The Effects of Quantization Noise and Sensor Nonideality on Digital Differentiator-Based Rate Measurement

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***Abstract*—This paper focuses on the nature of the rate error which arises when a first-order digital differentiator is applied to the output of a uniform quantizer for the purpose of rate estimation. The quantizer input is assumed to be a constant-rate signal which is subject to a uniformly distributed noise source. New formulae are presented for the corresponding rms and spectral error characteristics. The results are applicable to the signal processing of sensor outputs, exemplified by the shaft encoder-based velocity estimation of an almost uniformly rotating mass. Both computer-generated finite data sets and experimental data derived from encoder-based shaft velocity measurements are utilized to verify the theoretical results. The results described are also applicable to a more general class of systems which involve the digital differentiation of quantized, noise-affected signals, such as first-order sigma–delta modulators with nominally constant input.**

***Index Terms*— Differentiation, digital measurements, optical transducers, optical velocity measurement, quantization, sigma–delta modulation, tachometers.**

I. Introduction

SIGNAL quantization can represent a significant limiting factor in the resolution of many sensor-based data-acquisition and control systems. When a knowledge of the rate of change of a digitally represented measurand is of importance, the quality of output of the digital differentiator is especially prone to be adversely affected by quantization. Additional rate estimation error is introduced when the signal is subject to added electrical noise prior to quantization, or equivalently, when finite manufacturing tolerances introduce variations in the quantizer threshold values from those expected of a regular unit.

Commonly encountered examples of the phenomena under investigation in this paper are seen in the generation of velocity estimates for a rotating mass using regularly sampled digital position information, usually obtained by counting the output transitions of an optical shaft encoder, or similar transducer. Various velocity estimators, incorporating observers [1], auxiliary counting hardware [2], [3], or various filtering algorithms [4], [5] have been designed. However, the simplicity of the simple first-order digital differentiator leads to its widespread use in pulse-count tachometers and motion-control systems [6]. Such a rate estimator is considered in this paper.

Manuscript received November 16, 1998.

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Publisher Item Identifier S 0018-9456(98)09702-2.

When the system under investigation rotates at very close to a uniform rate, as is common in speed-sensitive industrial applications, the effects of quantization on the differentiator output can exceed the actual velocity variation of the system under test. Many additional error sources are introduced by imperfections in incremental encoders [7]. In the context of a digital differentiator, the predominant error is likely to be the differential nonlinearity caused by random variation of any particular encoder transition location from that expected of an ideal system. Experimentation with an M/T-type digital tachometer [2], [3] permits accurate estimation of the transition noise. The probability density function (pdf) of the differential error (the difference between the errors associated with the two transition locations used in any particular velocity calculation), as shown in Fig. 1 for the encoder under test, indicates that the noise present in the real data will be satisfactorily modeled by a triangularly distributed differential error, which is consistent with uniformly distributed transition locations. (A truncated Gaussian assumption is also reasonable but would complicate the analysis unnecessarily.) The phasing errors associated with quadrature decoding and the presence of point defects are other error types which are often adequately modeled by a differential nonlinearity. Code-wheel eccentricity or shaft misalignment can give rise to an integral nonlinearity (termed "position error" in [7]) corresponding to a low-frequency variation in transition locations from their expected values over a mechanical revolution. This variation, which is found to be very close to sinusoidal in many systems, can be incorporated into an extended model, if required.

The initial analytical results on exact characterization of quantized systems such as pulse-code modulators were presented by Clavier *et al.* in 1947 [8]. Many of the more recent results have followed from extensive investigation of sigma–delta modulators. The two IEEE Press volumes [9], [10] contain many of the standard reference papers in this field. The analysis presented below employs a similar methodology to that utilized (for the analysis of quantization noise) by Gray in [11] (reprinted in [9] and, somewhat modified, in [10]). Digital differentiation is intrinsic to the operation of the single-loop sigma–delta modulator. The quantization error resulting from application of a dc input to such a converter is identical to that which results when a constant-rate signal is uniformly quantized, the difference between two successive discrete samples of the quantizer output providing an estimate of the actual signal rate. Such a converter was analyzed in [12].

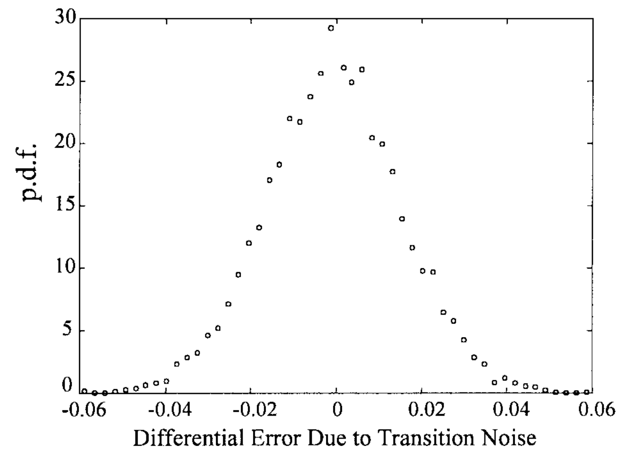


Fig. 1. Experimental determination of the pdf of the differential error in calculated velocity because of transition noise in an incremental shaft encoder.

Previous work on the application of dithering signals to whiten the effects of quantization noise [13]–[17] has similarities to the consideration of transition noise in sensors. However, the lack of accessibility of the quantizer input signal to a digital sensor constitutes an important difference between such sensors and analog-to-digital converters where the input signal can be manipulated by analog prefiltering or user-selected dither signals.

II. Definition of System Characteristics (Sensor Model)

It is assumed that a signal, p, undergoing a constant rate of change, v, is subject to an additive noise process, $\{w\}$, which is uniformly distributed over $[-\epsilon, +\epsilon]$. The resultant signal, $y = p + w$, is operated on by a regular quantizer, as shown in Fig. 2. For simplicity, units of bits and bits per sample-time are used for the discrete samples of the signal and its rate, respectively, so that the signal at sample i is defined by $p_i = p_0 + v\,i$, p_0 being the signal value at the time-origin. When v is irrational, $\{\langle p_i \rangle\}$, the process corresponding to the fractional part of p_i, and $\{\langle y \rangle\}$ are both uniformly distributed over $[0, 1)$ (henceforth termed *uniform covering*) when an infinite sample is considered. The quantizer is defined by

$$\begin{aligned} q(y) &= \lfloor y + \tfrac{1}{2} \rfloor = y + \tfrac{1}{2} - \langle y + \tfrac{1}{2} \rangle, \\ e'(y) &= \langle y + \tfrac{1}{2} \rangle - \tfrac{1}{2} \end{aligned} \tag{1}$$

where $e'(y)$ represents the *quantization error* [13]. Regular samples of the quantizer output generate a corresponding process $\{e'_i\}$, where e'_i is defined by $e'_i = y_i - q(y_i)$ for sample i.

Possible noise sources include stochastic noise corruption of an input signal prior to the digitization of said analog signal, and the presence of transition noise in a sensor, as described above. Both types of noise source can be represented by a single model

$$p_d = q(p + w) = q(y) = \lfloor p + w \rfloor + 1/2 \tag{2}$$

where, in the case of transition noise, w represents the negative of the deviation of the position at which an output transition occurs relative to the nominal value. (The consequences of intentionally adding a dither signal in the form of an independent, identically distributed process, as commonly implemented in control and audio–visual systems, can equally be evaluated using this model.) The *quantization noise*, e, (the overall noise associated with the conversion process [13]) is given by $e = p - q(y) = e' - w$.

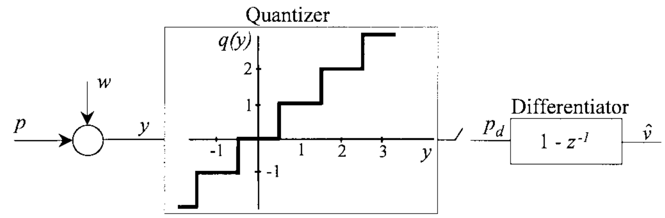


Fig. 2. Model of rate estimator using a digital differentiator. The estimator output is discretized, yielding p_d, prior to obtaining a velocity estimate, $\hat{v}$, by digital differentiation.

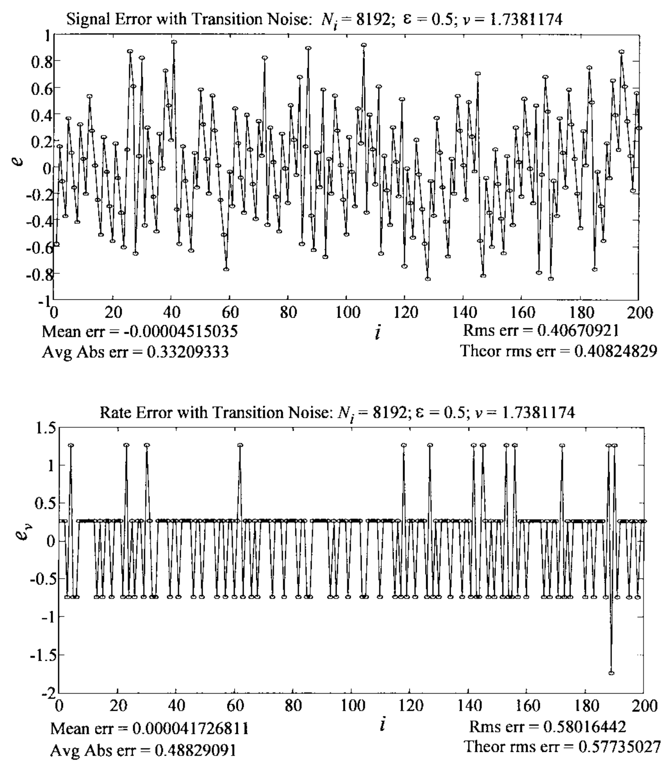


Fig. 3. Typical plots of signal error, e, and rate error, e_v, for a constant-rate signal ($v \approx 1.738\,1174$) with transition noise ($\epsilon = 0.5$).

Assuming a first-order differentiator, the rate computation provides an estimate of v given by

$$\begin{aligned} \hat{v}_i &= p_{d(i)} - p_{d(i-1)} \\ &= (p_i - e_i) - (p_{i-1} - e_{i-1}) \\ &= v - (e_i - e_{i-1}) \end{aligned} \tag{3}$$

so that the velocity estimate error, e_v, at sample i, is

$$e_{v(i)} = v - \hat{v}_i = e_i - e_{i-1}. \tag{4}$$

Fig. 3 shows typical time-domain plots of computer-generated finite data sets ($N_i = 8192$ data points) of the signal and rate errors, illustrating the effect of transition noise. (The actual rms error is found to be well predicted using the theoretical formulae derived below.)

III. Derivation of Formula for RMS Quantization Noise of the Rate Error

The rms quantization noise and the power spectrum associated with the rate error process $\{e_{v(i)}\}$ can be obtained through knowledge of the corresponding autocorrelation function $R_v(r)$. It is convenient to concentrate initially on $R_e(r)$, the autocorrelation of the error signal, the significant relationships between the signal error and rate error being

$$R_v(r) = 2R_e(r) - R_e(r+1) - R_e(r-1), \tag{5}$$

$$S_v(f) = 2(1 - \cos(2\pi k v))S_e(f) \tag{6}$$

where $S_e(f)$ and $S_v(f)$ are the spectral densities corresponding to the signal and rate error processes.

The quantization nonlinearity is made tractable for mathematical analysis through use of the Fourier series representation of the quantization error function, e', where the quantizer input, y, is the independent variable

$$\begin{aligned} e' &= e'(y) \\ &= \langle y \rangle - \frac{1}{2} \\ &= \sum_{k=1}^{\infty} -\frac{1}{\pi k} \sin(2\pi k y) \\ &= \sum_{k \neq 0} -\frac{1}{2\pi j k} e^{2\pi j k y} \end{aligned} \tag{7}$$

while the corresponding expansion for $e'(y)^2$ can be shown to equal

$$e'(y)^2 = \frac{1}{12} + \sum_{k\neq 0} \frac{1}{2(\pi k)^2} e^{2\pi j k y}. \tag{8}$$

A. Characteristic Functions

Initially, the quantization error process $\{e'\}$ is explored in a manner similar to that presented in [11], by recognizing that the characteristic function of a stationary, stochastic process, $\{w\}$, can be expressed as

$$\Phi_w(x) = E\{e^{2\pi j x w_i}\} \tag{9}$$

where $E\{\cdot\}$ represents the time-independent expectation. For the uniform distribution described in Section II

$$\Phi_w(x) = \frac{\sin 2\pi x\epsilon}{2\pi x \epsilon}. \tag{10}$$

Because $\{p\}$ and $\{w\}$ are independent processes, the one-dimensional characteristic function of y_i is

$$\begin{aligned} \overline{\Phi}_y(k) &= \overline{E}\left\{e^{2\pi j k(p_i+w_i)}\right\} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} e^{2\pi j k(p_i+w_i)} \\ &= \overline{\Phi}_p(k)\,\Phi_w(k), \end{aligned} \tag{11}$$

where the characteristic function for the signal p is obtained using Ljunj's expectation function for quasistationary processes, $\overline{E}\{\cdot\}$, [14]. When the signal is uniform covering, it is easily shown (through use of Weyl's ergodic theorem for fractional operators, as illustrated in [11]) that the one-dimensional (1-D) characteristic function, $\overline{\Phi}_p(k)$, is given by

$$\overline{\Phi}_p(k) = \begin{cases} 1, & k = 0; \\ 0, & k \neq 0. \end{cases} \tag{12}$$

Similarly, the two-dimensional (2-D) characteristic function is

$$\begin{aligned} \overline{\Phi}_y^r(k, h) &= \overline{E}\left\{e^{2\pi j k(p_i+w_i)}\, e^{2\pi j h(p_{i+r}+w_{i+r})}\right\} \\ &= \overline{\Phi}_p^r(k, h)\,\Phi_w(k)\,\Phi_w(h), \quad r \neq 0 \end{aligned} \tag{13}$$

because the two noise processes are independent when $r \neq 0$. Again, when $\{\langle p_i \rangle\}$ is uniform covering, $\overline{\Phi}_y^r(k, h)$ reduces to

$$\overline{\Phi}_p^r(k, h) = \begin{cases} e^{2\pi j k r v}, & k = -h; \\ 0, & \text{otherwise}; \end{cases} \tag{14}$$

upon making substitutions of the form $p_i = p_0 + i v$, and applying Weyl's theorem as above. When $r = 0$

$$\begin{aligned} \overline{\Phi}_y^0(k, h) &= \overline{E}\left\{e^{2\pi j k(p_i+w_i)}\, e^{2\pi j h(p_i+w_i)}\right\} \\ &= \overline{\Phi}_p(k+h)\,\Phi_w(k+h). \end{aligned} \tag{15}$$

B. Quantization Error

The autocorrelation of the quantization error process is

$$\begin{aligned} R_{e'}(r) &= \sum_{k\neq 0} \sum_{h\neq 0} \frac{1}{2\pi j k} \frac{1}{2\pi j h} \\ &\quad \cdot \overline{E}\left\{e^{2\pi j k\,(p_i+w_i)}\, e^{2\pi j h\,(p_{i+r}+w_{i+r})}\right\} \\ &= \sum_{k\neq 0} \sum_{h\neq 0} -\frac{1}{4\pi^2 k h}\, \overline{\Phi}_p^r(k, h) \\ &\quad \cdot \Phi_w(k)\,\Phi_w(h), \qquad r \neq 0. \end{aligned} \tag{16}$$

The expression sums to zero unless $k + h = 0$, under the assumption that $\{p_i\}$ is uniform covering. Making use of (10) and (14)

$$R_{e'}(r) = \sum_{k\neq 0} \frac{1}{4\pi^2 k^2} e^{2\pi j k \langle r v \rangle} \frac{\sin^2(2\pi k\epsilon)}{(2\pi k \epsilon)^2} \tag{17}$$

$$= \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2\pi k \langle r v \rangle)}{k^2} \frac{\sin^2(2\pi k \epsilon)}{(2\pi k \epsilon)^2}. \tag{18}$$

Therefore

$$R_{e'}(r) = \frac{1}{8\pi^4 \epsilon^2} \sum_{k=1}^{\infty} \frac{\cos(2\pi k \langle r v \rangle)\, \sin^2(2\pi k \epsilon)}{k^4}. \tag{19}$$

The mean-squared quantizer error, which equates to $R_{e'}(0)$, is given by

$$R_{e'}(0) = \frac{1}{12} + \sum_{k\neq 0} \frac{1}{2(\pi k)^2}\, \overline{\Phi}_p(k) \Phi_w(k) = \frac{1}{12} \tag{20}$$

because $\overline{\Phi}_p(k) = 0,\ \forall k \neq 0$.

The mean-squared quantizer error is independent of the magnitude of the transition noise because it is simply the error corresponding to the quantization of $y = p + w$ where $\langle p+w \rangle$ is uniform covering. A closed-form expression for the autocorrelation of the signal quantization error, $R_{e'}(r)$, $r \neq 0$, is derived in Appendix I as

$$\begin{aligned} R_{e'}(r) = \frac{1}{96\epsilon^2} \big[& \left(\langle rv + 2\epsilon \rangle^2 + \langle rv - 2\epsilon \rangle^2 - 2 \langle rv \rangle^2 \right) \\ & - 2\left(\langle rv + 2\epsilon \rangle^3 + \langle rv - 2\epsilon \rangle^3 - 2 \langle rv \rangle^3 \right) \\ & + \left(\langle rv + 2\epsilon \rangle^4 + \langle rv - 2\epsilon \rangle^4 - 2 \langle rv \rangle^4 \right) \big], \quad r \neq 0. \end{aligned} \tag{21}$$

C. Quantization Noise

The mean-squared signal quantization noise is

$$\overline{E}\{e^2\} = \overline{E}\{(e')^2\} + E\{w^2\} - 2R_{e'w}(0) \tag{22}$$

where $R_{e'w}(\cdot)$ represents the cross-correlation between the two processes

$$\begin{aligned} R_{e'w}(r) &= \overline{E}\{e'_i \, w_{i+r}\} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{k\neq 0} -\frac{1}{2\pi jk} w_{i+r} \, e^{2\pi jk\,(p_i + w_i)} \\ &= \sum_{k \neq 0} -\frac{1}{2\pi jk} \overline{E}\left\{ w_{i+r} \, e^{2\pi jk\,(p_i+w_i)} \right\} \\ &= \sum_{k \neq 0} -\frac{1}{2\pi jk} \left(E\{ w_{i+r} \, e^{2\pi jk\, w_i} \} \, \overline{\Phi}_p(k) \right) \end{aligned} \tag{23}$$

because of the independence of $\{p_i\}$ and $\{w_i\}$. When $r \neq 0$, it is clear that $R_{e'w}(r) = 0$. Additionally, (12) guarantees that $R_{e'w}(0) = 0$, if $\{\langle p_i \rangle\}$ is uniform covering. It is noteworthy that this result is not conditional on the nature of the added noise or dither, the only requirement being that it is independent of the input signal.

When $r \neq 0$, $R_e(r)$ which is defined by $\overline{E}\{(e'_i - w_i)\,(e'_{i+r} - w_{i+r})\}$, reduces to $R_{e'}(r)$, irrespective of the properties of $\{p_i\}$, due to the nature of the transition noise source. The autocorrelation of the quantization noise is given by

$$R_e(r) = R_{e'}(r) + R_w(r) - 2R_{e'w}(r). \tag{24}$$

The autocorrelation of the uniform transition noise is zero for $r \neq 0$, while $R_w(0) = \epsilon^2/3$. The cross-correlation between the quantization error and the transition noise is zero when the fractional part of the input signal is uniform covering, so that

$$R_e(r) = \begin{cases} R_{e'}(0) + \dfrac{\epsilon^2}{3} = \dfrac{1}{12} + \dfrac{\epsilon^2}{3}, & r = 0; \\ R_{e'}(r), & r \neq 0. \end{cases} \tag{25}$$

The autocorrelation of the rate error is obtained by making use of (5), and the even nature of the autocorrelation function

$$R_v(r) = \begin{cases} \dfrac{1}{6} + \dfrac{2\epsilon^2}{3} - 2R_{e'}(1), & r = 0; \\ 2R_{e'}(1) - \dfrac{1}{6} - \dfrac{2\epsilon^2}{3}, & r = 1; \\ 2R_{e'}(r) - R_{e'}(r+1) - R_{e'}(r-1), & r > 1. \end{cases} \tag{26}$$

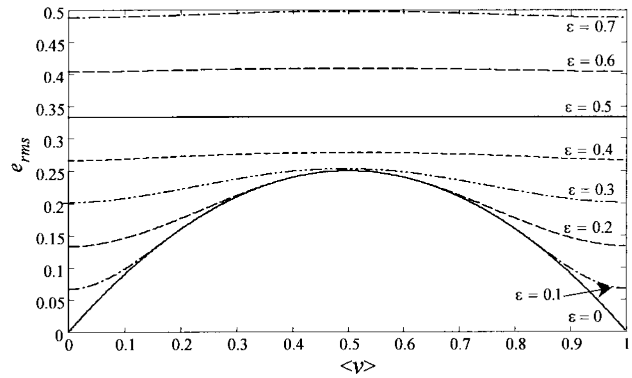


Fig. 4. The rms error of the digital differentiator, as a function of fractional input rate and transition noise magnitude.

The mean-squared rate error is directly obtainable from $R_v(0)$, utilizing (21)

$$\begin{aligned} e_{ms} &= \frac{2\epsilon^2}{3} + \frac{1}{6} - 2R_{e'}(1) \\ &= \frac{2\epsilon^2}{3} + \frac{1}{6} - \frac{1}{48\epsilon^2} \big[\left(\langle v + 2\epsilon \rangle^2 + \langle v - 2\epsilon \rangle^2 - 2\langle v \rangle^2 \right) \\ &\quad - 2\left(\langle v + 2\epsilon \rangle^3 + \langle v - 2\epsilon \rangle^3 - 2\langle v \rangle^3 \right) \\ &\quad + \left(\langle v + 2\epsilon \rangle^4 + \langle v - 2\epsilon \rangle^4 - 2\langle v \rangle^4 \right) \big] \end{aligned} \tag{27}$$

so that the rms error associated with the digital differentiator, $e_{\rm rms}$, is the square-root of this quantity. Two special cases are noteworthy: 1) when $\epsilon = 0$, (27) reduces to $e_{\rm rms} = \sqrt{\langle v \rangle (1 - \langle v \rangle)}$ and 2) when $\epsilon = 1/2$, $e_{\rm rms} = 1/\sqrt{3}$ independent of v. Fig. 4 illustrates the variation in this error measure as a function of both the fractional part of the rate, $\langle v \rangle$, and the maximum variation of the added transition noise, ϵ. It should be noted that small levels of noise have an insignificant effect on the overall rms quantization noise, particularly when $\langle v \rangle \approx 0.5$ (i.e., at those rates for which quantization noise is large). The increased rms noise when $\epsilon = 0.5$ is the cost associated with ensuring that the quantizer output exhibits a white noise property.

IV. Power Density Spectrum of the Rate Error

An expression for the spectral density of the signal error is derived in Appendix II as

$$\begin{aligned} S_e(f) = &\frac{\epsilon^2}{3} + \frac{1}{12} - \frac{\langle 2\epsilon \rangle^2}{48\epsilon^2} \left[1 - 2\langle 2\epsilon \rangle + \langle 2\epsilon \rangle^2 \right] \\ &+ \sum_{k \neq 0} \frac{1}{4\pi^2 k^2} \frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2} \delta(f - \langle kv \rangle) \end{aligned} \tag{28}$$

where the Dirac delta function indicates a finite power contribution at those frequencies given by $f = \langle kv \rangle$. The rate error spectrum is derived using (6) to be

$$\begin{aligned} S_v(f) = 2[1 - \cos(2\pi f)] \Bigg[&\frac{\epsilon^2}{3} + \frac{1}{12} - \frac{\langle 2\epsilon \rangle^2}{48\epsilon^2} \\ &\cdot [1 - 2\langle 2\epsilon \rangle + \langle 2\epsilon \rangle^2] + \sum_{k\neq 0} \frac{1}{4\pi^2 k^2} \\ &\cdot \frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2} \delta(f - \langle kv \rangle) \Bigg]. \end{aligned} \tag{29}$$

The power spectrum is seen to consist of two distinct components, white noise represented by those terms independent of k, being added to almost periodic components at discrete frequencies, the magnitudes of the latter also being dependent on the amount of transition noise. The power density is seen to be infinite at the discrete frequencies, $\langle kv \rangle$. The term $(\langle 2\epsilon\rangle^2/48\epsilon^2)[1-2\langle 2\epsilon\rangle+\langle 2\epsilon\rangle^2]$ represents the variation of the continuous component of the spectrum from that which would be expected if the quantization noise were independent of the transition noise.

Given the large standard deviation associated with each component of the noise spectrum, visualization is greatly assisted by defining an *integrated spectrum*

$$\begin{aligned} IS_v(f) &= \int_0^{f'} S_v(f')\,df' \\ &= \left[2f - \frac{\sin(2\pi f)}{\pi}\right] \\ &\quad \cdot \left[\frac{\epsilon^2}{3} + \frac{1}{12} - \frac{\langle 2\epsilon\rangle^2}{48\epsilon^2}\left(1 - 2\langle 2\epsilon\rangle + \langle 2\epsilon\rangle^2\right)\right] \\ &\quad + \sum_{\langle kv\rangle \in [0,\, f]} \frac{\sin^2(2\pi k\epsilon)}{16\pi^4 k^4 \epsilon^2}. \end{aligned} \tag{30}$$

(Note that the discrete line intensity components are eliminated if $\epsilon = 0.5$, thereby illustrating the utility of a uniformly distributed dither signal of this magnitude.)

V. Experimental Verification

Computer-generated data sets with representative noise and rate parameters were used to verify the theoretical spectral estimates derived in the previous section. The spectra of the rate error process and the integrated spectrum of the associated signal error process for a typical data set are shown in Fig. 5. The theoretically estimated spectra match the FFT outputs closely. [The spectra are appropriately discretized, the power spectral density $S_v(f)$ being replaced by a discretized power spectrum $P_v(f)$, the first 40 ($1 \le k \le 40$) discrete line intensities being included.] Some spectral leakage is evident in the FFT output shown in Fig. 5(c). Fig. 6 is generated using output from a real encoder (with a resolution of 125 cycles per revolution). The operation of the encoder is well modeled by appropriate choices of the transition noise parameter ϵ. Quadrature decoding obviously increases the rate by a factor of four (when expressed in transitions per sample time), but the equivalent transition noise (as represented by ϵ) also increases substantially.

VI. Conclusions

The errors associated with the digital differentiation of a signal which is subject to both quantization error and stochastic noise have been analyzed in this paper. In particular, new formulae for the important constant-rate case have been presented and verified. These formulae have been found to be of great practical use in determining quantitative figures of merit for optical shaft encoders, through application of (27) and (30). In particular, the inversion of (27) is routinely used by the authors to yield an estimate of the amount of transition noise inherent in the encoder when applied to rate estimate data generated by a sensor rotating at very close to uniform speed. An extended sensor model incorporating integral nonlinearity can also be derived using results from this paper. This model will be discussed in a future publication.

The work reported in this paper is also of importance because it indicates that the methods of analysis commonly applied to quantized data acquisition systems and sigma–delta modulators have relevance to sensor modeling. Conversely, the results derived in this paper can be applied to such systems. For

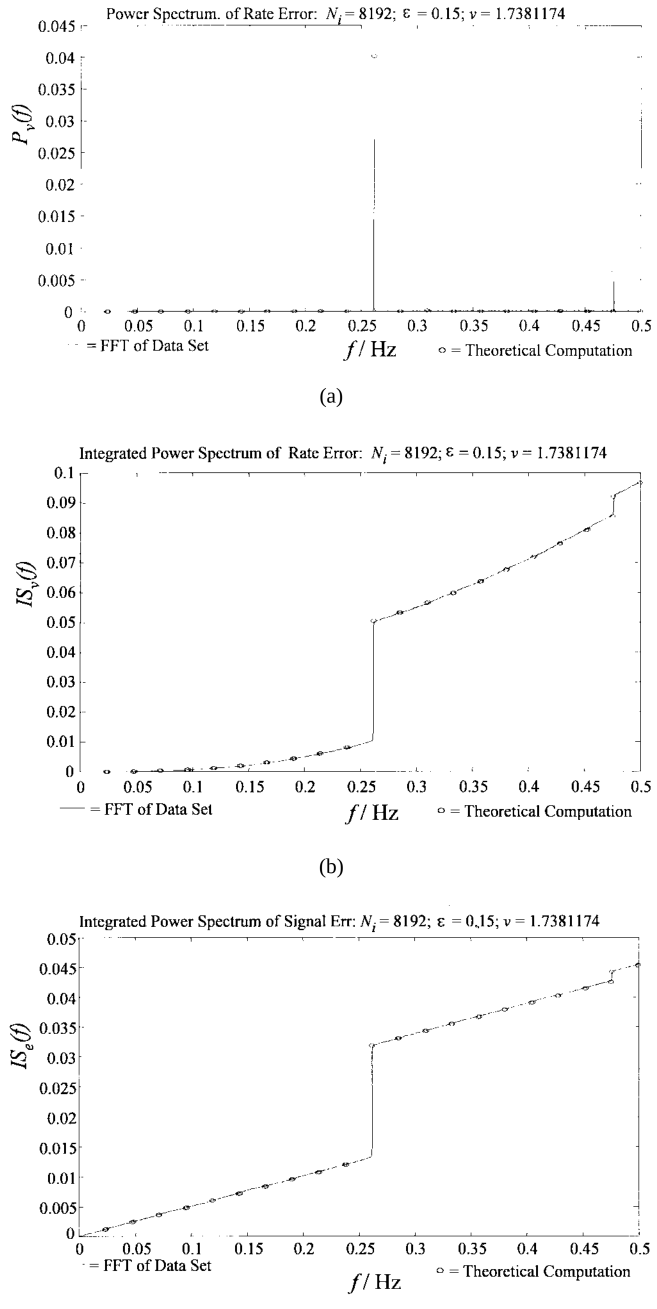
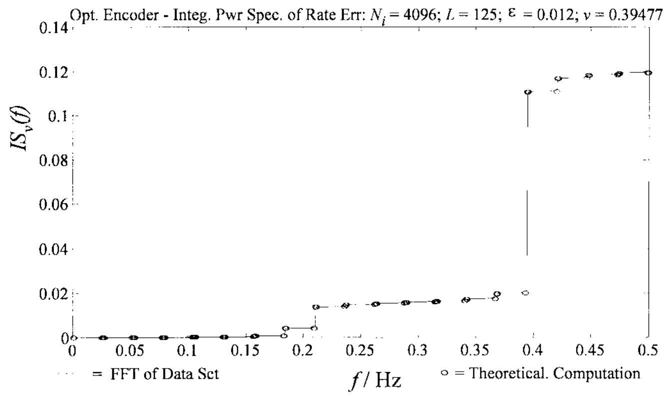
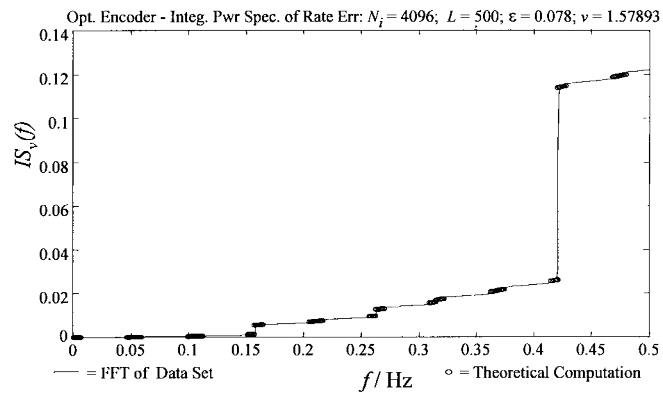


Fig. 5. Spectra of (a) rate error spectrum, $P_v(f)$; (b) its integrated spectrum, $IS_v(f)$; and (c) the corresponding integrated spectrum of the signal error, $IS_e(f)$, for typical computer generated data ($v \approx 1.738\,1174$; $\epsilon = 0.15$).



(a)



(b)

Fig. 6. Comparison of the frequency content of a 125 cycle per revolution encoder: (a) without and (b) with quadrature decoding.

example, the analysis presented above can be directly related to the operation of a first-order sigma–delta modulator, when the input voltage is nominally constant but has an added uniformly distributed noise component.

Appendix I
Derivation of Formulae for Autocorrelation Functions

A closed-form solution for the autocorrelation term given in (19)

$$R_{e'}(r) = \frac{1}{8\pi^4\epsilon^2}\sum_{k=1}^{\infty}\frac{\cos(2\pi k\langle rv\rangle)\,\sin^2(2\pi k\epsilon)}{k^4}, \quad r \neq 0$$

is derived using the identities (see [15, Formula 1.443.6])

$$\sum_{k=1}^{\infty}\frac{\cos kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48}, \quad 0 \le x \le 2\pi \tag{31}$$

$$\cos A\,\sin^2 B = \frac{\cos A}{2} - \frac{\cos(A+2B)+\cos(A-2B)}{4}. \tag{32}$$

The assignments $A = k\,(2\pi\langle rv\rangle)$ and $B = k\,(2\pi\epsilon)$ imply that

$$\cos(A+2B) = \cos(2\pi k\langle rv+2\epsilon\rangle), \tag{33}$$

$$\cos(A-2B) = \cos(2\pi k\langle rv-2\epsilon\rangle) \tag{34}$$

because $\langle\,\langle rv\rangle + 2\epsilon\rangle = \langle rv+2\epsilon\rangle$. Use of (31) allows (19) to be rewritten as

$$\begin{aligned} R_{e'}(r) = \frac{1}{32\pi^4\epsilon^2}\Bigg[2\bigg(&\frac{\pi^4}{90} - \frac{\pi^2\,(2\pi\langle rv\rangle)^2}{12} + \frac{\pi\,(2\pi\langle rv\rangle)^3}{12} \\ &- \frac{(2\pi\langle rv\rangle)^4}{48}\bigg) - \bigg(\frac{\pi^4}{90} - \frac{\pi^2\,(2\pi\langle rv+2\epsilon\rangle)^2}{12} \\ &+ \frac{\pi\,(2\pi\langle rv+2\epsilon\rangle)^3}{12} - \frac{(2\pi\langle rv+2\epsilon\rangle)^4}{48}\bigg) \\ &- \bigg(\frac{\pi^4}{90} - \frac{\pi^2\,(2\pi\langle rv-2\epsilon\rangle)^2}{12} + \frac{\pi\,(2\pi\langle rv-2\epsilon\rangle)^3}{12} \\ &- \frac{(2\pi\langle rv-2\epsilon\rangle)^4}{48}\bigg)\Bigg]. \end{aligned} \tag{35}$$

Algebraic simplification yields (21).

Appendix II
Derivation of Formula for Signal Error Spectrum for Constant-Rate Signal with Transition Noise

The power spectrum of the signal error, e, is

$$\begin{aligned} S_e(f) &= \sum_{r=-\infty}^{\infty} R_e(r)e^{-2\pi jrf} \\ &= \frac{\epsilon^2}{3} + \sum_{r=-\infty}^{\infty} R_e'(r)e^{-2\pi jrf} \end{aligned} \tag{36}$$

utilizing (25). The autocorrelation $R_{e'}(r)$ can be expressed as a variation of (17)

$$R_{e'}(r) = \sum_{k\neq 0}\frac{1}{4\pi^2k^2}\,e^{2\pi k\langle rv\rangle}\,\frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2}, \quad r \neq 0. \tag{37}$$

Substituting (37) into (36) yields

$$\begin{aligned} S_e(f) &= \frac{\epsilon^2}{3} + R_{e'}(0) + \sum_{r\neq 0}\sum_{k\neq 0} \\ &\quad\cdot \frac{1}{4\pi^2k^2}\left(e^{2\pi k\langle rv\rangle}\,\frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2}\right)e^{-2\pi jrf} \\ &= \frac{\epsilon^2}{3} + R_{e'}(0) - \sum_{k\neq 0}\frac{1}{4\pi^2k^2}\,\frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2} + \sum_{r=-\infty}^{\infty} \\ &\quad\cdot \sum_{k\neq 0}\frac{1}{4\pi^2k^2}\left(e^{2\pi k\langle rv\rangle}\,\frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2}\right)e^{-2\pi jrf} \end{aligned} \tag{38}$$

where the first summation term of this equation is introduced so that the second can be altered to include all $r \in \mathcal{Z}$. Changing the order of summation gives

$$\begin{aligned} S_e(f) &= \frac{\epsilon^2}{3} + \frac{1}{12} - \sum_{k\neq 0}\frac{1}{4\pi^2k^2}\,\frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2} + \sum_{k\neq 0} \\ &\quad\cdot \frac{1}{4\pi^2k^2}\,\frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2}\left(\sum_{r=-\infty}^{\infty} e^{2\pi r\langle kv\rangle}\,e^{-2\pi jrf}\right) \end{aligned} \tag{39}$$

where use is made of the fact that $e^{2\pi k\langle rv\rangle} = e^{2\pi\langle krv\rangle} = e^{2\pi r\langle kv\rangle}$. Examining the first summation term in isolation, and using the identity

$$\sum_{k=1}^{\infty} \frac{\sin^2(2\pi k\epsilon)}{k^4} = \sum_{k=1}^{\infty} \frac{1-\cos(4\pi k\epsilon)}{2k^4} = \frac{1}{2}\left[\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{\cos((4\pi\epsilon)\, k)}{k^4}\right] \quad (40)$$

and (31) leads to the result

$$\sum_{k\neq 0} \frac{1}{4\pi^2 k^2} \frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2} = 2\sum_{k=1}^{\infty} \frac{1}{4\pi^2 k^2} \frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2} = \frac{\langle 2\epsilon\rangle^2}{48\epsilon^2}\left[1 - 2\langle 2\epsilon\rangle + \langle 2\epsilon\rangle^2\right], \quad \forall\epsilon > 0. \quad (41)$$

The power spectral components are now defined by

$$\begin{aligned} S_e(f) &= \frac{\epsilon^2}{3} + \frac{1}{12} - \frac{\langle 2\epsilon\rangle^2}{48\epsilon^2}\left[1 - 2\langle 2\epsilon\rangle + \langle 2\epsilon\rangle^2\right] \\ &\quad + \sum_{k\neq 0} \frac{1}{4\pi^2 k^2} \frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2} \left(\sum_{r=-\infty}^{\infty} e^{2\pi r(\langle kv\rangle - f)}\right) \\ &= \frac{\epsilon^2}{3} + \frac{1}{12} - \frac{\langle 2\epsilon\rangle^2}{48\epsilon^2}\left[1 - 2\langle 2\epsilon\rangle + \langle 2\epsilon\rangle^2\right] \\ &\quad + \sum_{k\neq 0} \frac{1}{4\pi^2 k^2} \frac{\sin^2(2\pi k\epsilon)}{(2\pi k\epsilon)^2}\, \delta(f - \langle kv\rangle). \end{aligned} \quad (42)$$

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