|\gamma_{21}x+\gamma_{22}|+\cdots+|\gamma_{n1}x+\gamma_{n2}|$ occurs at a vertex point of x even though $n \to \infty$. Thus, the proof of Lemma 4 is completed. ∎

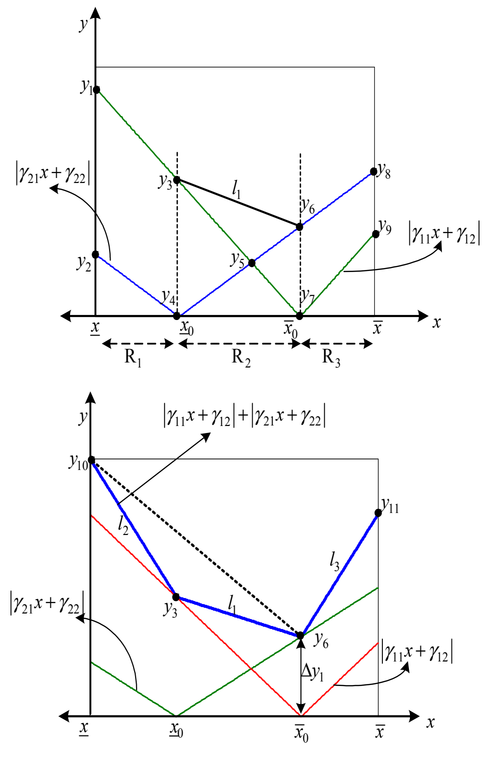

Fig. 3. Supplement figures for Appendix proofs.

Proof: (Lemma 5): Lemma 4 shows that, in the following equation

$$y = |\gamma_{11}x + \gamma_{12}| + \cdots + |\gamma_{n1}x + \gamma_{n2}|, \qquad (12)$$

$\gamma_{i2}, i = 1, \cdots, n$, can be any real values. So, in (6), if the following substitution is used:

$$\gamma_{i2}^1 + \cdots + (\gamma_{i1}^m x^m + \gamma_{i2}^m) := \gamma_{i2}, i = 1, \cdots, n$$

then, (6) is the same form as (5). Therefore, $\max\{y\}$ in (6) occurs at a vertex point of x^1(i.e., $x^1 \in \{\underline{x^1}, \overline{x^1}\}$) by Lemma 4, because all elements of $\{x^j\}$, $j = 1, \cdots, m$, are independent each other. Next, let us place $\gamma_{i1}^j x^j + \gamma_{i2}^j, j \in \{1, \cdots, m\}$, $i = 1, \cdots, n$ to the foremost in each absolute bracket like:

$$\begin{aligned} y = &|(\gamma_{11}^j x^j + \gamma_{12}^j) + \sum_{k=1,k\neq j}^{m} (\gamma_{11}^k x^k + \gamma_{12}^k)| \\ &+ \cdots \\ &+|(\gamma_{n1}^j x^j + \gamma_{n2}^j) + \sum_{k=1,k\neq j}^{m} (\gamma_{n1}^k x^k + \gamma_{n2}^k)|. \qquad (13) \end{aligned}$$

By denoting $\gamma_{l2}^j + \sum_{k=1,k\neq j}^{m} (\gamma_{l1}^k x^k + \gamma_{l2}^k) := \xi_l$, where $l = 1, \cdots, n$, the right-hand side of above equation is changed as:

$$y = |\gamma_{11}^j x^j + \xi_1| + \cdots + |\gamma_{n1}^j x^j + \xi_n|, \qquad (14)$$

where $\xi_i, i = 1, \cdots, n$ could be any real values. This is the same form as (5), so the maximum value of y of (14) occurs at a vertex point of x^j (i.e.,$x^j \in \{\underline{x^j}, \overline{x^j}\}$) by Lemma 4. Here, note that the maximum value of y, which occurs at a vertex point of x^j, is just with respect to x^j. Let us denote this maximum value as y_j^*. Now, it is required to show that the maximum value of y with respect to all intervals (i.e.,$\{x^j\}, j = 1, \cdots, m$) occurs at one of the vertex vectors such as

$$X^v = [\{\underline{x^1}, \overline{x^1}\}, \{\underline{x^2}, \overline{x^2}\}, \cdots, \{\underline{x^m}, \overline{x^m}\}]. \quad (15)$$

Denote this maximum value as $\overline{y}^*$. Note that $\overline{y}^* \neq y_j^*$. So, it is necessary to prove that, when the maximum value of y occurs at a vertex of x^j with fixed j, other interval parameters (i.e., $x^k, k \neq j$,) should be at vertices also (in this case, $\overline{y}^* = y_j^*$). Even though the maximum value of y occurs at a vertex of x^j, the other intervals $x^k, k \neq j$, might not be at vertex points (in this case, $\overline{y}^* \neq y_j^*$). Let us assume that, when the maximum value of y occurs at a vertex of x^j, the other interval parameter $x^k, k \neq j$ is not at a vertex point (i.e., x^k is an element of open set $x^k \in (\underline{x^k}, \overline{x^k})$). Let us change (14) using $\xi_i := \gamma_{i1}^k x^k + \xi_i'$ as:

$$y = |\gamma_{11}^k x^k + \gamma_{11}^j x^j + \xi_1'| + |\gamma_{21}^k x^k + \gamma_{21}^j x^j + \xi_2'| + \cdots + |\gamma_{n1}^k x^k + \gamma_{n1}^j x^j + \xi_n'|, \quad (16)$$

where $\xi_i', i = 1, \cdots, n$ could be any real values. Because (16) and (14) are same equations, the maximum value of y still occurs at a vertex of x^j. So, $\max\{y\} = y_j^*$, but $\max\{y\} \neq y_k^*$ and $\max\{y\} \neq \overline{y}^*$, where y_k^* is the maximum value with respect to x^k. However, by Lemma 4, y of (16) can be maximized more with respect to x^k. In other words, even though the current maximum value of (16) is y_j^*, when x^k is at one of vertex points, y_j^* can be increased more. Just by comparing the following two values:

$$y = \begin{cases} y_j^* \;, & \text{if } x^k \in (\underline{x^k}, \overline{x^k}), x^j = \{\underline{x^j}, \overline{x^j}\} \\ y_{jk}^* \;, & \text{if } x^k \in \{\underline{x^k}, \overline{x^k}\}, x^j = \{\underline{x^j}, \overline{x^j}\} \end{cases} \quad (17)$$

it is found that $\max\{y_{jk}^*\} \geq \max\{y_j^*\}$ by Lemma 4. Then, the maximum value of y of (16) with respect to k and j occurs at one of $\{\{\underline{x^k}, \overline{x^k}\}, \{\underline{x^j}, \overline{x^j}\}\}$. Finally, since $k \in \{1, \cdots, m\}$, the following is true by induction:

$$\max\{y\} = y_{123\cdots m}^*, \quad \text{when } x^i = \{\underline{x^i}, \overline{x^i}\}, \quad i = 1, \cdots, m, \quad (18)$$

where $y_{123\cdots m}^*$ is the maximum value with respect to all interval parameters. Then, from the relationship $\overline{y}^* = y_{123\cdots m}^*$, the maximum value of y occurs at one of the vertex vectors:

$$X^v = [\{\underline{x^1}, \overline{x^1}\}, \{\underline{x^2}, \overline{x^2}\}, \cdots, \{\underline{x^m}, \overline{x^m}\}]$$

Thus, the proof of Lemma 5 is completed. ∎

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