CISOIS Interval computation technical report series-2:
Impulse response boundary calculation based on power
of interval matrix
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Interval computation or interval algebra has been well defined and steadily studied [1], [2], [3], [4], [5], [6]. In interval algebra or in robust control area, various research topics and numerous results have been reported and introduced. For example, the Hurwitz (Schur) stability of an interval matrix, the Hurwitz (Schur) stability of an interval polynomial, the Hurwitz (Schur) stability of an interval polynomial matrices, the Hurwitz (Schur) stability of an matrix polytopes, the control applications of an parameter intervals, and etc have been studied in numerous literatures. However, still some important research topics have not been properly studied and the solutions have not been addressed. In this 2005 technical report series of Utah State University, The Center for Self-Organizing and Intelligent Systems (CSOIS), we address some important interval problems and provide solutions.

In “CSOIS Interval computation technical report series-1: Exact boundary calculation of maximum singular value of an interval matrix (USU-CSOIS-TR-05-02)”, we provide solution for calculating the exact boundary of maximum singular value of an interval matrix. In fact, even though in existing literatures, the eigenvalue boundary problems have been widely studied, the maximum singular value problem has not been properly studied. In “CSOIS Interval computation technical report series-2: Impulse response boundary calculation based on power of interval matrix (USU-CSOIS-TR-05-03)”, we suggest using the vertex matrices for calculating the power of interval matrix as a specified order. Although some results have been reported for checking the asymptotical property of the power of interval matrix, the boundaries of power of interval matrix at specified order has not been reported. In this report, for the first time, we provide some algorithms for this power of interval matrix. In “CSOIS Interval computation technical report series-3: Linear Independency of Interval Vectors and Its Applications to Robust Controllability Tests (USU-CSOIS-TR-05-04)”, we define the linear (in)dependency of interval vectors, then we provide some conditions for checking this linear (in)dependency property of interval vectors. Furthermore, we use this result for checking the robust controllability and observability of interval system. In “CSOIS Interval computation technical report series-4: New Sufficient Schur Stability Conditions of Interval Polynomial Matrix (USU-CSOIS-TR-05-05)”, we provide sufficient conditions of interval polynomial matrix system. Although the suggested method could be conservative, due to the simplicity of the algorithm, it can be effectively used in various control problems.

Sooner or later, in this CSOIS Interval computation technical report series, we will add a survey work for interval computations related with robust control problems (robust stability, controllability, interval application for design, and etc.). Also, in the near future, we will address some more interesting interval problems under the terminology “interval model conversion”. For example, we will provide solution for the following Lyapunov equation:

$$PA + A^T P = -Q$$

where $A \in A^I$, and we want to find the exact boundary of $P$ when $Q$ is fixed.

REFERENCES

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Abstract

In this report, the impulse response boundary of discrete-time, linear, time-invariant systems with interval uncertainty is studied. The key problem of determining the impulse response of such systems is to find the boundaries of the power of an interval matrix at a specified order. To estimate the bounds of the power of an interval matrix, interval perturbation theory is used. Two different methods are developed. The first method uses first-order perturbation theory based on eigenpair-decomposition and the second method finds the sensitivity of the perturbation from the nominal matrix to the power of the interval matrix to develop a computation based on the vertex matrices of an interval matrix. Through numerical examples the usefulness of the suggested method is illustrated.

Index Terms

Linear time invariant, interval uncertainty, power of interval matrix, matrix perturbation theory.

I. INTRODUCTION

Interval computation techniques are popularly used in robust stability analysis of uncertain systems described in terms of interval parameters and interval matrices. In the past two decades, a great amount of research effort has been applied to the analysis of the interval matrix. For example, Hurwitz stability [1], [2], Schur stability [3], [4], and the eigenvalue boundary problem [5], [6] have been studied. However, there is no available result for determining the impulse response bounds of discrete-time, linear, time-invariant systems with interval uncertainty in their state-space description. In control engineering practice, for example, in designing the learning gain matrix in iterative learning control [7], it is important to estimate an accurate or less conservative boundary or range of the impulse responses of an interval system. Also, in [8], it was shown that the interval impulse response could be used for robust control. However, [8] did not provide the source of interval uncertainties in impulse response. Indeed, even though impulse response is widely used for control design purpose [9], [10], the uncertain ranges of impulse response have not been properly addressed. In this report, if the interval uncertainty in state-space form is given, it will be then shown that the key problem of determining the impulse response of such an uncertain system is to find the power of an interval matrix. In fact, in robust control, the power of an interval matrix can be effectively used in the analysis of the controllability, observability, or impulse response of the interval system. However, very limited effort has been devoted to calculating these boundaries. Some existing results can be found in [11], [5], [12], [13] where the convergence problem of the powers of an interval matrix was studied and it was proved that the power of an interval matrix converges to zero if the maximum spectral radius of the interval matrix is less than 1. However, the boundaries of the power of an interval matrix at a specified order of power has not been addressed in [11], [5], [12], [13]. Some useful analysis of the power of an interval matrix at a specified order can be found in [14], where it was concluded that computing the boundaries of the power of an interval matrix is an NP-hard problem.

In this report, we suggest two different methods for the calculation of the power of an interval matrix with application to determining the impulse response of systems with interval uncertainty. The first method is based on our early work [15] and the second one is a newly-developed method explained in detail in this report. We show that the second method reduces the conservatism of the first method. Our ultimate goal is to first find the boundaries of the power of an interval matrix at a specified order and then to determine the boundary of the impulse responses of the discrete-time, linear, time-invariant (LTI) system subject to interval uncertainty. Our approaches are developed
based on matrix perturbation theory, which has been shown to be effective in analyzing interval systems [16], [17], [18].

This report is organized as follows. In Section II, the upper and lower boundaries of the power of an interval matrix are estimated. Simulation examples are given in Section III and conclusions are given in Section IV.

II. THE POWER OF AN INTERVAL MATRIX FOR SOLVING IMPULSE RESPONSE OF UNCERTAIN LTI SYSTEMS

Let the nominal LTI discrete system, with relative degree 1 without loss of generality, be given by

\[
x(t + 1) = Ax(t) + Bu(t) \\
y(t) = Cx(t),
\]

where \(A, B,\) and \(C\) are the matrices of compatible dimensions in the state space model; \(x(t), u(t),\) and \(y(t)\) are the state, input, and output vectors, respectively. The impulse response of the system (1) is given by:

\[
T(z) = C(zI - A)^{-1}B = h_1z^{-1} + h_2z^{-2} + h_3z^{-3} + \cdots.
\]

So, the impulse responses are simply calculated by \(h_k = CA^{k-1}B\) which converts the impulse response problem into a problem of finding the power of a matrix \(A\) at a specified order \(k\). That is, if \(A^{k-1}\) is known, it is easy to find \(h_k\). In the case of deterministic systems (1), there is no computational issue. However, let us suppose that \(A\) is subject to interval uncertainty. That is, the parameters of \(A\) take values within some known interval. For convenience, we use the superscript \(I\) such as \(A^I\) to represent the model uncertainty in \(A\). Since we only consider interval uncertainty in this report, we call \(A^I\) an “interval matrix”. Then, the impulse response problem is to find the uncertain boundaries of \(h_k^I \in [h_k, \bar{h}_k]\) from \(A^I, B,\) and \(C\) through the power of plant interval matrix \(A^I\). In other words, the key issue of the impulse response determination is to find the power of the interval matrix at a specified order. This is the problem addressed by this report.

A. Basic definitions

The following definitions are used throughout this report.

**Definition 2.1:** The \(n \times n\) interval matrix \(A^I\), which is a set, is defined as

\[A^I = \{ A = [a_{ij} : a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}] ] \},\]

where \(\underline{a}_{ij}\) and \(\overline{a}_{ij}\) are the lower and upper boundaries associated with the elements \(a_{ij}\) of \(A^I\). Or, by introducing the lower-boundary and upper-boundary matrices \(\underline{A}, \overline{A}\), where elements of \(\underline{A}\) are \(\underline{a}_{ij}\) and elements of \(\overline{A}\) are \(\overline{a}_{ij}\), the interval matrix can also be defined as \(\underline{A} \leq A \leq \overline{A}, \forall A \in A^I\), in an element-wise sense.

**Definition 2.2:** The nominal matrix \(A^o\) is defined as

\[A^o = \left[ a^o_{ij} : a^o_{ij} = \frac{a_{ij} + \overline{a}_{ij}}{2} \right].\]

**Definition 2.3:** The interval perturbation matrix set, \(\Delta A^I\), is defined as

\[\Delta A^I = \{ \Delta A = [\Delta a_{ij} : \Delta a_{ij} \in [\underline{a}_{ij} - a^o_{ij}, \overline{a}_{ij} - a^o_{ij}] ] \}\]

or \(\Delta A^I = \{ \Delta A : \Delta A = A - A^o, \forall A \in A^I \} \).

**Definition 2.4:** The vertex matrix set \(A^v\) is a subset of the interval matrix set \(A^I\) defined as:

\[A^v = \{ A \in A^I : A = [a_{ij} \in \{a_{ij}, \overline{a}_{ij}\}] \} \].

In the next subsection, matrix perturbation theory is utilized to find the uncertain eigenvalue and eigenvector matrix sets, which will be used to estimate the range of the power of a given interval matrix.

B. Eigenpair decomposition method

The following assumption is necessary to find the diagonal eigenvalue matrix.

**Assumption 2.1:** Each matrix \(A \in A^I\) is diagonalizable and can be decomposed as: \(A = X\Lambda X^{-1}\), with \(\Lambda = \text{diag}(\lambda_i)\), where \(\lambda_i\) are the eigenvalues of \(A\), \(X \in \mathbb{C}^{n \times n}\), and \(\lambda_i \in \mathbb{C}\).
Throughout this report, we call $X$ the left eigenvector matrix, $\Lambda$ the eigenvalue matrix, and $Y := X^{-1}$ the right eigenvector matrix. Let us define the interval matrix sets as:

$$\mathcal{L}_A = \left\{ \Lambda : A = X\Lambda X^{-1}, A \in \mathcal{A} \right\}; \quad \mathcal{X}_A = \left\{ X : A = X\Lambda X^{-1}, A \in \mathcal{A} \right\}; \quad \mathcal{Y}_A = \left\{ Y = X^{-1} : X \in \mathcal{X}_A \right\}$$

We now further define two interval power sets:

Definition 2.5: The set of the power of the interval matrix is defined as

$$\mathcal{P}^1 = \left\{ A^k : A \in \mathcal{A} \right\},$$

where $k$ is the order of power.

Definition 2.6: The set of the generalized power of the interval matrix is defined as

$$\mathcal{P}^2 = \left\{ A = X\Lambda^k Y : \Lambda \in \mathcal{L}_A, X \in \mathcal{X}_A, Y \in \mathcal{Y}_A \right\},$$

where $k$ is the order of power.

Then, it is required to estimate the accurate boundaries of $\mathcal{P}^1$ to calculate the impulse response. However, since $\mathcal{A}$ is infinite set, it seems like it is impossible to find the boundary of $\mathcal{P}^1$. However, it is obvious that $\mathcal{P}^1 \subseteq \mathcal{P}^2$, so we can estimate the boundaries of the set $\mathcal{P}^1$ by estimating the boundaries of the set $\mathcal{P}^2$. Specifically, since the boundaries of $\mathcal{P}^2$ can be estimated by using three different sets: $\mathcal{L}_A$, $\mathcal{X}_A$, and $\mathcal{Y}_A$, the lower and upper boundaries of $\mathcal{P}^1$ can be subsequently estimated. That is, the original interval system set lies inside a “bigger” interval system set according to: $\mathcal{P}^1 \subseteq \mathcal{P}^2$. Therefore, the remaining work is to estimate the boundaries of $\mathcal{L}_A$, $\mathcal{X}_A$, and $\mathcal{Y}_A$, respectively. This problem was discussed in [15] in detail where the left eigenvector/right eigenvector matrices and the eigenvalue matrix were found using the first-order perturbation theory [19]. However, Assumption 2.1 is restrictive and the result could be conservative. Hence, In the following subsection, as a main work of this report, we develop a new method using the perturbation of the nominal $A$ matrix. This method is called the sensitivity transfer method.

\[ C. \text{ Sensitivity transfer method} \]

In this subsection, a new method is developed for the calculation of the power of an interval matrix. This method first computes the sensitivity of the perturbation on the nominal $A$ matrix and then applies this sensitivity to the power of matrix $A^k$. The set of the power of the interval matrix $\mathcal{P}^1$ defined in Definition 2.5 can be rewritten as:

$$A^k = \left\{ P = \underbrace{AAA\cdots A}_{k} : A \in \mathcal{A} \right\}. \quad (3)$$

Then, from the relationship $A^k = \underbrace{AAA\cdots A}_{k}$, we can have

$$\frac{\partial A^k}{\partial a_{ij}} = \frac{\partial A}{\partial a_{ij}} \underbrace{(A \cdots A)}_{k-1} + A \frac{\partial A}{\partial a_{ij}} \underbrace{(A \cdots A)}_{k-2} + \cdots + \underbrace{(A \cdots A)}_{k-1} \frac{\partial A}{\partial a_{ij}}. \quad (4)$$

Here, by observing that

$$\frac{\partial A}{\partial a_{ij}} = I_{ij} \quad (5)$$

where $I_{ij}$ is a matrix whose $i^{th}$ row and $j^{th}$ column element is 1 and the other elements are all zeroes, we have

$$\frac{\partial A^k}{\partial a_{ij}} = I_{ij}(A \cdots A) + AI_{ij}(A \cdots A) + \cdots + (A \cdots A)I_{ij}. \quad (6)$$

So, we have the perturbed sensitivity ($\partial A^k$) of $A^k$ by amount of the uncertain change ($\partial a_{ij}$) of $a_{ij}$ such as

$$\partial A^k = \partial a_{ij} \left( I_{ij}(A \cdots A) + AI_{ij}(A \cdots A) + \cdots + (A \cdots A)I_{ij} \right). \quad (7)$$
For convenience, let us use the following notation

$$\prod_{ij} := \left( I_{ij}(A \cdots A) + AI_{ij}(A \cdots A) + (A \cdots A)I_{ij}(A \cdots A) + \cdots + (A \cdots A)I_{ij} \right)$$

which simplifies (7) as $\partial A^k = \partial a_{ij} \prod_{ij}$. Hence, we find that when there is a perturbation amount of $\partial a_{ij}$ in $a_{ij}$, there is a perturbation effect to $A^k$ by amount of $\partial A^k$, which is related by the sensitivity transfer matrix $\prod_{ij}$. Here, noticing that each element of $A$ perturbs $A^k$, we develop a method for bounding the uncertainty of $A^k$. Using the notation $P \in P^+ = A^k = [P, P]$, we make the following proposition:

**Proposition 2.1:** Given the order of power $k$, the upper and lower boundaries associated with the elements of $P$ occur at the power of one of the vertex matrices $A^v$ ($A^v \in A^u$).

**Proof:** Let us pick arbitrary $i_1$ and $j_1$, and fix all $a_{pq}$, where $p, q = 1, \ldots, n$, and $p \neq i_1$ or $q \neq j_1$, to specified values such as $a_{pq} = a^*_p q \in [a_p q, a_p q]$. Then, from $\partial A^k = \partial a_{i_1 j_1} \prod_{i_1 j_1}$, the $k^{th}$ row and $l^{th}$ column element of $\partial A^k$ is determined by $\partial a_{i_1 j_1}$ and $\left( \prod_{i_1 j_1} \right)_{kl}$. Noticing that $\partial a_{i_1 j_1} = [-\triangle a_{i_1 j_1}, \triangle a_{i_1 j_1}]$, the positive (negative) maximum of $\partial A^k$ occurs at $\triangle a_{i_1 j_1}$ if $\left( \prod_{i_1 j_1} \right)_{kl}$ is of a positive value. Otherwise, the positive (negative) maximum of $\partial A^k$ occurs at $-\triangle a_{i_1 j_1}$, $\triangle a_{i_1 j_1}$.

In Proposition 2.1, we considered general non-symmetric square interval matrix. However, if the sufficient conditions given in appendix are not satisfied, Proposition 2.1 should be used. Hence, there is a trade-off between Proposition 2.1 and Proposition 2.2.

Remark 2.1: Proposition 2.2 was developed based on an assumption that the signs of $\prod_{ij}$ do not change. In appendix, we provide some sufficient conditions, which can be used for the purpose of reducing the computational amount. However, remind that if the sufficient conditions given in appendix are not satisfied, Proposition 2.1 should be used. Hence, there is a trade-off between Proposition 2.1 and Proposition 2.2.
However, we can extend these results to symmetric interval matrix. This work is direct by repeating (4), (5), (6), and (7).

In this section, two different methods for finding the boundaries of the power of interval matrix were presented. The first method is based on our previous work [15] and the second method is a new method developed in this report for the first time to overcome the conservatism of the first method [15]. However, note that the computational amount of the first method is significantly less than the second method. Now, it is straightforward to find the impulse response of the uncertain system (1) because the boundaries of $A^{k-1}$, $A \in A^I$, can be estimated from the suggested methods. In other words, the boundaries of $h^T_k$, i.e., $[\underline{h}_k, \overline{h}_k]$, can be simply calculated by matrix multiplication $CP^{k-1}B$, where $P^{k-1} \in A^{k-1}$, because the upper and lower boundary matrices of $A^{k-1}$ have been estimated. In the next section, the effectiveness of the suggested methods is illustrated through numerical examples.

III. ILLUSTRATIVE EXAMPLES

To verify the usefulness of the suggested methods, Monte Carlo-type random tests are performed. The obtained results are considered as the “true” range of the impulse response of the interval LTI system, for comparison to the bounds computed by our suggested methods. Consider the following uncertain discrete time LTI interval system:

\[
\begin{align*}
x(t + 1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t),
\end{align*}
\]

where $B = [2, 0.5]^T$; $C = [1, 0]$ and the following two different $A$ matrices are tested:

- Case-1 (symmetric and unstable): $a_{11} = -1.05$; $a_{12} = 0.55$; $a_{21} = 0.55$; $a_{22} = 0.55$
- Case-2 (non-symmetric and stable): $a_{11} = 0.75$; $a_{12} = -0.40$; $a_{21} = 0.25$; $a_{22} = 0.55$.

In Case-1, nominal eigenvalues are $-1.977$, $0.9977$. So, the nominal system is initially unstable; whereas in Case-2, nominal eigenvalues are $0.6500 + 0.3000i$ and $0.6500 - 0.3000i$. So, the nominal system is stable. In both cases, it is supposed that there is 10 percent interval model uncertainty in the $A$ matrix parameters. So, in Case-1, $A^I$ is

$$
A^I = \begin{bmatrix}
-1.155, -0.945 & 0.495, 0.605 \\
0.495, 0.605 & 0.765, 0.935
\end{bmatrix}
$$

and in Case-2, $A^I$ is

$$
A^I = \begin{bmatrix}
0.675, 0.825 & -0.44, -0.36 \\
0.225, 0.275 & 0.495, 0.605
\end{bmatrix}
$$

Left figure of Fig. 1 shows the test result of Case-1. Since the system is unstable, the impulse responses diverge as $k$ increases. In this figure, four different test results are shown: the $x$-dot dashed lines are the upper/lower boundaries computed from Intlab [20]; the $o$-dashed lines show the upper/lower boundaries computed from the eigenpair-decomposition method; the $o$-solid lines represent the upper/lower boundaries computed from the sensitivity transfer method; and the thick solid vertical bars are the random test results. Clearly, the sensitivity transfer method accurately bounds the upper/lower boundaries of the impulse responses, while even if the eigenpair-decomposition method is better than Intlab, it is much more conservative than the sensitivity transfer method. Right figure of Fig. 1 shows the test result of Case-2. From this figure, it is also shown that the sensitivity transfer method accurately bounds the upper/lower boundaries of the impulse responses. In the early phase, Intlab performs better than the eigenpair-decomposition based method. However, as $k$ increases, the eigenpair-decomposition method performs better than Intlab. Note that for the sensitivity transfer method, we used Proposition 2.2 based on Lemma 5.1 and Lemma-5.2. However, the condition of Lemma 5.2 was satisfied only for the power $k = 1, \ldots, 4$ of Case-1 and the condition of Lemma 5.1 was satisfied only for for the power $k = 1, \ldots, 5$ of Case-2. Hence, for the higher order power of interval matrix, we used Proposition 2.1. Now, from Fig. 1, it is clear that the sensitivity transfer method suggested in this report very accurately bounds the impulse responses of the uncertain interval system in both stable and unstable systems.

\footnote{From numerous numerical tests, we have found that Lemma 5.1 and Lemma-5.2 are particularly effective for stable system and lower order impulse response.}
IV. CONCLUSION

In this report, two systematic approaches for calculating the impulse response of an LTI system with interval uncertainty were presented. The first method, called eigenpair-decomposition method, uses the first order perturbation theory and the second method, called “sensitivity transfer method”, uses the sensitivity of the perturbation of the nominal $A$ matrix for calculating the power of the interval matrix based on the vertex matrices of $A^T$. From the numerical examples for both stable and unstable systems, we found that the sensitivity transfer method very accurately estimates the interval ranges of the impulse responses of the interval uncertain system. In summary, the main contribution of this report includes (i) the power of an interval matrix can be reliably estimated; (ii) the impulse response of LTI uncertain interval systems can be calculated accurately, and (iii) using the suggested methods for the power of an interval matrix, various robust stability problems such as robust controllability, robust observability, robust realization and etc. can be solved effectively.

V. APPENDIX

In this appendix, we provide sufficient conditions of Proposition 2.2. Let us write the sensitivity transfer matrix $\prod_{ij}$ such as:

$$\prod_{ij} = \sum_{p=1}^{k} A^{p-1} I_{ij} A^{k-p}$$

(9)

where $A \in A^T$. For convenience, let us equalize $A$ such as $A = A^o + \Delta$ where $\Delta \in \Delta A^T$. Then, using $A^k = (A^o + \Delta)^k$, and denoting $O^k := (A^o + \Delta)^k - (A^o)^k$, we obtain:

$$\prod_{ij} = \sum_{p=1}^{k} (A^o + \Delta)^{p-1} I_{ij} (A^o + \Delta)^{k-p}$$

$$= \sum_{p=1}^{k} \left[ O^{p-1} + (A^o)^{p-1} \right] I_{ij} \left[ O^{k-p} + (A^o)^{k-p} \right]$$

(10)

Then, rearranging (10) yields

$$\prod_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} = \sum_{p=1}^{k} \left\{ O^{p-1} I_{ij} O^{k-p} + O^{p-1} I_{ij} (A^o)^{k-p} + (A^o)^{p-1} I_{ij} O^{k-p} \right\}.$$

(11)
Defining absolute matrix such as \(|A| := \|[a_{ij}]\|\), we have

\[
\left| \prod_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right| = \left| \sum_{p=1}^{k} \left\{ (O^{p-1} I_{ij} O^{k-p} + O^{p-1} I_{ij} (A^o)^{k-p} + (A^o)^{p-1} I_{ij} O^{k-p}) \right\} \right|
\]

\[
\leq \sum_{p=1}^{k} \left\{ \left| (O^{p-1} I_{ij} O^{k-p}) \right| + \left| (O^{p-1} I_{ij} (A^o)^{k-p}) \right| + \left| (A^o)^{p-1} I_{ij} O^{k-p} \right| \right\}
\]

\[
\leq \sum_{p=1}^{k} \left\{ \left[ (|A^o| + |\Delta|)^{p-1} - |A^o|^p \right] I_{ij} \left[ (|A^o| + |\Delta|)^{k-p} - |A^o|^{k-p} \right] 
+ \left[ (|A^o| + |\Delta|)^{p-1} - |A^o|^p \right] I_{ij} \left( A^o \right)^{k-p} 
+ \left( A^o \right)^{p-1} I_{ij} \left[ (|A^o| + |\Delta|)^{k-p} - |A^o|^{k-p} \right] \right\}
\]

(12)

where we used the inequality \(|O^k| \leq (|A^o| + |\Delta|)^k - |A^o|^k\), which can be derived after several algebraic manipulations. Now, defining \(\Delta^* := A - A^o = A^o - \bar{A}\) and using inequality \(|\Delta| \leq \Delta^*\), we obtain

\[
\left| \prod_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right| \leq \sum_{p=1}^{k} \left\{ \left[ (|A^o| + \Delta^*)^{p-1} - |A^o|^p \right] I_{ij} \left[ (|A^o| + \Delta^*)^{k-p} - |A^o|^{k-p} \right] 
+ \left[ (|A^o| + \Delta^*)^{p-1} - |A^o|^p \right] I_{ij} \left( A^o \right)^{k-p} 
+ \left( A^o \right)^{p-1} I_{ij} \left[ (|A^o| + \Delta^*)^{k-p} - |A^o|^{k-p} \right] \right\}
\]

(13)

Finally, denoting the right-hand side of (13) as \(R^*\) and denoting \(L := \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p}\), we make the following lemma.

**Lemma 5.1:** If \(L \geq R^*\) in element-wise, the signs of \(\prod_{ij}\) in element-wise do not change.

**Proof:** From \(\left| \prod_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right| \leq R^* \leq L = \left| \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right|\), we have the inequality:

\[
\left| \prod_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right| \leq \left| \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right|
\]

Hence element-wisely, if \(\sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \geq 0\), then \(0 \leq \prod_{ij} \leq 2 \left( \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right)\); else if \(\sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} < 0\), then \(-2 \left( \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right) \leq \prod_{ij} < 0\). Therefore, the signs of \(\prod_{ij}\) are same to the signs of \(\sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p}\). This completes the proof. \(\blacksquare\)

When the commutative property \(A^o \Delta = \Delta A^o\) holds, the less conservative condition can be derived. Note the commutative property is satisfied when \(A\) is a symmetric interval matrix, and the symmetric interval matrix system has been an important research topic as shown in [21], [22]. For this result, we use \((A^o + \Delta)^m = \sum_{u=0}^{m} m C_u (A^o)^{m-u} \Delta^u\) where \(m C_u = \frac{m!}{u!(m-u)!}\). Now, from the following relationship:

\[
\Pi_{ij} = \sum_{p=1}^{k} \left[ \sum_{u=0}^{p-1} C_u (A^o)^{p-1-u} \Delta^u \right] I_{ij} \sum_{v=0}^{k-p} C_v (A^o)^{k-p-v} \Delta^v
\]

\[
= \sum_{p=1}^{k} \left[ (A^o)^{p-1} + \sum_{u=1}^{p-1} C_u (A^o)^{p-1-u} \Delta^u \right] I_{ij} \left( A^o \right)^{k-p} + \sum_{v=1}^{k-p} C_v (A^o)^{k-p-v} \Delta^v
\]

(14)

we have

\[
\Pi_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} = \sum_{p=1}^{k} \left[ (A^o)^{p-1} \right] I_{ij} \left[ \sum_{v=1}^{k-p} C_v (A^o)^{k-p-v} \Delta^v \right]
\]
Using the commutative property, we simplify (15) such as:

\[
\Pi_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij}(A^o)^{k-p} = \sum_{p=1}^{k} \sum_{u=1}^{k-p} k_p C_u \left[ I_{ij} (A^o)^{k-p-1} \Delta^u \right] I_{ij} \left[ (A^o)^{k-p} \right] \\
+ \sum_{p=1}^{k} \sum_{u=1}^{k-p} k_p C_u \left[ (A^o)^{p-1} \Delta^u I_{ij} (A^o)^{k-p} \right] \\
+ \sum_{p=1}^{k} \sum_{u=1}^{k-p} \left( p-1 C_u (k-p C_v) \right) (A^o)^{p-1} \Delta^u I_{ij} (A^o)^{k-p-v} \Delta^v
\]

(15)

Using the commutative property, we simplify (15) such as:

\[
\Pi_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij}(A^o)^{k-p} = \sum_{p=1}^{k} \sum_{u=1}^{k-p} k_p C_u \left[ I_{ij} (A^o)^{k-p-1} \Delta^v \right] I_{ij} \left[ (A^o)^{k-p} \right] \\
+ \sum_{p=1}^{k} \sum_{u=1}^{k-p} \left( p-1 C_u (k-p C_v) \right) (A^o)^{p-1} \Delta^u I_{ij} (A^o)^{k-p-v} \Delta^v
\]

(16)

Hence, we obtain the following inequality:

\[
\Pi_{ij} - \sum_{p=1}^{k} (A^o)^{p-1} I_{ij}(A^o)^{k-p} \leq \sum_{p=1}^{k} \left\{ \sum_{v=1}^{k-p} k_p C_u \left[ I_{ij} (A^o)^{k-p-1} \right] (\Delta^v)^u \right\} \\
+ \sum_{u=1}^{p-1} \left[ p-1 C_u (A^o)^{u} I_{ij} (A^o)^{k-p-u-1} \right] \\
+ \sum_{u=1}^{p-1} \left( p-1 C_u (k-p C_v) (A^o)^{u+v} I_{ij} (A^o)^{k-p-u-1} \right)
\]

(17)

Now, denoting the right-hand side of (17) as \( S^* \), we produce the following lemma for symmetric interval matrix.

**Lemma 5.2:** For symmetric interval matrix, if \( L \geq S^* \) in element-wise, the signs of \( \Pi_{ij} \) do not change.

**The following remark is provided for some special cases.**

- In Proposition 2.2, in the case of \( A > 0 \) element-wisely for all \( A \in \mathcal{A}^I \) or \( A < 0 \) element-wisely for all \( A \in \mathcal{A}^I \), the signs of \( \Pi_{ij} \) do not change.
- When \( \mathcal{A}^I \) is an symmetric interval matrix and it satisfies the property \( \text{sign}(AA) = \text{sign}(A) \) for all \( A \in \mathcal{A}^I \), the signs of \( \Pi_{ij} \) do not change.

**REFERENCES**


