

designs in Section III demonstrate. It can be easily programmed either with or without exploiting symmetry in the impulse response coefficients, and yields optimal filter realizations while allowing for designer-specified frequency weighting.

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Design of High-Order Digital Differentiators Using L_1 Error Criteria

Chang-Kann Chen and Ju-Hong Lee

Abstract—This paper considers the optimal design of high order digital differentiators in the L_1 sense. Conventionally, using L_1 error criterion for this design problem results in a nonlinear optimization problem since the corresponding objective function contains an absolute error function. We first reformulate the design problem as a linear programming problem in the frequency domain. To avoid the requirement of huge computation load and storage space when using linear programming based algorithms, we present a method based on a modification of Karmarkar's algorithm to solve the design problem so that an analytical weighted least-squares (WLS) solution formula can be obtained. This leads to a very efficient procedure for the considered design problem. Computer simulations show that the designed differentiators can achieve more accurate wideband differentiation than those designed by using L_2 and Chebyshev (minimax) error criteria.

I. INTRODUCTION

Considering the FIR filter design for high order digital differentiators, the design task is to find a linear-phase FIR digital filter with magnitude response $H(\omega)$ which approximates a desired n th order digital differentiator with frequency response $D(\omega) = (\omega/2\pi)^n$ in some optimal sense. Several methods [1]–[5] employ either L_2 or Chebyshev criterion to obtain the associated error measure for minimization in the filter design process. Reference [1] modified the well-known McClellan-Parks program [4] to design higher order FIR digital differentiators by directly minimizing the approximation error $[(\omega/2\pi)^n - H(\omega)]$ in the minimax (Chebyshev) sense. However, [2], [3] have pointed out that the method of [1] usually produces design results with relatively large deviation over the frequency band, especially at low frequencies or even fails to converge. They then considered the design problem by presenting two methods based on L_2 criterion. Reference [2] employed the eigenfilter approach to obtain the least-squares design results with smaller deviation in most of the frequency band but the deviation gets severer in the region near the bandedge. Reference [3] proposed a least-squares approach employing a number of derivative constraints at the dc point. The design problem was formulated as a constrained quadratic programming problem. It was claimed in [3] that the method produces good error performance at low frequencies with acceptable peak error levels at high frequencies.

For designing the digital filters without flat frequency bands, such as digital differentiators, the relative approximation error is more widely used than the approximation error for minimization during the design process. The iterative Remez algorithm of [4] has been utilized by [5] to design optimal linear-phase FIR digital differentiators in a minimax relative error sense. The relative approximation error used by [5] is $[D(\omega) - H(\omega)]/D(\omega)$, i.e., the weighting function used is the inverse of the desired magnitude response $D(\omega)$. Many first order wideband differentiators have been successfully designed and reported in [5]. However, it is our design experience that minimizing

Manuscript received January 25, 1993; revised October 16, 1993 and September 9, 1994. This work was supported by the National Science Council under Grant NSC83-0404-E002-009. This paper was recommended by Associate Editor Y. C. Lim.

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IEEE Log Number 9409266.

the relative approximation error in the minimax sense often causes numerical problem when the order of differentiators is greater than three.

In this paper, we present a new technique for the optimal design of high order digital differentiators in the L_1 sense. The design problem is formulated as a weighted L_1 minimization problem with several equality constraints for achieving maximally accurate differentiation response. References [6] and [7] have shown that the L_1 minimization problems are equivalent to linear programming (LP) problems. Therefore, LP based algorithms can be used for solving the L_1 minimization problems. However, high computing cost is generally inevitable when using LP based design algorithms to solve the minimization problems with modest size of free parameters. To alleviate this difficulty, we propose a design method based on the affine-scaling variant of Karmarkar's algorithm which has been considered for efficiently solving LP problems [8]. The proposed method is efficient due to the fact that the coefficient vector of the designed differentiator can be expressed by an analytical WLS solution formula. Hence, finding the optimal differentiator coefficients requires to solve only a WLS solution at each iteration.

II. FORMULATION OF L_1 DESIGN PROBLEM

The magnitude response of an ideal n th-order digital differentiator is given by

$$D(\omega) = (\omega/2\pi)^n; \quad \text{for } |\omega| \leq \omega_p, \quad (1)$$

where ω_p denotes the passband edge frequency. Depending upon the values of n and ω_p , four types of digital differentiators are considered in the literature, namely, full-band odd-order, full-band even-order, nonfull-band odd-order, and nonfull-band even-order digital differentiators. The problem considering the choice of a suitable linear-phase FIR digital filter for designing each of the above digital differentiators has been studied in [2].

Considering the response of a linear-phase FIR digital filter for designing digital differentiators, we can express it without the linear phase term as follows [10]

$$H(\omega) = \sum_{k=1}^M a(k) \text{trig}(\omega, k), \quad (2)$$

where $\text{trig}(\omega, k)$ denotes an appropriate trigonometrical function and $a(k)$ is a function of the filter coefficients, $h(k)$. If the impulse response of the filter is symmetrical and the filter length N is odd, then $M = (N + 1)/2$, $a(1) = h((N - 1)/2)$, $a(k) = 2h((N + 1)/2 - k)$ for k not equal to 1, and $\text{trig}(\omega, k) = \cos(\omega(k - 1))$. If the impulse response of the filter is symmetrical and the filter length N is even, then $M = N/2$, $a(k) = 2h(N/2 - k)$, and $\text{trig}(\omega, k) = \cos(\omega(k - 1/2))$. If the impulse response of the filter is asymmetrical and the filter length N is odd, then $M = (N - 1)/2$, $a(k) = 2h((N - 1)/2 - k)$, and $\text{trig}(\omega, k) = \sin(\omega k)$. If the impulse response of the filter is asymmetrical and the filter length N is even, then $M = N/2$, $a(k) = 2h(N/2 - k)$, and $\text{trig}(\omega, k) = \sin((\omega k - 1/2))$. The design problem of high-order digital differentiators is then equivalent to finding $a(k)$ such that $H(\omega)$ approximates the desired frequency response $(\omega/2\pi)^n$ in some optimal sense. Consider that the approximation error $E(\omega)$ is evaluated on a dense grid of frequency linearly distributed from $\omega = 0$ to $\omega = \omega_p$. Then we can form a set of L linear equations as follows

$$E(\omega_i) = (\omega_i/2\pi)^n - \sum_{k=1}^M a(k) \text{trig}(\omega_i, k) \quad (3)$$

with $\omega_1 = 0$ and $\omega_L = \omega_p$. In matrix form, (3) can be written as

$$\mathbf{E} = \mathbf{D} - \mathbf{T}\mathbf{a}, \quad (4)$$

where

$$\begin{aligned} \mathbf{E} &= [E(\omega_1) E(\omega_2) \cdots E(\omega_L)]^T, \\ \mathbf{D} &= [\omega_1^n \omega_2^n \cdots \omega_L^n]^T / (2\pi)^n, \\ \mathbf{a} &= [a(1) a(2) \cdots a(M)]^T, \\ \mathbf{T} &= \begin{bmatrix} \text{trig}(\omega_1, 1) & \text{trig}(\omega_1, 2) & \cdots & \text{trig}(\omega_1, M) \\ \text{trig}(\omega_2, 1) & \text{trig}(\omega_2, 2) & \cdots & \text{trig}(\omega_2, M) \\ \vdots & \vdots & \ddots & \vdots \\ \text{trig}(\omega_L, 1) & \text{trig}(\omega_L, 2) & \cdots & \text{trig}(\omega_L, M) \end{bmatrix}. \end{aligned} \quad (5)$$

In general, $L = 8N$ grid points of frequency are adequate if the filter length is N . Therefore, the objective function to be minimized for L_1 design is given by

$$J = \sum_{i=1}^L W_{L1}(\omega_i) |E(\omega_i)| \quad (6)$$

where $W_{L1}(\omega_i)$, for $i = 1, 2, \dots, L$, are the weight values of the weighting function $W_{L1}(\omega)$ at the frequency grid points. Furthermore, additional constraints can be imposed to make the designed differentiators possess some desired properties. For example, linear constraints at the dc point of the frequency response can be imposed so that the designed differentiators show the maximally accurate property [3]. In general, the maximally accurate requirement can be imposed at any frequency point ω_0 . The required constraints which must be imposed at ω_0 are given as follows

$$H(\omega_0) = \sum_{k=1}^M a(k) \text{trig}(\omega_0, k) = \left(\frac{\omega_0}{2\pi}\right)^n, \quad (7)$$

$$\begin{aligned} \left. \frac{d^q H(\omega)}{d\omega^q} \right|_{\omega=\omega_0} &= \sum_{k=1}^M a(k) \left. \frac{d^q \text{trig}(\omega, k)}{d\omega^q} \right|_{\omega=\omega_0} \\ &= \frac{n(n-1) \cdots (n-q+1)}{(2\pi)^n} \omega_0^{n-q} \end{aligned} \quad (8)$$

for $q = 1, 2, \dots, Q$, where Q is the highest order derivative imposed at ω_0 . In matrix form, these linear constraints can be written as

$$\mathbf{C}\mathbf{a} = \mathbf{K}, \quad (9)$$

where \mathbf{C} is a $(Q + 1) \times M$ constraint matrix given by

$$\mathbf{C} = \begin{bmatrix} \text{trig}(\omega_0, 1) & \text{trig}(\omega_0, 2) \cdots & \text{trig}(\omega_0, M) \\ \left. \frac{d \text{trig}(\omega, 1)}{d\omega} \right|_{\omega=\omega_0} & \left. \frac{d \text{trig}(\omega, 2)}{d\omega} \right|_{\omega=\omega_0} \cdots & \left. \frac{d \text{trig}(\omega, M)}{d\omega} \right|_{\omega=\omega_0} \\ \left. \frac{d^Q \text{trig}(\omega, 1)}{d\omega^Q} \right|_{\omega=\omega_0} & \left. \frac{d^Q \text{trig}(\omega, 2)}{d\omega^Q} \right|_{\omega=\omega_0} \cdots & \left. \frac{d^Q \text{trig}(\omega, M)}{d\omega^Q} \right|_{\omega=\omega_0} \end{bmatrix}. \quad (10)$$

and \mathbf{K} is a $(Q + 1) \times 1$ constraint vector given by

$$\mathbf{K} = \begin{bmatrix} \left(\frac{\omega_0}{2\pi}\right)^n & \frac{n\omega_0^{n-1}}{(2\pi)^n} & \frac{n(n-1)\omega_0^{n-2}}{(2\pi)^n} \\ \cdots & \frac{n(n-1) \cdots (n-Q+1)\omega_0^{n-Q}}{(2\pi)^n} \end{bmatrix}^T. \quad (11)$$

The corresponding design problem becomes a constrained L_1 minimization problem as follows

$$\begin{aligned} \text{Minimize} \quad & J = \sum_{i=1}^L W_{L1}(\omega_i) |E(\omega_i)| \\ \text{Subject to} \quad & \mathbf{C}\mathbf{a} = \mathbf{K}. \end{aligned} \quad (12)$$

(12) represents a highly nonlinear minimization problem. However, it can be reformulated as the following equivalent minimization

problem

$$\begin{aligned} \text{Minimize } & J = \sum_{i=1}^L W_{L1}(\omega_i) E_a(\omega_i) \\ \text{Subject to } & \mathbf{C}\mathbf{a} = \mathbf{K} \\ & |E(\omega_i)| \leq E_a(\omega_i), i = 1, 2, \dots, L, \end{aligned} \quad (13)$$

where $E_a(\omega)$ denotes the upper bound of $|E(\omega)|$. As shown in [9], the values of the upper bound function $E_a(\omega)$ at frequency grid points ω_i are variables to be found during the optimization process since the approximation errors $E(\omega_i)$ can only be computed at each iteration. However, we will present an analytical formula for computing this upper bound during the design process in Section IV. Let $\mathbf{E}_a = [E_a(\omega_1) E_a(\omega_2) \cdots E_a(\omega_L)]^T$, $\mathbf{W}_{L1} = [W_{L1}(\omega_1) W_{L1}(\omega_2) \cdots W_{L1}(\omega_L)]^T$, and \mathbf{o} be an $M \times 1$ zero vector. For notational simplicity, we rewrite (13) as follows

$$\begin{aligned} \text{Maximize } & \mathbf{b}^T \mathbf{w} \\ \text{Subject to } & \mathbf{A}^T \mathbf{w} \leq \mathbf{c}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \mathbf{b} &= [\mathbf{o}^T \quad \mathbf{W}_{L1}^T]^T, \quad \mathbf{w} = [-\mathbf{a}^T - \mathbf{E}_a^T]^T, \\ \mathbf{c} &= [-\mathbf{D}^T - \mathbf{K}^T \quad \mathbf{D}^T \mathbf{K}^T]^T, \\ \mathbf{A} &= \begin{bmatrix} \mathbf{T}^T & \mathbf{C}^T & -\mathbf{T}^T & -\mathbf{C}^T \\ \mathbf{I} & \mathbf{O}^T & \mathbf{I} & \mathbf{O}^T \end{bmatrix}, \end{aligned} \quad (15)$$

and \mathbf{I} is an $L \times L$ identity matrix and \mathbf{O} is an $(Q+1) \times L$ zero matrix. We note that (14) represents a dual form of the following standard LP problem [11]

$$\begin{aligned} \text{Minimize } & \mathbf{c}^T \mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0, \end{aligned} \quad (16)$$

where \mathbf{x} is the parameter vector to be found. Conventional simplex LP algorithms can be employed to solve (14). However, the size of the LP problem is about $(2L + 2Q + 2) \times (M + L)$ and L is approximately equal to $16M$. Therefore, it is very time consuming even if the filter length is moderate if we directly apply LP based algorithms to solve (14).

III. A MODIFIED KARMARKAR'S LP ALGORITHM

Here, we briefly describe the modification of Karmarkar's LP algorithm which has been discussed in [8]. This algorithm was originally developed to solve the primal form minimization problem as shown in (16). Assume that we have a current solution \mathbf{x} which satisfies $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} > 0$. The corresponding feasible region is transformed to place \mathbf{x} near its center. Let $\mathbf{D} = \text{diag}(\mathbf{x})$ denote the diagonal matrix containing the components of \mathbf{x} , $\mathbf{A} = \mathbf{A}\mathbf{D}$, and $\mathbf{c} = \mathbf{D}\mathbf{c}$. Then $\mathbf{x} = \mathbf{D}^{-1}\mathbf{x} = \mathbf{1} = [1 \cdots 1]^T$ is a feasible solution of the following affine scaling version of (16)

$$\begin{aligned} \text{Minimize } & \mathbf{c}^T \mathbf{x} \\ \text{Subject to } & \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0, \end{aligned} \quad (17)$$

and \mathbf{x} is also at the center of the feasible region corresponding to (17). To further reduce the objective function $\mathbf{c}^T \mathbf{x}$ while satisfying the constraints, we update \mathbf{x} in the descent direction as follows

$$\mathbf{d} = -(\mathbf{I} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A})\mathbf{c}. \quad (18)$$

Then we determine the required step size λ to make $\mathbf{c}^T(\mathbf{x} + \lambda\mathbf{d})$ smallest, while keeping $\mathbf{x} + \lambda\mathbf{d} > 0$. It is our experience that an appropriate λ can be set to $0.99/v$ if v is the absolute value of the smallest entry of \mathbf{d} . Finally, we compute $\mathbf{x} = \mathbf{D}(\mathbf{x} + \lambda\mathbf{d})$ and use it as the trial solution for the next iteration. Based on the above description, an algorithm for solving (16) can be summarized as follows.

Algorithm 1:

- Step 1* Select an initial guess \mathbf{x} such that $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} > 0$.
- Step 2* Construct the diagonal matrix $\mathbf{D} = \text{diag}(\mathbf{x})$.
- Step 3* Compute the vector $\mathbf{w} = (\mathbf{A}\mathbf{D}^2\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{D}^2\mathbf{c}$ which is in a WLS solution form.
- Step 4* Compute the descent direction vector $\mathbf{d} = -\mathbf{D}^2(\mathbf{c} - \mathbf{A}^T\mathbf{w})$.
- Step 5* Set the step size $\lambda = |0.99/\min(\mathbf{D}^{-1}\mathbf{d})|$.
- Step 6* Update the parameter vector $\mathbf{x} = \mathbf{x} + \lambda\mathbf{d}$.
- Step 7* If a preset stopping criterion is satisfied, stop the process. Otherwise, go to Step 2.

We note that the main computation load of *Algorithm 1* is dominated by the calculation of the WLS solution for the parameter vector \mathbf{w} required in Step 3. This algorithm iteratively searches the optimal solution for (16). It has been shown in [8] and [9] that the parameter vector \mathbf{w} also converges to the optimal solution of (14) as the parameter vector \mathbf{x} converges to the optimal solution of (16).

IV. THE PROPOSED DESIGN METHOD

In this section, we present an efficient method based on *Algorithm 1* to solve the digital differentiator design problem formulated in (14). This method provides a systematic procedure to iteratively search the solution optimal in the L_1 sense. The detail of the design procedure is described step by step as follows

Step 1: Let the $(2L + 2Q + 2) \times 1$ parameter vector \mathbf{x} be decomposed as $\mathbf{x} = [\mathbf{x}_1^T \quad \mathbf{x}_{1c}^T \quad \mathbf{x}_2^T \quad \mathbf{x}_{2c}^T]^T$, where \mathbf{x}_1 and \mathbf{x}_2 are of size $L \times 1$, \mathbf{x}_{1c} and \mathbf{x}_{2c} are of size $(Q+1) \times 1$. Then we choose an initial guess for \mathbf{x} to satisfy the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} > 0$. It is an appropriate manner to choose the initial \mathbf{x} by setting $\mathbf{x}_1 = \mathbf{x}_2 = (1/2)\mathbf{W}_{L1}$ and $\mathbf{x}_{1c} = \mathbf{x}_{2c} = (1/2)\mathbf{1}$.

Step 2: Compute the associated WLS solution vector $\mathbf{w} = (\mathbf{A}\mathbf{D}^2\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{D}^2\mathbf{c}$. As shown in Appendix, instead of directly performing the matrix inversion $(\mathbf{A}\mathbf{D}^2\mathbf{A}^T)^{-1}$ which is of size $(M+L) \times (M+L)$, we propose an efficient procedure for computing \mathbf{w} as follows

- (2.1) Construct the diagonal matrices $\mathbf{D}_1 = \text{diag}(\mathbf{x}_1)$ and $\mathbf{D}_2 = \text{diag}(\mathbf{x}_2)$.
- (2.2) Construct the diagonal matrices $\mathbf{D}_{1c} = \text{diag}(\mathbf{x}_{1c})$ and $\mathbf{D}_{2c} = \text{diag}(\mathbf{x}_{2c})$.
- (2.3) Compute the diagonal matrix $\mathbf{W} = (\mathbf{D}_1^2 + \mathbf{D}_2^2) - (\mathbf{D}_1^2 - \mathbf{D}_2^2)^2(\mathbf{D}_1^2 + \mathbf{D}_2^2)^{-1}$.
- (2.4) Compute the diagonal matrix $\mathbf{W}_c = \mathbf{D}_{1c}^2 + \mathbf{D}_{2c}^2$.
- (2.5) Compute the coefficient vector $\mathbf{a} = (\mathbf{T}^T\mathbf{W}\mathbf{T} + \mathbf{C}^T\mathbf{W}_c\mathbf{C})^{-1}(\mathbf{T}^T\mathbf{W}\mathbf{D} + \mathbf{C}^T\mathbf{W}_c\mathbf{K})$.
- (2.6) Compute the corresponding error vector $\mathbf{E} = \mathbf{D} - \mathbf{T}\mathbf{a}$.
- (2.7) Compute the associated error bound vector $\mathbf{E}_a = (\mathbf{D}_1^2 + \mathbf{D}_2^2)^{-1}(\mathbf{D}_1^2 - \mathbf{D}_2^2)\mathbf{E}$.
- (2.8) Finally, we form the desired parameter vector $\mathbf{w} = [-\mathbf{a}^T - \mathbf{E}_a^T]^T$.

Step 3: Note that $\mathbf{W}_{L1}^T|\mathbf{E}|$ represents the weighted L_1 norm error associated with the current coefficient vector \mathbf{a} . This weighted error is approximately equal to $\mathbf{W}_{L1}^T\mathbf{E}_a$ when the iteration process approaches the optimal solution. Our design experience shows that it is suitable to stop the design procedure if $(\mathbf{W}_{L1}^T\mathbf{E}_a)/(\mathbf{W}_{L1}^T|\mathbf{E}|) > 0.999$. Otherwise, we go to *Step 4*.

TABLE I
THE COEFFICIENTS OF THE DESIGNED
DIFFERENTIATOR USING THE PROPOSED METHOD

n	h(n)	n	h(n)
0	-0.00001477452516	8	-0.00025419188471
1	0.00002998211507	9	0.00034443585237
2	-0.00004649166916	10	-0.00048423032569
3	0.00006510495606	11	0.00071808258782
4	-0.00008695508521	12	-0.00114962807672
5	0.00011361937013	13	0.00203998940861
6	-0.00014746599087	14	-0.00356015325897
7	0.00019223916555	15	0.00496822660347

Step 4: Compute the appropriate descent direction vector $\mathbf{d} = -\mathbf{D}^2(\mathbf{c} - \mathbf{A}^T \mathbf{w})$. However, instead of finding \mathbf{d} directly, we can compute the descent direction vectors \mathbf{d}_1 , \mathbf{d}_{1c} , and \mathbf{d}_{2c} corresponding to \mathbf{x}_1 , \mathbf{x}_{1c} , and \mathbf{x}_{2c} , respectively as follows

$$\mathbf{d}_1 = -\mathbf{W}\mathbf{E}/2, \quad \mathbf{d}_{1c} = \mathbf{D}_{1c}^2(\mathbf{C}\mathbf{a} - \mathbf{K}),$$

and

$$\mathbf{d}_{2c} = -\mathbf{D}_{2c}^2(\mathbf{C}\mathbf{a} - \mathbf{K}). \quad (19)$$

Step 5: Determine the required step size λ as follows in (20) shown at the bottom of the page.

Step 6: Update the current parameter vectors \mathbf{x}_1 , \mathbf{x}_2 , \mathbf{x}_{1c} , and \mathbf{x}_{2c} according to

$$\mathbf{x}_1 = \mathbf{x}_1 - \lambda \mathbf{d}_1, \quad \mathbf{x}_2 = \mathbf{W}_{L1} - \mathbf{x}_1, \quad \mathbf{x}_{1c} = \mathbf{x}_{1c} - \lambda \mathbf{d}_{1c},$$

and

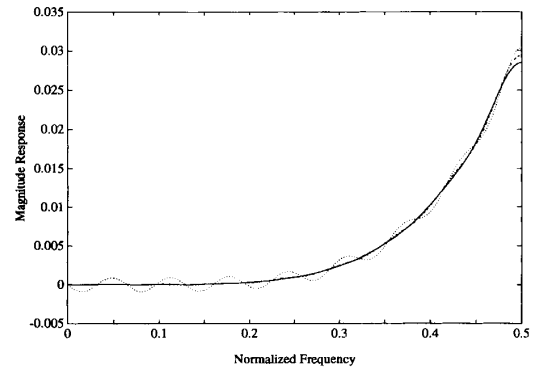
$$\mathbf{x}_{2c} = \mathbf{x}_{2c} - \lambda \mathbf{d}_{2c}. \quad (21)$$

Then go to *Step 2*.

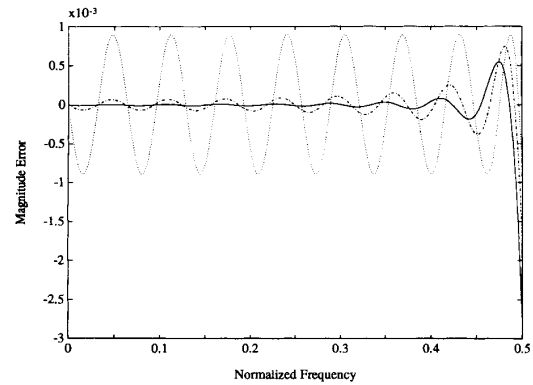
From the above description, we note that the convergence of the proposed design method is ensured by the existence of the inverse of $(\mathbf{T}^T \mathbf{W} \mathbf{T} + \mathbf{C}^T \mathbf{W}_c \mathbf{C})$. Ideally, $\mathbf{T}^T \mathbf{W} \mathbf{T}$ is a positive definite matrix and $\mathbf{C}^T \mathbf{W}_c \mathbf{C}$ is a positive semi-definite matrix. Therefore, $(\mathbf{T}^T \mathbf{W} \mathbf{T} + \mathbf{C}^T \mathbf{W}_c \mathbf{C})$ is a positive definite matrix and hence its inverse exists. In practice, it is our experience that taking the inverse of $(\mathbf{T}^T \mathbf{W} \mathbf{T} + \mathbf{C}^T \mathbf{W}_c \mathbf{C})$ may cause numerical problems when the singular values of $\mathbf{C}^T \mathbf{W}_c \mathbf{C}$ spread too large or the condition number (the ratio of the largest and the smallest eigenvalues) of the diagonal matrix \mathbf{W} is too large. However, this numerical difficulty can be avoided by increasing the machine precision.

V. A DESIGN EXAMPLE

Here, an example of designing high order digital differentiators using the proposed method is presented for illustration. The simulations are performed on a 117 MHz 80486 personal computer using the MATLAB programming language. The coefficients of the designed differentiator using the proposed method are listed in Table I for the



(a)



(b)

Fig. 1. The frequency responses of the designed differentiator for the example. Solid line: The proposed method, dot line: The Chebyshev design, dash-dot line: The eigenfilter method.

design example. Due to the symmetry of coefficients, we only list $h(n)$ for $n = 0, 1, 2, \dots, 15$.

The Design Example: In this case, the design of a full-band and fifth-order digital differentiator with filter length $N = 32$ is considered. The error weighting function $W_{L1}(\omega)$ is set to 1 over the entire frequency band. We perform the design using several error criteria, namely L_1 , L_2 , and Chebyshev criteria. The design results are shown in Fig. 1. The results using the proposed method (i.e., the L_1 design) are obtained after 14 iterations (about 7 seconds in CPU). The L_2 design results are obtained using the eigenfilter approach of [2]. Although Chebyshev design provides equiripple magnitude error response, it has the largest error ringing in the lower frequency band. In contrast, the proposed method based on L_1 error criterion produces the best wideband accurate differentiation over almost the entire frequency band.

APPENDIX

Since the parameter vector \mathbf{w} is given by

$$\mathbf{w} = [-\mathbf{a}^T - \mathbf{E}_a^T]^T = (\mathbf{A}\mathbf{D}^2\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{D}^2\mathbf{c}, \quad (22)$$

$$\lambda = \frac{0.99}{\text{MAX}\{\text{MAX}(\mathbf{D}_1^{-1}\mathbf{d}_1), \text{MAX}(-\mathbf{D}_2^{-1}\mathbf{d}_1), \text{MAX}(\mathbf{D}_{1c}^{-1}\mathbf{d}_{1c}), \text{MAX}(\mathbf{D}_{2c}^{-1}\mathbf{d}_{2c})\}}. \quad (20)$$

we have

$$\begin{aligned} & \begin{bmatrix} -\mathbf{a} \\ -\mathbf{E}_a \end{bmatrix} \\ &= \begin{pmatrix} \mathbf{T}^T & \mathbf{C}^T & -\mathbf{T}^T & -\mathbf{C}^T \\ \mathbf{I} & \mathbf{O}^T & \mathbf{I} & \mathbf{O}^T \end{pmatrix} \begin{bmatrix} \mathbf{D}_1^2 & & & \\ & \mathbf{D}_{1c}^2 & & \\ & & \mathbf{D}_2^2 & \\ & & & \mathbf{D}_{2c}^2 \end{bmatrix} \\ & \quad \times \begin{pmatrix} \mathbf{T} & \mathbf{I} \\ \mathbf{C} & \mathbf{O} \\ -\mathbf{T} & \mathbf{I} \\ -\mathbf{C} & \mathbf{O} \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{T}^T & \mathbf{C}^T & -\mathbf{T}^T & -\mathbf{C}^T \\ \mathbf{I} & \mathbf{O}^T & \mathbf{I} & \mathbf{O}^T \end{bmatrix} \\ & \quad \times \begin{bmatrix} \mathbf{D}_1^2 & & & \\ & \mathbf{D}_{1c}^2 & & \\ & & \mathbf{D}_2^2 & \\ & & & \mathbf{D}_{2c}^2 \end{bmatrix} \begin{bmatrix} -\mathbf{D} \\ -\mathbf{K} \\ \mathbf{D} \\ \mathbf{K} \end{bmatrix}, \quad (23) \end{aligned}$$

where $\mathbf{D}_1 = \text{diag}(\mathbf{x}_1)$, $\mathbf{D}_2 = \text{diag}(\mathbf{x}_2)$, $\mathbf{D}_{1c} = \text{diag}(\mathbf{x}_{1c})$, and $\mathbf{D}_{2c} = \text{diag}(\mathbf{x}_{2c})$. From (23), we obtain

$$\begin{aligned} & \begin{bmatrix} \mathbf{a} \\ \mathbf{E}_a \end{bmatrix} = \\ & \begin{bmatrix} \mathbf{T}^T(\mathbf{D}_1^2 + \mathbf{D}_2^2)\mathbf{T} + \mathbf{C}^T(\mathbf{D}_{1c}^2 + \mathbf{D}_{2c}^2)\mathbf{C} & \mathbf{T}^T(\mathbf{D}_1^2 - \mathbf{D}_2^2) \\ & (\mathbf{D}_1^2 - \mathbf{D}_2^2)\mathbf{T} & & (\mathbf{D}_1^2 + \mathbf{D}_2^2) \end{bmatrix}^{-1} \\ & \quad \times \begin{bmatrix} \mathbf{T}^T(\mathbf{D}_1^2 + \mathbf{D}_2^2)\mathbf{D} + \mathbf{C}^T(\mathbf{D}_{1c}^2 + \mathbf{D}_{2c}^2)\mathbf{K} \\ & (\mathbf{D}_1^2 - \mathbf{D}_2^2)\mathbf{D} \end{bmatrix} \quad (24) \end{aligned}$$

After some algebraic manipulations, the coefficient vector \mathbf{a} is given by

$$\begin{aligned} \mathbf{a} = & \{ \mathbf{T}^T(\mathbf{D}_1^2 + \mathbf{D}_2^2)\mathbf{T} + \mathbf{C}^T(\mathbf{D}_{1c}^2 + \mathbf{D}_{2c}^2)\mathbf{C} \\ & - \mathbf{T}^T(\mathbf{D}_1^2 - \mathbf{D}_2^2)(\mathbf{D}_1^2 + \mathbf{D}_2^2)^{-1}(\mathbf{D}_1^2 - \mathbf{D}_2^2)\mathbf{T} \}^{-1} \\ & \times \{ \mathbf{T}^T[(\mathbf{D}_1^2 + \mathbf{D}_2^2) - (\mathbf{D}_1^2 - \mathbf{D}_2^2)] \\ & \times (\mathbf{D}_1^2 + \mathbf{D}_2^2)^{-1}(\mathbf{D}_1^2 - \mathbf{D}_2^2)\mathbf{D} \\ & + \mathbf{C}^T(\mathbf{D}_{1c}^2 - \mathbf{D}_{2c}^2)\mathbf{K} \}. \quad (25) \end{aligned}$$

Let

$$\mathbf{W} = (\mathbf{D}_1^2 + \mathbf{D}_2^2) - (\mathbf{D}_1^2 - \mathbf{D}_2^2)(\mathbf{D}_1^2 + \mathbf{D}_2^2)^{-1}(\mathbf{D}_1^2 - \mathbf{D}_2^2)$$

and

$$\mathbf{W}_c = \mathbf{D}_{1c}^2 - \mathbf{D}_{2c}^2. \quad (26)$$

Substituting (26) into (25) yields

$$\mathbf{a} = (\mathbf{T}^T \mathbf{W} \mathbf{T} + \mathbf{C}^T \mathbf{W}_c \mathbf{C})^{-1} (\mathbf{T}^T \mathbf{W} \mathbf{D} + \mathbf{C}^T \mathbf{W}_c \mathbf{K}). \quad (27)$$

Moreover, from (24), the associated error bound vector \mathbf{E}_a during the design process is given by

$$\mathbf{E}_a = (\mathbf{D}_1^2 + \mathbf{D}_2^2)^{-1}(\mathbf{D}_1^2 - \mathbf{D}_2^2)(\mathbf{D} - \mathbf{T}\mathbf{a}). \quad (28)$$

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On Some Properties of N -D Discrete-Time BR Systems and Time-Delay BR Systems

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Abstract—Based on N -D state equations, we present sufficient condition for N -D discrete-time systems to be structurally stable bounded real (BR) and its circuits theoretical interpretations. N -D discrete-time systems satisfying this condition can be embedded into a certain augmented lossless bounded real (LBR) system and energy balanced realization is possible. These results can be extended straightforward to time-delay systems which are BR independent of delay (BR i. o. d.). But in the case BR property is dependent on delay (BR d. o. d.), such extension is impossible.

I. INTRODUCTION

In recent years, the idea of BR and LBR property [1], [3] has been used to the various design problems, such as low coefficient sensitivity filter design [2], [3], H_∞ control systems design [4] and learning control systems design [5]. The aim of this paper is to extend the idea of BR and LBR property of one-dimensional (1-D) systems to N -dimensional (N -D) systems. N -D BR property has relation with i) low sensitivity multidimensional digital filter design, ii) lumped and distributed elements network synthesis, iii) robust controller design

Manuscript received August 10, 1992; revised February 27, 1994 and June 12, 1994. This work was supported by The Institute of Science and Technology, Meiji University under Grant 1992 Academic Year Intensive Research. This paper was recommended by Associate Editor P. Agathoklis.

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IEEE Log Number 9409267.