



**CSOIS Interval computation technical report series-1:  
Exact boundary calculation of maximum singular value of an  
interval matrix**

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# CSOIS Interval Computation Technical Report Series

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Interval computation or interval algebra has been well defined and steadily studied [1], [2], [3], [4], [5], [6]. In interval algebra or in robust control area, various research topics and numerous results have been reported and introduced. For example, the Hurwitz (Schur) stability of an interval matrix, the Hurwitz (Schur) stability of an interval polynomial, the Hurwitz (Schur) stability of an interval polynomial matrices, the Hurwitz (Schur) stability of an matrix polytopes, the control applications of an parameter intervals, and etc have been studied in numerous literatures. However, still some important research topics have not been properly studied and the solutions have not been addressed. In this 2005 technical report series of Utah State University, The Center for Self-Organizing and Intelligent Systems (CSOIS), we address some important interval problems and provide solutions.

In “CSOIS Interval computation technical report series-1: Exact boundary calculation of maximum singular value of an interval matrix (USU-CSOIS-TR-05-02)”, we provide solution for calculating the exact boundary of maximum singular value of an interval matrix. In fact, even though in existing literatures, the eigenvalue boundary problems have been widely studied, the maximum singular value problem has not been properly studied. In “CSOIS Interval computation technical report series-2: Impulse response boundary calculation based on power of interval matrix (USU-CSOIS-TR-05-03)”, we suggest using the vertex matrices for calculating the power of interval matrix as a specified order. Although some results have been reported for checking the asymptotical property of the power of interval matrix, the boundaries of power of interval matrix at specified order has not been reported. In this report, for the first time, we provide some algorithms for this power of interval matrix. In “CSOIS Interval computation technical report series-3: Linear Independency of Interval Vectors and Its Applications to Robust Controllability Tests (USU-CSOIS-TR-05-04)”, we define the linear (in)dependency of interval vectors, then we provide some conditions for checking this linear (in)dependency property of interval vectors. Furthermore, we use this result for checking the robust controllability and observability of interval system. In “CSOIS Interval computation technical report series-4: New Sufficient Schur Stability Conditions of Interval Polynomial Matrix (USU-CSOIS-TR-05-05)”, we provide sufficient conditions of interval polynomial matrix system. Although the suggested method could be conservative, due to the simplicity of the algorithm, it can be effectively used in various control problems.

Sooner or later, in this CSOIS Interval computation technical report series, we will add a survey work for interval computations related with robust control problems (robust stability, controllability, interval application for design, and etc.). Also, in the near future, we will address some more interesting interval problems under the terminology “interval model conversion”. For example, we will provide solution for the following Lyapunov equation:

$$PA + A^T P = -Q$$

where  $A \in A^I$ , and we want to find the exact boundary of  $P$  when  $Q$  is fixed.

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# CSOIS Interval computation technical report series-1: Exact boundary calculation of maximum singular value of an interval matrix

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## Abstract

In this report, we present a method for calculating maximum singular value of an interval matrix. First, we provide an algorithm for calculating the maximum singular value of a square interval matrix. Then, based on this algorithm, we extend the result to non-square interval matrix case. Through numerical examples, the validity of the suggested method is illustrated.

## Index Terms

Maximum singular value, interval matrix.

## I. INTRODUCTION

A great amount of literatures is available for interval matrix and its stability conditions [1], [2], [3], [4]. In particular, the Hurwitz stability [5], [6], the Schur stability [7], [8], and the eigenvalue boundary problem with perturbation [4], [3] have been well studied and formulated. However, the maximum singular value of an interval matrix has not been fully investigated. Even though boundaries of singular values of an interval matrix have been studied in [9], the sign of eigenvectors was limited to be unchanged with interval perturbation. Hence, the algorithm developed in [9] was based on some restrictive assumptions. In this report, we provide a generalized method, which can be used for finding the maximum singular value of a general interval matrix. In Section II, we provide our main result for square matrix. Then, in Section III, we extend the result to non-square matrix. In Section IV, numerical examples are offered for demonstration purpose and conclusion is given in Section V.

## II. MAIN RESULTS

For our main results, we make use of Hertz's idea for finding extreme eigenvalues of a *symmetric* interval matrix [2]. In this report, let us consider a real square non-symmetric interval matrix such as:

$$A^I = [a_{ij}^I], \quad a_{ij}^I := [\underline{a}_{ij}, \overline{a}_{ij}], \quad i, j = 1, \dots, n \quad (1)$$

where  $a_{ij}^I$  is an element of interval matrix  $A^I$ ,  $\underline{a}_{ij}$  is the lower boundary of an interval  $a_{ij}^I$ , and  $\overline{a}_{ij}$  is the upper boundary of an interval  $a_{ij}^I$ . If we define the lower boundary matrix and the upper boundary matrix as  $\underline{A} = [\underline{a}_{ij}]$ ;  $\overline{A} = [\overline{a}_{ij}]$ , the interval matrix can then be written as  $A^I := [A^o - \Delta, A^o + \Delta]$ , where the center matrix and the radius matrix are defined as

$$A^o = \frac{1}{2}(\overline{A} + \underline{A}); \quad \Delta = \frac{1}{2}(\overline{A} - \underline{A}).$$

In fact, the upper boundary of singular values of an interval matrix can be found as (in descending order)  $\sigma_i(A^I) = \sqrt{\lambda_i((A^I)^T \otimes A^I)}$  where  $\otimes$  represents multiplication of interval matrices,  $\sigma$  is the singular value, and  $\lambda$  is the eigenvalue. However, as commented in [9], the results of this method will be quite conservative. In this report, we suggest using the following relationship between singular values and eigenvalues:

$$\sigma_i(A) = \text{Positive} \left( \lambda_i \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} \right) = \text{Positive}(\lambda_i(H)), \quad A \in A^I \quad (2)$$

where  $\text{Positive}(\cdot)$  considers only the positive part of  $(\cdot)$ . Obviously,  $H$  is a symmetric matrix and it is a member of the symmetric interval matrix

$$H^I = \begin{bmatrix} 0 & (A^I)^T \\ A^I & 0 \end{bmatrix}.$$

Hence, if we make use of the results of [2], there will be a way to find maximum singular value of  $A^I$ . In the sequel, we briefly summarize our main idea and results. Based on [2], since  $H$  and  $H^I$  are symmetric matrices, we have relationship:

$$\lambda = x^T H x = x^T \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} x \quad (3)$$

where  $x$  is eigenvector corresponding to  $\lambda$  and  $x^T x = 1$ . Let us divide  $x$  into two parts such as  $x^T = [y^T, z^T]$ . Then,  $y_i = x_i$ ,  $i = 1, \dots, n$  and  $z_i = x_{n+i}$ ,  $i = 1, \dots, n$ , and from (3), we obtain  $\lambda = 2 \left( \sum_{i=1}^n \sum_{j=1}^n a_{ji} y_i z_j \right)$ . Therefore, value of  $\lambda$  depends on signs of  $y_i$  and  $z_j$ . That is, the maximum of  $\lambda$  occurs at one of vertex points of  $a_{ij}$ , which is given as:

$$a_{ij} = \begin{cases} a_{ij} = \overline{a_{ij}} & \text{if } y_i z_j \geq 0 \\ a_{ij} = \underline{a_{ij}} & \text{if } y_i z_j < 0 \end{cases} \quad (4)$$

Now, since  $y$  and  $z$  are length- $n$  vectors, we have totally  $2^n$  different sign patterns for  $y$  and  $2^n$  different sign patterns for  $z$ . For example, when  $n = 3$ , sign patterns of  $y$  and  $z$  could be  $+++$ ,  $++-$ ,  $+ - +$ ,  $+ - -$ ,  $- + +$ ,  $- + -$ ,  $- - +$ ,  $- - -$ . In this case, we have in total  $2^3 \times 2^3 = 64$  combinations as shown in Table I and Table II.

TABLE I  
32 SIGN PATTERNS WITH  $\text{sign}(y_1) = +$  FOR  $3 \times 3$  MATRIX.

y	z	y	z
+++	+++	+ - +	+++
	++-		++-
	+ - +		+ - +
	+ - -		+ - -
	- + +		- + +
	- + -		- + -
	- - +		- - +
	- - -		- - -
++-	+++	+ - -	+++
	++-		++-
	+ - +		+ - +
	+ - -		+ - -
	- + +		- + +
	- + -		- + -
	- - +		- - +
	- - -		- - -

However, Table I and Table II produce the same vertex matrix set of  $A^I$ . Therefore, in our purpose, it will be enough to check total  $2^5$  vertex matrices corresponding Table I. These vertex matrices can be found easily. For example, in Table I, for sign pattern  $+ - -$  of  $y$  and for sign pattern  $+ - +$  of  $z$ , the sign of vertex matrix is defined by  $zy^T$  such as:

$$\begin{bmatrix} + \\ - \\ + \end{bmatrix} \begin{bmatrix} + & - & - \end{bmatrix} = \begin{bmatrix} + & - & - \\ - & + & + \\ + & - & - \end{bmatrix}, \quad (5)$$

TABLE II  
32 SIGN PATTERNS WITH  $\text{sign}(y_1) = -$  FOR  $3 \times 3$  MATRIX.

y	z	y	z
- + +	+++	-- +	+++
	++-		++-
	+ - +		+ - +
	+ - -		+ - -
	- + +		- + +
	- + -		- + -
	-- +		-- +
	---		---
- + -	+++	-- -	+++
	++-		++-
	+ - +		+ - +
	+ - -		+ - -
	- + +		- + +
	- + -		- + -
	-- +		-- +
	---		---

which provides the corresponding vertex matrix for calculation of maximum singular value:

$$\begin{bmatrix} \overline{a_{ij}} & \underline{a_{ij}} & \overline{a_{ij}} \\ \underline{a_{ij}} & \overline{a_{ij}} & \overline{a_{ij}} \\ \overline{a_{ij}} & \underline{a_{ij}} & \underline{a_{ij}} \end{bmatrix}. \quad (6)$$

In the following algorithm, based on above discussion, for a general size  $n$  square interval matrix, we can develop a generalized method:

- **Step-1:** Produce the set of  $\pm 1$  vectors with  $y_1 = 1$  of length  $n$  such as

$$Y = \{y \in R^n : y_1 = 1, |y_j| = 1, \text{ for } j = 2, \dots, n\}.$$

- **Step-2:** Produce the set of  $\pm 1$  vectors of length  $n$  such as

$$Z = \{z \in R^n : |z_j| = 1, \text{ for } j = 1, \dots, n\}.$$

- **Step-3:** Make  $n \times n$  diagonal matrix  $T_y$  defined by  $(T_y)_{ii} = y_i$  and  $(T_y)_{ij} = 0$  for  $i \neq j, i, j = 1, \dots, n$  where  $y \in Y$ .
- **Step-4:** Make  $n \times n$  diagonal matrix  $T_z$  defined by  $(T_z)_{ii} = z_i$  and  $(T_z)_{ij} = 0$  for  $i \neq j, i, j = 1, \dots, n$  where  $z \in Z$ .
- **Step-5:** Produce a matrix set  $\mathcal{S}^v := \{A_{yz} : A_{yz} = \overline{A}^o + T_y \Delta T_z \forall y \in Y \text{ and } \forall z \in Z\}$ .
- **Step-6:** Find maximum singular values of all element of the finite set  $\mathcal{S}^v$  and select the largest one as the maximum singular value of the interval matrix  $A^I$ .

### III. MAXIMUM SINGULAR VALUE OF NON-SQUARE INTERVAL MATRIX

Results of the preceding section can be extended to the general non-square interval matrix case easily. Let us consider  $m \times n$  interval matrix  $A^I$ . Then,  $H^I$  is  $(m+n) \times (m+n)$  interval matrix. Now, introducing length- $n$  vector  $y$  and length- $m$  vector  $z$ , using the same procedure as in the square matrix case, we have  $\sigma(A) = 2 \left( \sum_{i=1}^n \sum_{j=1}^m a_{ji} y_i, z_j \right)$ . Then, we can find that there are total  $2^{m+n-1}$  possible combinations of vertex matrices

for maximum singular value of non-square interval matrix. For example, for  $3 \times 2$  matrix, we have total  $2^3 \times 2^1$  combinations as shown in Table III.

TABLE III  
16 SIGN PATTERNS FOR  $3 \times 2$  NON-SQUARE MATRIX.

y	z	y	z
++	+++	+-	+++
	++-		++-
	+ - +		+ - +
	+ - -		+ - -
	- + +		- + +
	- + -		- + -
	- - +		- - +
	- - -		- - -

In Table III, for example, for sign pattern  $+-$  of  $y$  and for sign pattern  $+ - +$  of  $z$ , the sign of vertex matrix is defined by  $zy^T$  such as:

$$\begin{bmatrix} + \\ - \\ + \end{bmatrix} \begin{bmatrix} + & - \end{bmatrix} = \begin{bmatrix} + & - \\ - & + \\ + & - \end{bmatrix}, \quad (7)$$

which provides the corresponding vertex matrix for maximum singular value:

$$\begin{bmatrix} \overline{a_{ij}} & \underline{a_{ij}} \\ \underline{a_{ij}} & \overline{a_{ij}} \\ \overline{a_{ij}} & \underline{a_{ij}} \end{bmatrix}. \quad (8)$$

#### IV. ILLUSTRATIVE EXAMPLES

##### A. Example-1: Non-square case

Let us test non-square case first. For non-square case, we use the existing example [9]:

$$A^I = \begin{bmatrix} [2, 3] & [1, 1] \\ [0, 2] & [0, 1] \\ [0, 1] & [2, 3] \end{bmatrix}. \quad (9)$$

Using the results given in Section III, we found the maximum singular value of  $A^I$  as 4.54306177572459, which is quite close to the value 4.543062 given in [9]. This result shows that our method can find the exact (without conservatism) upper boundary of maximum singular value of interval matrix. Note that the suggested scheme in this report does not require any assumption for calculating the upper boundary of maximum singular value of interval matrix.

##### B. Example-2: Square case

Next, for square matrix and to represent an exception of Deif's method [9], we test an interval matrix with the following center matrix

$$A^o = \begin{bmatrix} -3.33 & -2.24 & 0.06 \\ 1.03 & -0.34 & 1.09 \\ -2.02 & -1.02 & 2.27 \end{bmatrix} \quad (10)$$

and radius matrix

$$\Delta = \begin{bmatrix} 1.32 & 0.86 & 4.38 \\ 0.84 & 2.97 & 1.42 \\ 1.61 & 3.06 & 0.55 \end{bmatrix}. \quad (11)$$

Using the suggested method, we found that the maximum singular value of  $A^I$  is 9.8549, but from Deif's method, we have 9.7408. For demonstration purpose, we performed random tests. Figure 1 shows the Monte-Carlo type random tests. In these figure, dash-dot line is the calculated maximum singular value from the suggested method (9.8549) and solid line is the maximum singular value from Deif's method. Clearly there exist exceptions in the case of Deif's method, while the suggested method is bounding without an exception.

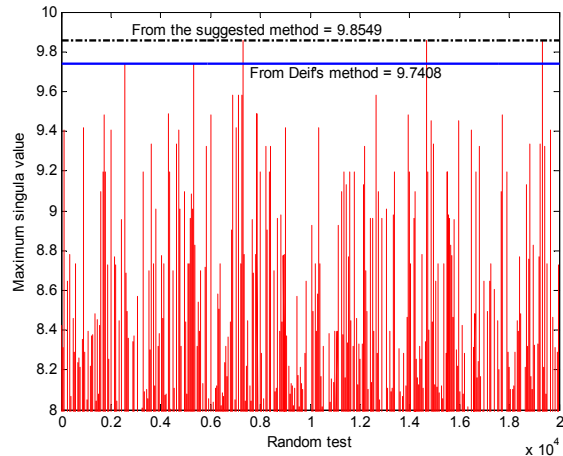


Fig. 1. Maximum singular values of randomly selected matrices and the calculated maximum singular values from the suggested method (dash-dot line) and from Deif's method (solid line).

## V. CONCLUSIONS

In this report, an algorithm for calculating the maximum singular value of a general non-square interval matrix and a square interval matrix has been suggested. Using the existing result [9], which was developed based on perturbation under some restrictive assumptions, we verified that the proposed method can calculate the maximum singular value accurately. Furthermore, from the created example, we have shown that the existing method does not find the maximum singular value some cases while the suggested method finds the maximum singular value without an exception.

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