Suppression of Overflow Limit Cycles in LDI All-Pass/Lattice Filters

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Abstract—In this paper it is shown that zero-input overflow limit cycles can be suppressed in a class of lossless digital integrator (LDI) all-pass filters, namely, that introduced in [1]. This holds for any filter order, provided that saturation overflow characteristics are used at the input of each delay and for certain restrictions for the multiplier values. The results are shown to apply also to lossless digital differentiator (LDD) filters. The restrictions of multiplier values have the effect of excluding certain combinations of poles within the unit circle, most of which are in the left half circle where corresponding LDD filters can be used. Asymptotic stability can be guaranteed for all second-order LDI and LDD filters.

Index Terms—All-pass filter, LDI filter, limit cycles, stability.

I. INTRODUCTION

Digital lattice filters consist of two parallel connected all-pass filters [2]. They can be designed as odd-order elliptic, Butterworth, or Chebyshev low-pass filters. A high-pass function can be obtained in the same structure by using the power complementary output. In the same fashion, two times odd-order bandpass and bandstop filters can be obtained.

The all-pass filters are often realized using wave digital (WD) filter techniques [3], [4]. The wave digital circulator filter, which consists of cascade connected all-pass adaptors, is a common choice. The advantage of this type of structure is that it has only one multiplier in the critical loop, provided that three-port adaptors are employed [5]. This allows for high computational speeds when implemented in hardware.

Another way of realizing the all-pass filters is via lossless digital integrator (LDI) structures which, like WD filters, are simulations of passive prototypes [6]–[10]. All-pass/lattice LDI filters (LDIF’s) were first introduced in [11]–[15], but the filters had, unfortunately, two multipliers in the critical loop. In [1] this was circumvented by changing the locations of the multipliers, thus reducing the critical loop to one multiplier only. Here, as well as in the related conference papers [16], [17], the structure is further modified by interchanging delayed and nondelayed integrators.

A shortcoming of the LDI filters has been that no general proof of the absence of limit cycles has appeared so far. (Such a proof for WD filters appeared in [18].) Bounds for granularity limit cycles in certain LDIF’s are, however, given in [19]. By invoking the main result of [20], it is shown in this paper that, with certain constraints for the multiplier values, the all-pass structure under consideration is free of zero-input overflow limit cycles. Having only one multiplier in the critical loop, this is an LDI filter class of interest for hardware implementation. Thus, while the result covers only a subset of all LDI filters, it is still significant.

In practice, the input signal is not equal to zero when most overflow events occur. Therefore, any analysis of the occurrence of zero-input overflow limit cycles accounts for a somewhat unrealistic situation. Although several such analyses have been published over the last two decades, it is really more relevant to analyze the stability of the forced response of the filter [21], [22]. However, this is considerably more difficult, since the forced system is not autonomous, and is therefore left for future research (in which the results presented in this paper very likely will prove useful).

This paper is organized as follows. The filter structure considered is presented in Section II. A state-space description is derived and it is shown how a Lyapunov function for the associated linear filter can be found. In Section III, this Lyapunov function is used to prove the absence of overflow limit cycles. Unscaled as well as scaled filters are treated and criteria for the multiplier values are obtained. It is shown that the theory can be extended also to lossless digital differentiator filters. Finally, in Section IV, restrictions for pole locations under the criteria found are investigated.

II. ALLPASS LDI FILTER STRUCTURE

Consider the structure in Fig. 1. It is identical to that in [1], except that delayed and nondelayed integrators are interchanged. This removes the extra delay from $u$ to $y$ present in the previous structure. The filter is assumed to be implemented with fixed-point arithmetic with sufficiently long word length to support the assumption that signal quantization can be neglected. Hence, the signal at any node can attain all values on $[-1, 1]$. In order to suppress overflow limit cycles, saturation arithmetic [23] is employed. The structure allows the saturation nonlinearities to be placed at the delay inputs, as indicated by the rings in Fig. 1.

A. State-Space Representation

Introducing state variables $x_1, x_2, \ldots$ at the output of each delay (except for the delay immediately following the input), moving from left to right in Fig. 1, and collecting the them in the vector $x$, the following state-space representation for $u = 0$ is obtained:

$$x(n+1) = Ax(n) \quad (1)$$

where $s(\cdot)$ is the saturation nonlinearity

$$s(v) = \begin{cases} 1, & v > 1 \\ v, & -1 \leq v \leq 1 \\ -1, & v < -1 \end{cases} \quad (2)$$

which is applied to each state variable

$$s(x) = [s(x_1), s(x_2), \ldots, s(x_N)]^T.$$ Furthermore

$$A = \begin{bmatrix} 1 - \alpha_1 & -\alpha_2 & 0 & 0 & 0 & 0 & \cdots \\ 1 - \alpha_1 & \beta_2 & 1 & -\alpha_4 & 0 & 0 & 0 & \cdots \\ 0 & -\alpha_3 & 1 & -\alpha_6 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \beta_4 & 1 & -\alpha_6 & 0 & \cdots \\ 0 & 0 & 0 & -\alpha_5 & 1 & \beta_6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$
where if $N$ is odd
\begin{equation}
\beta_i = 1 - \alpha_i - \alpha_{i+1}, \quad i = 2, 4, \ldots, N - 1
\end{equation}
and if $N$ is even
\begin{equation}
\beta_i = \begin{cases} 
1 - \alpha_i - \alpha_{i+1}, & i = 2, 4, \ldots, N - 2, \\
1 - \alpha \alpha_{N}, & i = N.
\end{cases}
\end{equation}

B. Lyapunov Function for the Associated Linear Filter

By the associated linear filter (ALF) to (1) it is meant an ideal filter with infinite number precision and range. That is
\begin{equation}
x(n+1) = Ax(n).
\end{equation}
When analyzing the stability of fixed-point filters using Lyapunov theory, the first step is to find a Lyapunov function for the ALF (cf. [18], [20], and [24]). For a linear system, the quadratic form
\begin{equation}
V(x(n)) = x^T(n)Px(n)
\end{equation}
is a Lyapunov function, which implies that the system is stable, if and only if $V(x(n)) > 0$, $x(n) \neq 0$, $V(0) = 0$, and $\Delta V(x(n)) = V(x(n+1)) - V(x(n)) = x^T(n) [A^T P A - P] x(n) = -x^T(n)Qx(n) \leq 0$, i.e., if the matrices $P$ and $Q = P - A^T P A$ are positive definite and positive semidefinite, respectively. While Lyapunov functions for stable linear systems always exist, finding an analytically parametrized matrix $P$ is, in general, a difficult task. Wave digital filters and Gray–Markel lattice filters have the unique property that a diagonal $P$ exists for any filter order. This not only makes finding $P$ straightforward, but it also directly follows that the nonlinear filter (1) is stable [18], [24] (see further the next section).

Unfortunately, for (6) with $A$ given by (3), it is easy to find values of $\alpha_i$ for which no diagonal $P$ yields a positive semidefinite $Q$. As a diagonal $P$ cannot be used, the logical next step is to expand $P$ to a structure with three diagonal bands (this being the second least complex structure). After several tries carried out with the assistance of MAPLE, the following matrix was found:
\begin{equation}
P = \begin{bmatrix}
p_1 & -\frac{p_1}{2} & 0 & 0 & 0 & 0 & \cdots \\
-\frac{p_1}{2} & p_2 & -\frac{p_2}{2} & 0 & 0 & 0 & \cdots \\
0 & -\frac{p_2}{2} & p_3 & -\frac{p_3}{2} & 0 & 0 & \cdots \\
0 & 0 & -\frac{p_3}{2} & p_4 & -\frac{p_4}{2} & 0 & \cdots \\
0 & 0 & 0 & -\frac{p_4}{2} & p_5 & -\frac{p_5}{2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{equation}
where
\begin{equation}
p_1 = 2 - \alpha_1, \quad p_2 = 2 \alpha_2
\end{equation}
and
\begin{equation}
p_i = \begin{cases}
\frac{p_{i-1}}{2} & i = 3, 5, 7, \ldots \\
\frac{p_{i+1}}{2} & i = 4, 6, 8, \ldots
\end{cases}
\end{equation}

Direct calculations then yield
\begin{equation}
Q = P - A^T PA = \begin{bmatrix}
\alpha_1 (\alpha_1 - 2)^2 & \alpha_1 \alpha_2 (\alpha_1 - 2) & 0 & \cdots \\
\alpha_1 \alpha_2 (\alpha_1 - 2) & \alpha_2 (\alpha_2 - 2) & 0 & \cdots \\
0 & 0 & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\end{equation}
giving
\begin{equation}
x^T(n)Qx(n) = \alpha_1 [(\alpha_1 - 2)x_1(n) + \alpha_2 x_2(n)]^2.
\end{equation}
Thus, $x^T(n)Qx(n) \geq 0$ if $\alpha_1 \geq 0$, so (7) with $P$ given by (8) is a Lyapunov function as long as $P$ is positive definite.

Remark 1: Notice that $\Delta V(x(n)) = -x^T(n)Qx(n) = 0$ when $\alpha_1 = 0$. Making $\alpha_1 = 0$ corresponds (see [1], [10]) to removing the resistance of the passive prototype, thus making the filter lossless.

III. STABILITY ANALYSIS

We shall now investigate whether (1) is asymptotically stable. Suppose that there exists a positive definite matrix $P$ such that
\begin{equation}
V(s(x)) = s^T(x)Ps(x) \leq x^T P x = V(x).
\end{equation}
Then
\begin{equation}
\Delta V(x(n)) = V(x(n+1)) - V(x(n)) = V(s(Ax(n))) - V(x(n)) \leq V(Ax(n)) - V(x(n)) = -x^T(n)Qx(n) \leq 0
\end{equation}
if $Q = P - A^T P A$ is positive semidefinite. Hence, if $V$ is a Lyapunov function for the ALF, it is also a Lyapunov function for the nonlinear filter (1). While (13) obviously holds for any diagonal $P$ with positive elements, it does not hold for a general positive definite matrix. Undoubtedly this is why a proof of the absence of limit cycles in LDI filters has, to the best knowledge of the authors, not appeared until now. Liu and Michel [20] provided the tools needed to handle general matrices. We restate Lemma 1 of [20]: an $N \times N$ positive definite matrix $P = (p_{ij})$ satisfies (13) if and only if
\begin{equation}
p_{ii} \geq \sum_{j=1, j \neq i}^{N} |p_{ij}|, \quad i = 1, \ldots, N.
\end{equation}
Now, applying (15) to (8), we obtain for $i = 1$
\begin{equation}
p_1 \geq \frac{p_2}{2} \Rightarrow 2 - \alpha_1 \geq \alpha_2 \Rightarrow \alpha_1 + \alpha_2 \leq 2.
\end{equation}
Even $i$ give equality in (15) as the nondiagonal elements both are equal to $-p_{i, i} / 2$ and, thus, add to $p_{ii}$, signs disregarded. For odd $i$, except for $i = 1$ and $i = N$
\begin{equation}
p_i \geq \frac{|p_{i-1}|}{2} + \frac{|p_{i+1}|}{2}
\end{equation}
but from (10) we have $p_{i-1} = \alpha_i p_i$ and $p_{i+1} = \alpha_{i+1} p_i$, so (17) can be written as
\begin{equation}
2p_i \geq (\alpha_i + \alpha_{i+1}) p_i \Rightarrow \alpha_i + \alpha_{i+1} \leq 2.
\end{equation}
Thus, (16) is a special case of (18). For \( i = N, N \) odd, (18) reduces to
\[
\alpha_N \leq 2.
\] (19)

It then remains to be shown that \( P \) is positive definite for the multiplier values satisfying (16), (18), and (19). We have
\[
x^T P x = \left( p_1 - \frac{p_2}{2} \right) x_1^2 + \sum_{i=2,4,6,\ldots} \frac{p_i}{2} \left( |x_{i-1} - x_i|^2 + |x_{i+1} - x_i|^2 \right)
\]
\[
= \left( \frac{2 - \alpha_1 - \alpha_2}{2} \right) x_1^2 + \sum_{i=2,4,6,\ldots} \frac{p_i}{2} \left( |x_{i-1} - x_i|^2 + |x_{i+1} - x_i|^2 \right)
\]
\[
= \left( \frac{2 - \alpha_1 - \alpha_2}{2} \right) x_1^2 + \sum_{i=2,4,6,\ldots} \frac{p_i}{2} \left( 2 - \alpha_i - \alpha_{i+1} \right) x_i^2
\] (20)

so \( V(x) \) is guaranteed to be positive for \( x \neq 0 \) if the inequalities in (16), (18), and (19) are sharpened to strict ones.

Since we have shown that \( \Delta V(x(n)) \leq 0 \), (1) is stable. However, from Theorem 2.2 of [20] it follows that (1) is, indeed, asymptotically stable. We have thus proven the following theorem.

**Theorem 1:** The system (1) with \( A : N \times N \) given by (3) is globally asymptotically stable with the equilibrium point \( x = 0 \) if \( \alpha_i > 0, i = 1, \ldots, N \) and
\[
N \text{ even: } \alpha_i + \alpha_{i+1} < 2, \quad i = 1, 3, \ldots, N - 1,
\]
\[
N \text{ odd: } \alpha_i + \alpha_{i+1} < 2, \quad i = 1, 3, \ldots, N - 2,
\]
\[
\alpha_N < 2.
\]

**Remark 2:** Usage of saturation overflow characteristics is a necessity for the proof of stability. Consider the general overflow function
\[
s(v) = \begin{cases} L, & v > 1 \\ v, & -1 \leq v \leq 1 \\ -L, & v < -1 \end{cases}
\] (21)

where \(-1 < L \leq 1\). (For example, \( L = 1 \) and \( L = 0 \) yield saturation and zeroing characteristic, respectively.) Lemma 2 of [20] generalizes (15) to
\[
(1 + L)|p_{ii}| \geq 2 \sum_{j=1, j \neq i}^{N} |p_{ij}|.
\] (22)

However, it follows from (15) that only \( L = 1 \) satisfies (22). Notice also that there is never any guarantee for freedom of overflow limit cycles (regardless of filter structure) if wrapping characteristics are used, i.e., \( L = -1 \) in (22).

**A. Scaling**

Scaling of digital filters is made in order to reduce the risk for overflow as well as to reduce the quantization noise level at the output [25]. The LDI all-pass filter structure can be scaled by the insertion of power-of-two multipliers in the branches interconnecting the integrators in Fig. 1, yielding the structure depicted in Fig. 2. This scaling is related to that used for wave digital filters (with multipliers \( k \) and \( 1/k \) inserted in the branches interconnecting two adaptors) [3] and, clearly, does not alter the transfer function of the ALF. Normally, (see [1] and Section IV) \( \delta_i = 2^{-\delta_i} < 1 \), i.e., scaling by \( \delta_i \) implies \( |h_i| \) binary right shifts.

![Fig. 2. LDI all-pass filter with power-of-two scaling multipliers](image)

Repeating steps similar to those above, the following theorem can be proven.

**Theorem 2:** The system (1), where \( A : N \times N \) is the state matrix for the ALF to the filter depicted in Fig. 2, is globally asymptotically stable with the equilibrium point \( x = 0 \) if \( \alpha_i > 0, i = 1, \ldots, N \); \( k_i > 0, i = 2, \ldots, N; k_1 = 1 \) (by definition) and
\[
N \text{ even: } \left\{ \begin{array}{l} \alpha_i + \alpha_{i+1} < 2, \quad i = 1, 3, \ldots, N - 1, \\ k_i + k_{i+1} \leq 2, \quad i = 2, 4, \ldots, N, \end{array} \right.
\]
\[
N \text{ odd: } \left\{ \begin{array}{l} \alpha_i + \alpha_{i+1} < 2, \quad i = 1, 3, \ldots, N - 2, \\ k_i + k_{i+1} \leq 2, \quad i = 2, 4, \ldots, N - 1. \end{array} \right.
\]

(Observe that for \( N = 2 \) the criterion for the scaling constants reduces to \( k_2 \leq 2 \).)

Theorem 2 is a generalization of Theorem 1. Notice the extra condition in Theorem 2, \( k_i + k_{i+1} \leq 2 \), which for Theorem 1 corresponds to that for even \( i, p_{ii} = |p_{i-1} - 1| + |p_{i+1} - 1| \). Since the scaling multipliers are normally smaller than or equal to unity, condition (23) is easily satisfied.

**B. LDD Filters**

By the substitution \( z \rightarrow -z \), the mirror image of the \( z \) plane in the imaginary axis is obtained. Hence, if \( H(z) \) is low pass, \( H(-z) \) is high pass. Making the above substitution in an LDI filter, a lossless digital differentiator (LDD) filter is obtained [7] in which delayed and nondelayed integrators become delayed and nondelayed differentiators, \((-1)/(z+1)\) and \(1/(z+1)\), respectively.

In a state-space model (with \( u = 0 \)), substituting \( z \rightarrow -z \) is equivalent to substituting \( A \rightarrow -A \). A Lyapunov function for an LDI filter is also a Lyapunov function for the corresponding LDD filter as \( P = (-A)^T P(-A) = P - A^T P A = Q \). Hence, for any LDI filter which is stable according to Theorems 1 or 2, the corresponding LDD filter is also stable.

LDD filters should (see [10]) be used rather than LDI filters in cases where the poles are located within the left half circle of the \( z \) plane in order to reduce coefficient sensitivities.

**IV. Restictions for Pole Locations**

**A. Second-Order Filters**

An implication of Theorem 1 is that the poles of the ALF cannot be placed arbitrarily within the unit circle. The characteristic polynomial for the second-order filter is
\[
\det (z I - A) = z^2 - (2 - \alpha_1 - \alpha_2) z + 1 - \alpha_1.
\] (23)

Poles at \( z = r e^{\pm \theta i}, r > 0 \) yield the polynomial \( z^2 - 2r \cos \theta z + r^2 \), so the criterion \( \alpha_1 + \alpha_2 < 2 \) implies that \( 2 \cos \theta > 0 \), i.e., complex-conjugated poles cannot be located within the left half circle \((-1 < \text{Re } z < 0)\) of the \( z \) plane if stability is to be guaranteed.
Now, introduce the scaling $k_2 = 2$. This satisfies the criterion $k_2 \leq 2$ in Theorem 2. Thus, for stability it is required that

$$\alpha_1 + \frac{\alpha_2}{2} < 2 \Rightarrow 2\alpha_1 + \alpha_2 < 4.$$  \hspace{1cm} (24)

This criterion allows poles located within the entire unit circle (24) together with $\alpha_1 > 0$, $\alpha_2 > 0$ coincide with the result of Schur–Cohn's test. Thus, with the scaling $k_2 = 2$, any second-order filter is asymptotically stable.

**B. Higher Filter Orders**

Except for second-order filters scaling does, unfortunately, contract rather than expand the region of poles in the $\zeta$ plane in which stability can be guaranteed. This since all $k_i$ have to be equal to or less than unity (for filter orders higher than two) in order to satisfy the criterion of Theorem 2, and thus $\alpha_1/k_i + \alpha_1 + 1/k_i + 1 = \alpha_1 + \alpha_1 + 1$. So, to satisfy the criteria Theorems 1 and 2, $\alpha_1$ must be smaller in the scaled case. However, most filters with poles in the right half circle satisfy Theorems 1 and 2. Consider, for example, a sixth-order elliptic bandpass filter with a passband ripple of 0.5 dB, a stopband ripple of 40 dB, and a group delay of 8. This result holds regardless of filter order.

Another good topic for further research is to search for a strict proof of the freedom of granularity limit cycles.

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Stability Analysis of 2-D State-Space Digital Filters with Overflow Nonlinearities

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Abstract—New criteria for the global asymptotic stability of two-dimensional (2-D) state-space digital filters subject to overflow nonlinearities are presented. A comparative evaluation of the present approach with an earlier one is made. Finally, the approach is extended to the global asymptotic stability of 2-D state-space digital filters with quantization nonlinearities.

Index Terms—Asymptotic stability, digital filter wordlength effects, multidimensional digital filters.

I. INTRODUCTION

Consider the state-space quarter-plane model of a two-dimensional (2-D) digital filter given by (1a)–(d)

\[
\mathbf{x}(k+1, l) = \mathbf{f}(\mathbf{y}(k, l))
\]

(1a)

where \( \mathbf{x}(k, l) \) is the state vector at time \( k \) and \( l \), \( \mathbf{f}(\cdot) \) represents the nonlinear function, and \( \mathbf{y}(k, l) \) is the input vector. The matrix \( \mathbf{A} \) is the system matrix, and \( \mathbf{P} \) is a positive definite diagonal matrix. The system is stable if \( \mathbf{D} - \mathbf{A}^T \mathbf{D} \mathbf{A} > 0 \) (2)

where \( \mathbf{D} > 0 \) and \( \mathbf{A} \) is the system matrix. The stability analysis is based on the sector information of the overflow nonlinearities as stated in (1c)

\[
\mathbf{f}(\mathbf{y}(k, l)) = \begin{bmatrix}
    f_1(y_1(k, l)) \\
    f_2(y_2(k, l)) \\
    \vdots \\
    f_n(y_n(k, l))
\end{bmatrix}^T
\]

(1c)

where \( f_i(y_i(k, l)) \) are the nonlinear functions. These conditions are sufficient for the stability of the system.

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Publisher Item Identifier S 1057-7122/00/$10.00 © 2000 IEEE