New design of full band differentiators based on Taylor series

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Abstract: The classical central difference approximations of the derivative of a function based on Taylor series are the same as type III maximally linear digital differentiators for low frequencies. A new finite difference formula is derived, which can be implemented as a fullband type IV maximally linear differentiator. The differentiator is compared with type III maximally linear and type IV equiripple minimax differentiators. A modification is proposed in the design to minimise the region of inaccuracy near the Nyquist frequency edge.

1 Introduction

Digital differentiators are one of the most important types of digital filters, and are widely used when the rate of change of the collected data is needed e.g. measuring the cooling rate from temperature, or speed of the moving objects from their displacements in radar systems etc. Design of an ideal non-recursive differentiator is not possible with a finite number of coefficients, because the Fourier series corresponding to the ideal differentiator is non-converging. Truncation of the Fourier series after a finite number of terms leads to the residual oscillations, named Gibbs [1] oscillations, on the amplitude response (amplitude against frequency) of the differentiator. Although the oscillations may be reduced by using various kinds of windows [1], more accurate and common designs for the differentiators and other FIR filters are the Chebyshev (also called minimax or equiripple) designs based on the McClellan–Parks algorithm [2]. These differentiators are optimal in the sense that, for a given set of specifications, they can be designed with a minimum number of coefficients. Some other common differentiators include the designs based on the modified Kaiser window [3], eigenfilters [4, 5], optimisation techniques [6] weighted least square technique [7], and a recent design with optimal noise attenuation [8].

Another important class of FIR filters having a maximally flat frequency response around $\omega = 0$ and $\omega = \pi$ was introduced by Herrmann [9] in 1971 while Jinaga *et al.* [10] gave explicit formulas for the coefficients of FIR filters of this class in 1984. Kumar *et al.* [11] established an explicit formula for the coefficients of a maximally linear, FIR digital differentiator and Carlsson [12] gave its simplified form, establishing its relationship to the classical approach to numerical differentiation based on polynomial interpolation.

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We observed that the coefficients of a maximally linear digital differentiator of order $2N+1$ are the same as the coefficients of Nth-order central difference approximation of the first-order derivative based on Taylor series expansion [13]. This relationship between the Taylor series and digital differentiators gives the intuition that some different treatment of Taylor series may lead to a different design, which may prove to be better than the currently available ones. In this paper, a finite difference formula to approximate the derivative of a function using even numbers of its equally spaced samples is developed, and a closed form expression for its coefficients is determined. The coefficients, which observe odd symmetry, can be used as a maximally linear full band type IV differentiator, and the performance of this differentiator is compared with those of commonly used type III maximally linear and type IV minimax differentiators.

2 Maximally linear digital differentiators

The relationship between the input and output of a maximally linear digital differentiator of order $2N+1$ can be written as

$$f_i^{(1)} = \frac{1}{T}\sum_{k=-N}^{N} d_k f_{k+i} \tag{1}$$

where f_i is the value of the input signal $f(t)$ at $t = i\,T$, and T is the sampling period. $f_i^{(1)}$ is the ith element of the output i.e. the first derivative of the function $f(t)$. The coefficients d_k are given as [12]

$$d_0 = 0, d_k = \frac{(-1)^{k+1} N!^2}{k(N-k)!(N+k)!}, k = \pm 1, \pm 2, \pm 3, \ldots, \pm N \tag{2}$$

It is interesting to note that the coefficients d are the same as those in the Nth-order central difference approximation [13] of the derivative of a function based on Taylor series.

This result gives the important information, based on the properties of central difference approximations, that maximally linear differentiators are exact for the polynomial signals $f(t) = \sum_{k=0}^{K} a_k t^k$ of order K less than $2N$, and the error term is of the order of T^{2N} for other signals. This relationship also provides an important link between the

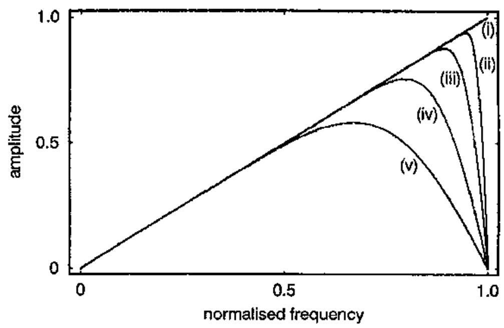

Fig. 1 *Effect of increased order on passband of maximally linear type III differentiators*

(i) ideal differentiator; (ii) $N = 500$; (iii) $N = 100$; (iv) $N = 20$; (v) $N = 5$

classical Taylor series and design of digital differentiators. As many different finite difference formulas can be obtained based on different treatments of Taylor series, new procedures can be developed to design the digital differentiators, which may prove to be better than the currently available designs. A new such design is presented in Section 3 in the present paper.

Accuracy of the maximally linear differentiators is quite high in a certain passband, which increases with the order $2N+1$ of the differentiator. This can be seen in Fig. 1, where the amplitude response of these differentiators is shown for several orders. It can be seen from Fig. 1, that even a very high order cannot give a full band differentiator. The reason is that these differentiators are of type III filters, as they consist of odd number of odd symmetric coefficients. Type III filters have the restriction that their amplitude response must be zero at the Nyquist frequency $\omega = 1$. Therefore, this design cannot give a full band differentiator for any finite number of coefficients, which are essential when the signal to be differentiated has components near the Nyquist frequency. To obtain a full-band differentiator, we need type IV filters, which have even number of odd symmetric coefficients, and do not have any restriction on the amplitude at $\omega = 1$. The design of such a fullband, type IV differentiator is presented in Section 3.

The phase response of a maximally linear type III differentiator is linear, hence having a constant group delay they do not introduce any nonlinearity in the phase of the input signal. This can be seen in Fig. 2, where the phase responses of these differentiators are shown for different orders. It is seen that for a filter of order $2N+1$, the unwrapped phase increases linearly from zero to $N\pi$.

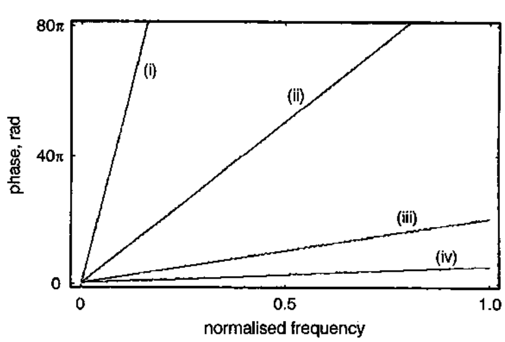

Fig. 2 *Phase response of maximally linear type III differentiators for different orders*

(i) $N = 500$; (ii) $N = 100$; (iii) $N = 20$; (iv) $N = 5$

3 Full band maximally linear differentiators

Taylor series expansion of a function $f(t)$ analytic at $t = t_0$ can be written as:

$$f(t) = f(t_0) + \sum_{m=1}^{\infty} \frac{(t - t_0)^m}{m!} f^{(m)}(t_0)$$

where $f^{(m)}(t_0)$ denotes the mth derivative of $f(t)$ at $t = t_0$. Now consider the equally spaced samples of $f(t)$ taken at $t = nT$, where $n = \pm(2k-1)/2$, $k = 1, 2, \ldots, N$, and T is the sampling period. If we expand $f(t)$ at each of these samples for $t_0 = 0$ and truncate the series after $2N$ terms, a set of $2N$ equations will be obtained which can be written as:

$$f_n - f_0 = \sum_{m=1}^{2N} \frac{(nT)^m}{m!} f_0^{(m)} + \sum_{m=2N}^{\infty} \frac{(nT)^m}{m!} f_0^{(m)},$$
$$n = (2k-1)/2; \; -N < k \leq N \qquad (3)$$

where f_n denotes the value of $f(t)$ at $t = nT$. The second summation on the right-hand side has terms of the order of T^{2N+1} and derivatives of function of order greater than $2N$ at $t = 0$. For smaller values of T, this term can be neglected and eqn. 3 can be written in matrix form as:

$$\boldsymbol{F} = \boldsymbol{A} \cdot \boldsymbol{D} \qquad (4)$$

$\boldsymbol{F}$ and $\boldsymbol{D}$ are the vectors of length $2N$, $\boldsymbol{A}$ is a $2N \times 2N$ square matrix, and these are defined as:

$$\boldsymbol{F} = [f_{1/2} - f_0 f_{-1/2} - f_0 \cdots f_{(2N-1)/2} - f_0 f_{-(2N-1)/2} - f_0]^T$$

$$\boldsymbol{D} = [f_0^{(1)} f_0^{(2)} f_0^{(3)} \cdots f_0^{(2N)}]^T$$

$$A = \begin{bmatrix} T/2 & (T/2)^2/2! & (T/2)^3/3! & \ldots & (T/2)^{2N}/2N! \\ -T/2 & (-T/2)^2/2! & (-T/2)^3/3! & & (-T/2)^{2N}/2N! \\ 3T/2 & (3T/2)^2/2! & (3T/2)^3/3! & & (3T/2)^{2N}/2N! \\ -3T/2 & (-3T/2)^2/2! & (-3T/2)^3/3! & & (-3T/2)^{2N}/2N! \\ \vdots & & & & \\ (2N-1)T/2 & ((2N-1)T/2)^2/2! & ((2N-1)T/2)^3/3! & & ((2N-1)T/2)^{2N}/2N! \\ -(2N-1)T/2 & (-(2N-1)T/2)^2/2! & (-(2N-1)T/2)^3/3! & & (-(2N-1)T/2)^{2N}/2N! \end{bmatrix}$$

From the set of equations described by eqn. 4, $f_0^{(1)}$ can be written as the ratio of two determinants $|\boldsymbol{A}_1|/|\boldsymbol{A}|$, where the matrix $\boldsymbol{A}_1$ is obtained by replacing the first column in matrix $\boldsymbol{A}$ by vector $\boldsymbol{F}$. Noting that these two matrices are the same except for the first column, and the power of T is the same in every element of a column in $\boldsymbol{A}$, $f_0^{(1)}$ can be written as

$$f_0^{(1)} = \frac{1}{T} \frac{|\boldsymbol{A}_1|_{T=1}}{|\boldsymbol{A}|_{T=1}} \qquad (5)$$

Calculating the determinant of $\boldsymbol{A}$ at $T = 1$, for different arbitrary orders $2N$, it can be shown that it is written in a closed form as:

$$|\boldsymbol{A}|_{T=1} = \frac{(2N-1)!!^2}{2^{2N}(2N)!} \qquad (6)$$

where the double factorial $a!! = a(a-2)(a-4)\ldots 1$. Similarly, the determinant of $\boldsymbol{A}_1$ is written as

$$|\boldsymbol{A}_1|_{T=1} = \sum_{k=1}^{N} c_{(2k-1)/2}(f_{(2k-1)/2} - f_{-(2k-1)/2}) \qquad (7)$$

where

$$c_{(2k-1)/2} = \frac{(-1)^{k+1}(2N-1)!!^4}{2^{2N}(2N)!(N+k-1)!(N-k)!(2k-1)^2};$$
$$k = 1, 2, 3, \ldots, N$$

From eqns. 5–7, the first derivative of $f(t)$ at $t=iT$, $f_i^{(1)}$, can be written in a closed form as:

$$f_i^{(1)} = \frac{1}{T}\sum_{k=-N+1}^{N} d_{(2k-1)/2} f_{(2k-1)/2+i} \tag{8}$$

where

$$\begin{aligned} d_{(2k-1)/2} &= -d_{-(2k-1)/2} \\ &= \frac{(-1)^{k+1}(2N-1)!!^2}{2^{2N-2}(N+k-1)!(N-k)!(2k-1)^2}; \\ &\qquad k = 1, 2, 3, \ldots, N \end{aligned} \tag{9}$$

To increase the computation efficiency, eqn. 9 can be implemented in the following iterative way:

$$\left.\begin{aligned} d_{1/2} &= -d_{-1/2} = \frac{(2N-1)!!^2}{2^{2N-1}N!(N-1)!}, \\ d_{(2k-1)/2} &= -d_{-(2k-1)/2} \\ &= -\frac{(N-k+1)(2k-3)^2}{(N+k-1)(2k-1)^2} d_{(2k-3)/2}; \\ &\qquad k = 2, 3, \ldots, N \end{aligned}\right\} \tag{10}$$

As we see, the coefficients d are even in number and have odd symmetry, so they may be used to construct a full band differentiator. The remainder terms in eqn. 3, which are neglected in the present design, have derivatives of order greater than $2N$; therefore differentiators given by eqn. 10 are exact for polynomial signals of order less than or equal to $2N$. Since only the polynomial signals are completely expressed in the frequency domain at $\omega=0$, i.e., $t^k \Leftrightarrow k!/(j\omega)^{k+1}$, the differentiator has a maximal linearity of order $2N$ at $\omega=0$. This simple proof of maximal linearity at $\omega=0$ holds for type III differentiators as well. An alternative proof for type III given by Carlsson [12], can be extended for type IV as well.

In Fig. 3, we have shown the amplitude and in Fig. 4 the phase response of this differentiator for order $2N=30$. It can be seen that the phase response is linear, as required for an ideal differentiator and amplitude response is also very close to an ideal one except in a small region close to $\omega=1$. As the order of the differentiator is increased, this region becomes narrower. This can be seen from the error curves shown in Fig. 5, drawn for different orders of the differentiator. Although the differentiator is fullband and its passband is the entire range of frequencies, we will call the region except this narrow region near $\omega=1$ the actual passband of the differentiator. It must be noted that, even in this narrow region, the error is not very large. However, it is quite large compared with that in the actual passband.

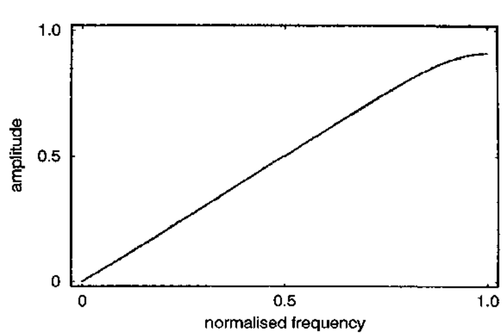

Fig. 3 *Amplitude of type IV Taylor series based differential or order 30*

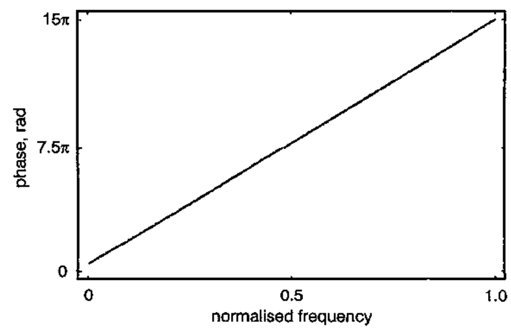

Fig. 4 *Phase response of type IV Taylor series based differentiator of order 30*

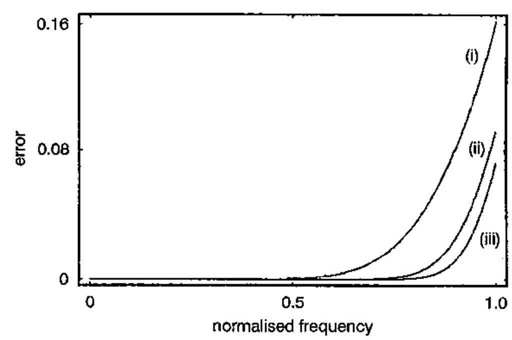

Fig. 5 *Error curves of type IV Taylor series based differentiators for different orders*

(i) $N=5$; (ii) $N=15$; (iii) $N=25$

4 Comparison with type III maximally linear differentiators

For low frequencies, a type IV maximally linear differentiator presented in this paper is as accurate as type III given by eqn. 2, but for high frequencies the former is much more reliable. The comparison of the two is shown in Fig. 6, where the magnitude responses of both are plotted for $N=15$. It can be seen that a type III differentiator cannot be used when the signal to be differentiated has components near the Nyquist frequency, whereas the type IV filter presented in this paper is good for the wider band. To understand the difference, suppose that a sinusoidal signal of 1 kHz sampled at 8 kHz is being differentiated separately using type III and type IV differentiators of orders 31 and 30 respectively ($N=15$). As the frequency of the signal is well below the Nyquist frequency (4 kHz), both

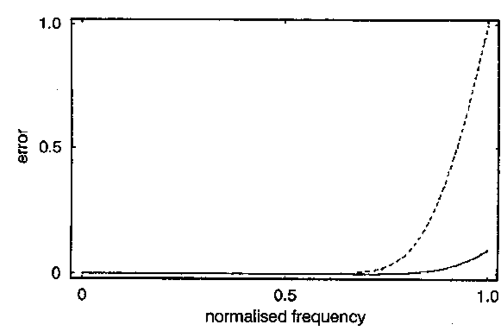

Fig. 6 *Comparison of maximally linear differentiators of type III and type IV of orders 31 and 30, respectively*

$N=15$

···· type III

—— type IV

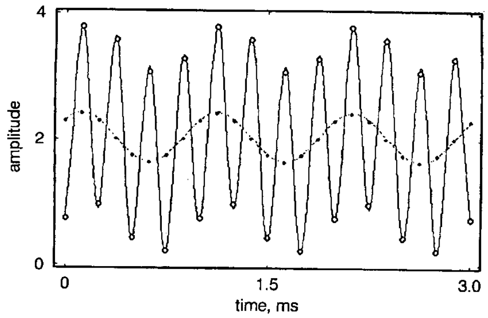

Fig. 7 *Example differentiation of sin(2000πt) + cos(8000πt) for sampling frequency $f_s = 8$ kHz with differentiators shown in Fig. 6*

—— actual derivative
---- derivative of sin(2000πt)
o output of type IV differentiator
• output of type III differentiator

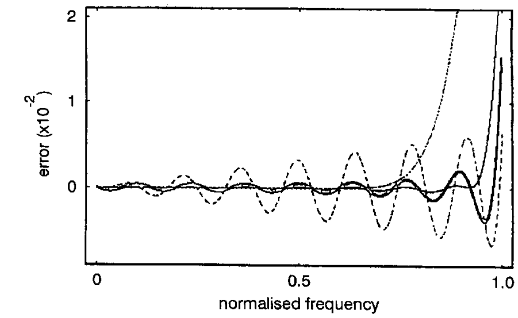

Fig. 8 *Comparison of minimax and Taylor series based differentiators of order 30*

—— Taylor ($\alpha = 100$)
—— Taylor ($\alpha = 10$)
···· Taylor ($\alpha = 1$)
--- minimax

differentiators will work well. The actual derivative of the signal is shown in Fig. 7 by a dotted line. Now suppose that another sinusoid of 4 kHz is added to the 1 kHz signal, making the actual derivative like that shown by solid line in the Figure. The outputs of type IV and type III differentiators are shown in the Figure with bold and smaller dots, respectively. It can be seen that a type IV differentiator works well but a type III differentiator will not be able to show any change to a 4 kHz signal in its output. In the Figure, all the signals are shown for 3 ms, and the amplitudes and phase of actual derivatives are adjusted to match with outputs of the differentiators.

5 Modification in the design

It can be noted from eqn. 8 that the values of the coefficients decrease very sharply as the index k increases. If the coefficients below a certain small value are neglected while implementing the filter, it will introduce a small ripple on the amplitude response, while the phase response will still be linear. This idea can be used to increase the actual passband for smaller orders at the expanse of accuracy. For example, if we design a differentiator for an order $2N_1$, and $2N_2 \ll 2N_1$, central coefficients are used in actual implementation, the filter of order $2N_2$ will be having an actual passband that corresponds to the filter of order $2N_1$. To implement this idea, we need not calculate all of the coefficients; rather it can be achieved by multiplying N in eqn. 9 by an integer $\alpha > 1$. Therefore, the iterative way to calculate the coefficients given in eqn. 10 is changed as

$$\left.\begin{aligned} d_{1/2} &= -d_{-1/2} = \frac{(2\alpha N-1)!!^2}{2^{2\alpha N-1}(\alpha N)!(\alpha N-1)!}, \\ d_{(2k-1)/2} &= -d_{-(2k-1)/2} \\ &= -\frac{(\alpha N-k+1)(2k-3)^2}{(\alpha N+k-1)(2k-1)^2} d_{(2k-3)/2}; \\ & \qquad k = 2, 3, \ldots, N \end{aligned}\right\} \quad (11)$$

For $\alpha = 1$, eqn. 11 is the same as eqn. 10. For $\alpha > 1$, the differentiators are not maximally linear and a very small ripple appears on their amplitude response; however, their actual passband is considerably increased. This is demonstrated in Fig. 8, where the error curve of differentiator of order 30 designed with eqn. 11 is plotted for $\alpha = 1$, 10 and 100. The error curve of a minimax differentiator of the same order is also shown. It can be seen that, as the value of α increases, the actual passband of the differentiator increases, but at the same time, the size of the ripple is also increased. Therefore, a tradeoff must be made between the desired accuracy and desired actual passband. However, for any value of α, the size of the ripple is much smaller in the actual passband than that on the amplitude response of the minimax differentiator. For very high values of α, the effect of increase in its value becomes very small; a value between 1 to 50 can be selected depending upon the application. It must be noted that, even in the narrow

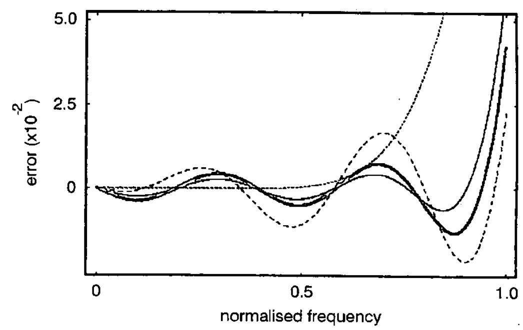

Fig. 9 *Comparison of minimax and Taylor series based differentiators of order 10*

—— Taylor ($\alpha = 100$)
—— Taylor ($\alpha = 10$)
···· Taylor ($\alpha = 1$)
--- minimax

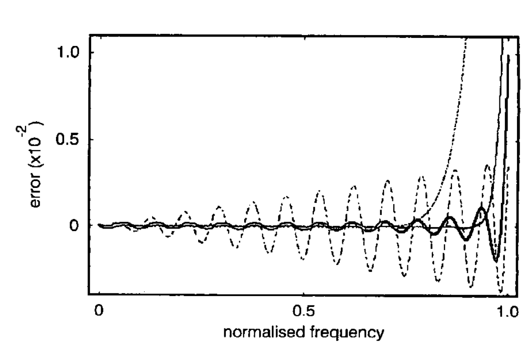

Fig. 10 *Comparison of minimax and Taylor series based differentiators of order 50*

—— Taylor ($\alpha = 100$)
—— Taylor ($\alpha = 10$)
···· Taylor ($\alpha = 1$)
--- minimax

region out of the actual passband, the error is not very high. The design is repeated for orders 10 and 50 and the error curves are shown in Fig. 9 and Fig. 10, respectively.

6 Comparison with minimax differentiators

Minimax differentiators are optimal in the Chebychev sense, whereas maximally linear ones are in the constrained least square sense. The amplitude response of Taylor series based differentiators matches more closely to ideal response than minimax differentiators in the actual passband. However, out of the actual passband (i.e. in the narrow region near $\omega = 1$) the minimax differentiators match more closely to the ideal response. This is clear from the design examples shown in Figs. 8, 9 and 10.

The superiority of minimax filters over all other designs lies in the fact that they have a minimum number of coefficients for a desired set of specifications. The design software [14] available for these filters is based on the Remez exchange procedure, which requires interpolation over a dense grid of frequencies and search of extremal frequencies during each iteration. Although the procedures to speed up this algorithm [15] and alternative ways to achieve the same design [16] have been proposed, as yet no explicit formula is available for the coefficients. In comparison, the Taylor series based differentiators proposed in this paper have an explicit formula for their coefficients and hence can be designed very easily even with the use of a simple calculator. Therefore, the Taylor series based designs are more efficient and easy to design.

The phase response of an ideal differentiator should be a constant at $\pi/2$, giving a constant group delay equal to zero. However, a linear phase response (i.e. a constant group delay not equal to zero) is also acceptable, because it does not induce any nonlinearity in the input signal. If required, a constant group delay not equal to zero can be made equal to zero by introducing a certain number of delay elements. The phase response for both Taylor series based and minimax differentiators is linear. Therefore, the group delay is constant in both, and they do not induce any nonlinearity in the phase of the input signal.

7 Conclusions

It is observed that the coefficients of maximally linear digital differentiators are the same as those of a central difference formula based on Taylor series. Based on this relationship, the Taylor series is used in a different way to obtain a different central difference formula with an even number of odd symmetric terms. The coefficients of these terms are used as type IV fullband differentiators, and their performance is compared with type III maximally linear and type IV minimax differentiators. The presented differentiators are found to be very close to ideal over the entire frequency range except in a narrow frequency band near the Nyquist frequency edge. A modification is proposed in the design to make this band narrower.

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