Design of Fractional Order Digital FIR Differentiators

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***Abstract*—In this letter, the design of a fractional order FIR differentiator is investigated. First, the fractional derivative of power function is defined. Then, the impulse response of fractional order differentiator is obtained by solving linear equations of Vandermonde form. Finally, one example is used to demonstrate that the fractional derivatives of digital signals are easily computed by using proposed filtering technique.**

***Index Terms*—Differentiator, finite impulse response (FIR), fractional calculus.**

I. Introduction

The DIGITAL differentiator is a very useful tool to determine and estimate the time derivatives of a given signal. For example, in radar and sonar applications, the velocity and acceleration are computed from position measurements using differentiators [1]. So far, several methods have been developed to design digital finite impulse response (FIR) and infinite impulse response (IIR) differentiators such as the eigenfilter method [2], the quadratic programming method [3], the Taylor series method [4] etc. An excellent survey of differentiator design has been presented in a tutorial paper [5].

On the other hand, the integer order n of derivative $D^n f(x) = (d^n f(x))/(dx^n)$ of function $f(x)$ has been generalized to fractional order $D^\nu f(x)$, where ν is a noninteger [6], [7]. The fractional calculus is now very useful in many fields of sciences and engineering including fluid flow, automatic control, electrical networks, electromagnetic theory, and probability [6]–[10]. Recently, it has also drawn the attention of researchers in the signal processing area [11], [12]. In this letter, a method will be presented to design a fractional order FIR differentiator such that the fractional derivatives of digital signals are easily computed using filtering technique.

II. Problem Statement

Let us start with the fractional derivative of the power function x^p. If q is a positive integer, the qth order derivative of x^p is given by

$$D^q x^p = p(p-1)(p-2)\cdots(p-q+1)x^{p-q} = \frac{p!}{(p-q)!}\,x^{p-q}. \tag{1}$$

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In [6], the integer order q is generalized to an arbitrary order ν by replacing $p!$ and $(p-q)!$ by gamma functions $\Gamma(p+1)$ and $\Gamma(p-\nu+1)$, so we have

$$D^\nu x^p = \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)}\,x^{p-\nu}. \tag{2}$$

When $p-\nu+1>0$, the gamma function can be computed by the following integral:

$$\Gamma(p-\nu+1) = \int_0^\infty t^{p-\nu}e^{-t}\,dt. \tag{3}$$

However, when $p-\nu+1\le 0$, the reflection property of gamma function

$$\Gamma(-x) = \frac{-\pi\csc(\pi x)}{\Gamma(x+1)} \tag{4}$$

can be used to compute $\Gamma(p-\nu+1)$. This property is described in [6, pg. 18]. Now let us study the fractional derivative of a given function $f(x)$ whose Taylor series expansion exists. Because function $f(x)$ can be transformed into a polynomial of x by using Taylor series expansion, i.e.,

$$f(x) = \sum_{p=0}^{\infty} a_p x^p \tag{5}$$

where $a_p = D^p f(x)|_{x=0}/p!$, the fractional derivative of $f(x)$ is given by

$$\begin{aligned} D^\nu f(x) &= \sum_{p=0}^{\infty} a_p D^\nu x^p \\ &= \sum_{p=0}^{\infty} a_p \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)}\,x^{p-\nu}. \end{aligned} \tag{6}$$

If $f(x) = g(xT)$, then sampling $g(x)$ at nT is equivalent to sampling $f(x)$ at integer n, where T is the sampling period. Due to this scaling property, it is enough to discuss the sampling of $f(x)$ and $D^\nu f(x)$ at $x=n$ in this letter. Let sequences $f(n)$ and $D^\nu f(n)$ be obtained by integer sampling. The problem is how to design a digital FIR filter

$$H(z) = \sum_{k=0}^{N} h(k)z^{-k} \tag{7}$$

such that its output is an νth order derivative $D^\nu f(n-\tau)$ when the input signal is $f(n)$, where τ is a prescribed delay. That is, $H(z)$ is a νth order digital differentiator. Note that the value of delay τ is not limited to an integer and can be a fractional value.

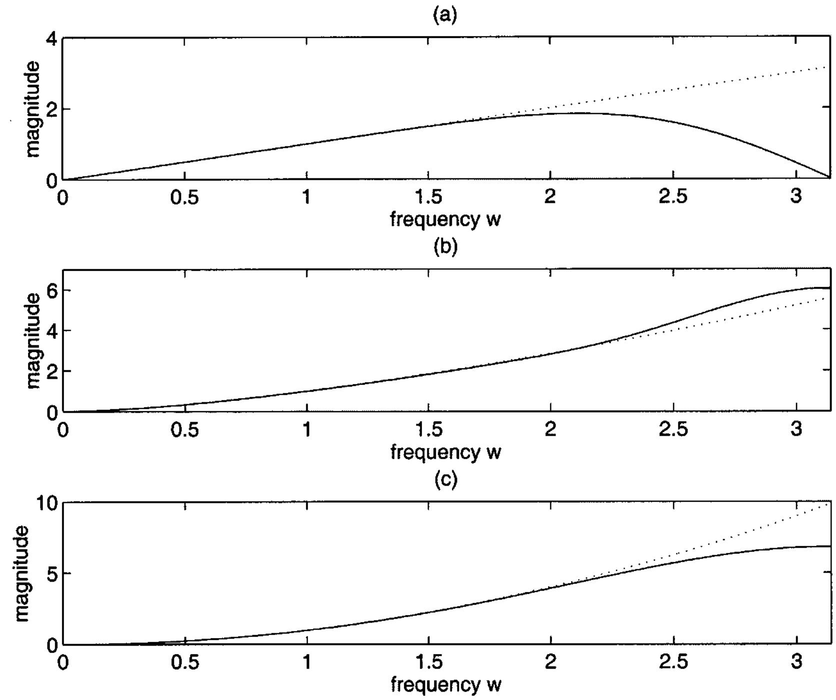

Fig. 1. Magnitude responses of the designed fractional order FIR differentiators. The solid lines are the designed magnitude responses and dotted lines are ideal responses ω^{ν}. (a) Order $\nu = 1$, (b) order $\nu = 1.5$, and (c) order $\nu = 2$.

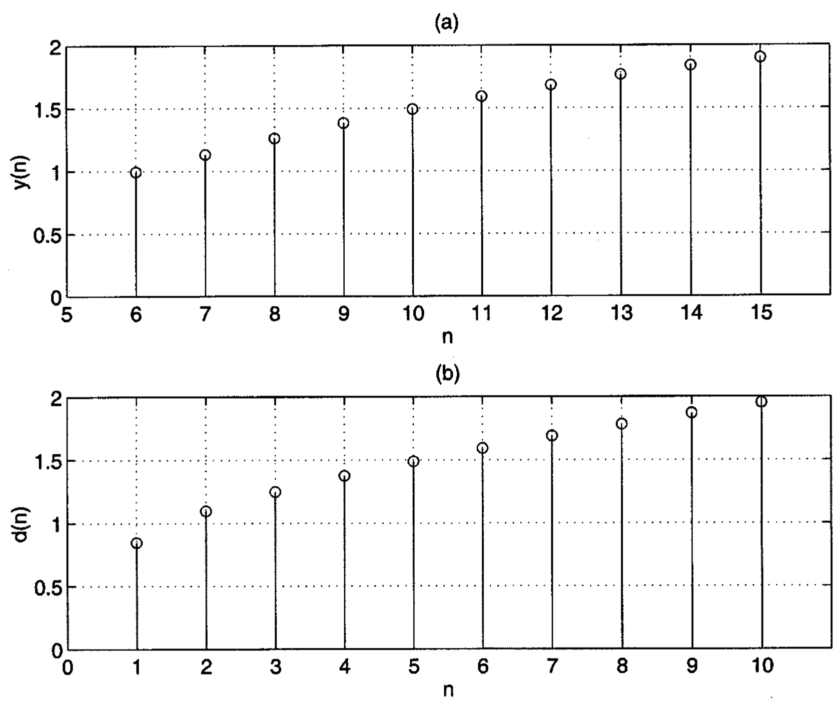

Fig. 2. (a) Output $y(n)$ of the designed fractional order differentiator. (b) Ideal output $d(n) = D^{1.5}f(n)$.

III. Design Method

When signal $f(n)$ passes through the FIR filter $H(z)$ with order N, its output $y(n)$ is given by

$$y(n) = \sum_{k=0}^{N} h(k)f(n-k). \tag{8}$$

From (5), we have $f(n-k) = \sum_{p=0}^{\infty} a_p(n-k)^p$, and output $y(n)$ can be rewritten as

$$y(n) = \sum_{p=0}^{\infty} a_p \sum_{k=0}^{N} h(k)(n-k)^p. \tag{9}$$

In order to make $y(n)$ be the νth order fractional derivative of $f(n)$, we need to find $h(n)$ such that $y(n) = D^{\nu}f(n-\tau)$. From

(6) and (9), teh filter coefficient $h(n)$ must satisfy the following equalities:

$$\sum_{k=0}^{N} h(k)(n-k)^p = \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)}(n-\tau)^{p-\nu} \qquad p = 0,\, 1,\, \cdots,\, \infty. \tag{10}$$

Equation (10) must be satisfied for all integers n. However, it is impossible to find $N+1$ filter coefficients $h(n)$ to satisfy the above infinite equalities simultaneously. Thus, we must choose $N+1$ equalities to determine $h(n)$ uniquely. This choice makes the designed FIR filter $H(z)$ be an approximate fractional order differentiator. Because it will suffer from numerical problem to choose big integers n and p, we take $n = N+1$ and $p = 0,\, 1,\, \cdots N$ to obtain these $N+1$ equalities. Based on this choice, (10) can be rewritten as the following matrix form:

$$\mathbf{A}\mathbf{h} = \mathbf{b} \tag{11}$$

where $\mathbf{h} = [h(0),\, h(1),\, \cdots,\, h(N)]^t$, $\mathbf{A}$, and $\mathbf{b}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ N+1 & N & N-1 & \cdots & 2 & 1 \\ (N+1)^2 & N^2 & (N-1)^2 & \cdots & 2^2 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ (N+1)^N & N^N & (N-1)^N & \cdots & 2^N & 1 \end{bmatrix} \tag{12}$$

$$\mathbf{b} = \left[\frac{\Gamma(1)}{\Gamma(1-\nu)}(N+1-\tau)^{-\nu}\ \frac{\Gamma(2)}{\Gamma(2-\nu)}(N+1-\tau)^{1-\nu} \cdots \frac{\Gamma(N+1)}{\Gamma(N+1-\nu)}(N+1-\tau)^{N-\nu}\right]^t. \tag{13}$$

The solution $\mathbf{h}_o$ is easily obtained by solving the linear equation in (11), i.e.,

$$\mathbf{h}_o = \mathbf{A}^{-1}\mathbf{b}. \tag{14}$$

Because matrix $\mathbf{A}$ is a Vandermonde matrix, the above solution can be calculated by a computation efficient algorithm in [13, p. 92].

IV. Design Example

Now an example performed with MATLAB language in an IBM PC computer is presented to evaluate the performance of the proposed method. Fig. 1 shows the magnitude responses of the designed differentiators for $N = 10$, delay $\tau = 5$, and orders $\nu = 1,\, 1.5,\, 2$. It is clear that the magnitude responses change smoothly from w to ω^2 at low frequency when order ν increases from 1 to 2. The approximation errors at high frequency are due that we only find $h(n)$ to satisfy the chosen $N+1$ equalities in (10). Moreover, let signal

$$f(n) = 1 + n + 0.25n^2 \tag{15}$$

pass through the designed differentiator with order $\nu = 1.5$, the output signal $y(n)$ is shown in Fig. 2(a). For comparison, the ideal output $d(n)$ given by

$$d(n) = D^{1.5}f(n) = \frac{\Gamma(1)}{\Gamma(-0.5)}n^{-1.5} + \frac{\Gamma(2)}{\Gamma(0.5)}n^{-0.5} + \frac{0.25\Gamma(3)}{\Gamma(1.5)}n^{0.5} \tag{16}$$

are also shown in Fig. 2(b). From the results in Fig. 2, we see that $y(n-5)$ is almost equal to $D^{1.5}f(n)$. Thus, the designed fractional order differentiator can compute the fractional derivative of a given signal accurately and easily.

V. Conclusions

In this letter, a novel approach has been presented to design fractional order FIR differentiator. One example is used to demonstrate the effectiveness of proposed design method. However, only a one-dimensional (1-D) differentiator is considered here. Thus, it is interesting to extend this method to design a two-dimensional (2-D) differentiator. This topic will be investigated in the future.

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