# Coefficients of FIR Digital Differentiators and Hilbert Transformers for Midband Frequencies

J. Le Bihan

*Abstract*—**An efficient algorithm for calculating the weighting coefficients of FIR digital differentiators and Hilbert transformers for midband frequency ranges is presented. New simple closed-form explicit and recursive formulas are derived in a straightforward manner. Moreover, a new simple recursive relation is established, relating the coefficients of two digital filters of adjacent ranks.**

## I. Introduction

As an important signal processing problem, numerical differentiation has been the subject of numerous investigations. The reason is that digital filters find extensive use in a large number of applications. This considerable interest in the design of suitable digital differentiating filters has encouraged the development of various techniques. The objective is to design these filters so that they meet given requirements with sufficient accuracy. For example, good filtering behavior can be expected for low frequencies [1], or for midband frequencies [2], [3].

The frequency response of an ideal digital differentiator (DD) is $D(\omega) = j\omega$ $(-\pi \le \omega \le \pi)$. The frequency response of an ideal digital Hilbert transformer (DHT) is $H(\omega) = -j$ $(0 \le \omega \le \pi)$ and $H(\omega) = j$ $(-\pi \le \omega \le 0)$. Various approximations of $D(\omega)$ and $H(\omega)$ can be found in the literature. Some designs are suitable for wideband frequency ranges but less appropriate for narrow bands of frequencies around $\omega = \pi/2$. Kumar and Dutta Roy [3] proposed efficient solutions for this latter situation by using as criteria of optimality maximal linearity for the differentiators and maximal flatness for the Hilbert transformers at $\omega = \pi/2$. These authors derived mathematical formulas for computation of the respective weighting coefficients required in the design. But these formulas present disadvantages. They give the coefficients as weighted sums of numerous other coefficients. Besides, the length of these sums grows with the rank of the filter.

In this brief, we obtain the coefficients of the differentiators and the Hilbert transformers through a new advantageous method. So we present new closed-form explicit formulas, and additionally provide immediate recursive relations for a straightforward calculation of the coefficients of any filter of a given rank. Moreover, we point out a new recursive relation showing that the general coefficient of a filter can be simply derived from three coefficients of the filter of previous rank.

FIR digital filters are considered for obvious advantages. Additional aims are the realization of a constant phase $\pi/2$ and a delay of a given integral number of samples. A FIR filter of order $N$ realizing a constant phase $\pi/2$ within an integral sample delay of $N/2$ must be of the form (antisymmetric impulse response) $\tilde{D}_{N/2}(z) = z^{-N/2}\sum_{k=0}^{N/2-1} d_k[z^{N/2-k} - z^{-(N/2-k)}]$. Evaluated along the unit circle, this equation becomes $\tilde{D}_{N/2}(e^{j\omega}) = e^{-j\omega N/2}\sum_{k=0}^{N/2-1} 2j\, d_k \sin[(N/2 - k)\omega]$. Putting $n = N/2$ ($N$ even) and $i = n - k$ in this equation gives $\tilde{D}_n(e^{j\omega}) = e^{-j\omega n}\sum_{i=1}^{n} 2jd_{n-i}\sin i\omega$. Leaving aside the delay term and setting $c_i^{(n)} = 2jd_{n-i}$, we obtain the following form of the DD approximation, as proposed in [3]

$$D_n(\omega) = \sum_{i=1}^{n} c_i^{(n)} \sin i\omega \tag{1}$$

where $2n$ is the filter's order. Let $n$ denote the filter's rank in the studied filter set. $D_n(\omega)$ presents the property of being antisymmetric, in particular $D_n(0) = 0$. Note that the coefficient of $\sin i\omega$ in $D_n(\omega)$ is $c_i^{(n)}$, of $\cos i\omega$ in $dD_n(\omega)/d\omega$ is $ic_i^{(n)}$ and of $\sin i\omega$ in $d^2D_n(\omega)/d\omega^2$ is $-i^2c_i^{(n)}$.

As in [3], we consider even values of $n$, which yields more efficient approximations than odd values of $n$. Without loss of generality, $D_n(\omega)$ can be expressed as follows [3]:

$$D_n(\omega) = \frac{\pi}{2}\sum_{i=1,\, i\,\mathrm{odd}}^{n-1} a_i^{(n)} \sin i\omega - \frac{1}{2}\sum_{i=2,\, i\,\mathrm{even}}^{n} b_i^{(n)} \sin i\omega. \tag{2}$$

Maximal linearity of $D_n(\omega)$ at $\omega = \pi/2$ requires $D_n(\omega)|_{\omega=\pi/2} = \pi/2$, $dD_n(\omega)/d\omega|_{\omega=\pi/2} = 1$ and $d^\nu D_n(\omega)/d\omega^\nu|_{\omega=\pi/2} = 0$ $(\nu = 2, 3, \cdots, n-1)$. Forcing these conditions on (2) leads to two sets of linear equations that can be solved by Crout's method. So after a number of calculations, the authors in [3] finally arrive at solutions for the weighting coefficients of the maximally linear DD, under the form of weighted sums, increasing with $n$, of other coefficients.

## II. Explicit and Recursive Formulas for the Weighting Coefficients of the DD

Let us consider $z = e^{j\omega}$ and let us note that $d/d\omega = jzd/dz$ and $d^2/d\omega^2 = -z(d/dz + zd^2/dz^2)$. We denote $\hat{D}_n(z) = D_n(\omega)|_{e^{j\omega}=z}$, $\hat{D}_n^{(1)}(z) = dD_n(\omega)/d\omega|_{e^{j\omega}=z}$ and $\hat{D}_n^{(2)}(z) = d^2D_n(\omega)/d\omega^2|_{e^{j\omega}=z}$. Here we propose to take directly into account the properties of symmetry and the conditions of maximal linearity, and show that the weighting coefficients can be advantageously derived by interpretation of these conditions with regard to the form of $\hat{D}_n(z)$, $\hat{D}_n^{(1)}(z)$, and $\hat{D}_n^{(2)}(z)$. From (1) $\hat{D}_n(z)$ can be written as

$$\hat{D}_n(z) = (2j)^{-1}\sum_{i=1}^{n} c_i^{(n)}(z^i - z^{-i}) \tag{3}$$

$z^n\hat{D}_n(z)$ is a polynomial of degree $2n$ in $z$ with coefficients of ranks $i$ and $2n-i$ of equal absolute values and opposite signs. $z^n\hat{D}_n^{(1)}(z)$ and $z^n\hat{D}_n^{(2)}(z)$ are polynomials of degree $2n$ in $z$ with coefficients of ranks $i$ and $2n-i$ of equal absolute values and identical or opposite signs, respectively. Thus, the coefficient of $z^i$ $(0 < i \le n)$ is $c_i^{(n)}/(2j)$, $ic_i^{(n)}/2$, and $-i^2c_i^{(n)}/(2j)$ in $\hat{D}_n(z)$, $\hat{D}_n^{(1)}(z)$, and $\hat{D}_n^{(2)}(z)$, respectively. Likewise, the coefficient of $z^{-i}$ $(0 < i \le n)$ is $-c_i^{(n)}/(2j)$, $ic_i^{(n)}/2$, and $i^2c_i^{(n)}/(2j)$ in $\hat{D}_n(z)$, $\hat{D}_n^{(1)}(z)$, and $\hat{D}_n^{(2)}(z)$, respectively.

It is obvious that $z = 1$ and $z = -1$ are zeros of $z^n\hat{D}_n(z)$ and $z^n\hat{D}_n^{(2)}(z)$. Further, properties of symmetry and conditions of

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The author is with Ecole Nationale d'Ingénieurs de Brest-Laboratoire RESO, Technopôle Brest-Iroise-CP 15-29608 Brest Cedex-France.

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TABLE I
WEIGHTING COEFFICIENTS FOR MAXIMALLY LINEAR MIDBAND FREQUENCY DIGITAL DIFFERENTIATORS $[a_i^{(n)}, b_i^{(n)}]$ AND MAXIMALLY FLAT MIDBAND FREQUENCY DIGITAL HILBERT TRANSFORMERS $[h_i^{(n)}]$

| n | NF1 | $a_1^{(n)}, -h_1^{(n)}$ | $a_3^{(n)}, -h_3^{(n)}$ | $a_5^{(n)}, -h_5^{(n)}$ | $a_7^{(n)}, -h_7^{(n)}$ | NF2 | $b_2^{(n)}$ | $b_4^{(n)}$ | $b_6^{(n)}$ | $b_8^{(n)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | | | | 1 | 1 | | | |
| 4 | 8 | 9 | 1 | | | 6 | 8 | 1 | | |
| 6 | 128 | 150 | 25 | 3 | | 30 | 45 | 9 | 1 | |
| 8 | 1024 | 1225 | 245 | 49 | 5 | 420 | 672 | 168 | 32 | 3 |

NF1 : normalizing factor for $a_i^{(n)}$, $h_i^{(n)}$, $i$ odd    NF2 : normalizing factor for $b_i^{(n)}$, $i$ even

maximal linearity mean that $z = j$ and $z = -j$ are zeros of order $n-2$ of $z^n \hat{D}_n^{(2)}(z)$.

According to the properties of the polynomial $z^n \hat{D}_n^{(2)}(z)$, the following relation holds

$$\hat{D}_n^{(2)}(z) = -\frac{1}{2j}(z - z^{-1})(z + z^{-1})^{n-2} \cdot \left[\frac{\pi}{2}\alpha_n - \frac{1}{2}\beta_n(z + z^{-1})\right] \quad (4)$$

where $\alpha_n$ and $\beta_n$ are real coefficients. This derives from these stated characteristics of $z^n \hat{D}_n^{(2)}(z)$: degree $2n$, zeros 1 and $-1$ of order 1, zeros $j$ and $-j$ of order $n-2$, coefficients of $z^{2n}$ and $z^0$ opposite imaginary quantities. So the term inside brackets, multiplied by $z$, represents, at this step, an arbitrary polynomial of degree 2, with identical coefficients of $z^2$ and 1.

By integrating $D_n^{(2)}(z)|_{z=e^{j\omega}}$ with respect to $\omega$, putting $e^{j\omega} = z$, and taking into account that $\hat{D}_n^{(1)}(j) = 1$, we obtain $D_n^{(1)}(z)$,

$$\hat{D}_n^{(1)}(z) = 1 + (z + z^{-1})^{n-1}\tfrac{1}{2} \cdot \left[\frac{\pi}{2}\frac{\alpha_n}{n-1} - \frac{1}{2}\frac{\beta_n}{n}(z + z^{-1})\right]. \quad (5)$$

Since $\hat{D}_n^{(1)}(z) = [\sum_{i=1}^{n} i\, c_i^{(n)} \cos i\omega]|_{e^{j\omega}=z}$, $\hat{D}_n^{(1)}(z)$ does not include any constant term. So (5) yields

$$\beta_n = \frac{4n}{\binom{n}{\frac{n}{2}}}. \quad (6)$$

Note that, by integrating the expression of $\hat{D}_n(z)$ can be easily found as a weighted sum of terms as $z^i - z^{-i}$

$$\hat{D}_n(z) = \frac{1}{2j}\left[\frac{\pi}{2}\frac{\alpha_n}{n-1}\sum_{i=1,\, i\,\text{odd}}^{n-1}\binom{n-1}{\frac{n-1-i}{2}}\frac{1}{i}(z^i - z^{-i}) - \frac{1}{2}\frac{\beta_n}{n}\sum_{i=2,\, i\,\text{even}}^{n}\binom{n}{\frac{n-i}{2}}\frac{1}{i}(z^i - z^{-i})\right]. \quad (7)$$

By integrating $\hat{D}_n^{(1)}(z)|_{z=e^{j\omega}} = 1 + 2^{n-2}\cos^{n-1}\omega\,\{(\pi/2)[\alpha_n/(n-1)] - (\beta_n/n)\cos\omega\}$ from $\omega = 0$ to $\omega = \pi/2$, we obtain $D_n(\pi/2) = (\pi/2) + 2^{n-2}\{(\pi/2)[\alpha_n/(n-1)]W_{n-1} - (\beta_n/n)W_n\}$, where $W_N = \int_0^{\pi/2}\cos^N\omega\, d\omega$ denotes the Wallis' integral. As $D_n(\pi/2) = \pi/2$, taking into account the expressions of $\beta_n$ (6), $W_{n-1}$ and $W_n$ [5] leads to

$$\alpha_n = \frac{n(n-1)}{2^{2(n-1)}}\binom{n}{\frac{n}{2}}. \quad (8)$$

According to the properties of the polynomial $z^n \hat{D}_n(z)$, comparison of $c_i/(2j)$ and the coefficient of $z^i$ or $-z^{-i}$ in (7) gives immediately closed-form explicit expressions of the DD coefficients, similar to those obtained in [4] through extensive algebraic manipulation

$$a_i^{(n)} = \frac{1}{2^{2(n-1)}}\frac{n}{i}\binom{n-1}{\frac{n-1-i}{2}}\binom{n}{\frac{n}{2}} \qquad i = 1, 3, \cdots, n-1 \quad (9)$$

$$b_i^{(n)} = 4\frac{1}{i}\frac{\binom{n}{\frac{n-i}{2}}}{\binom{n}{\frac{n}{2}}} \qquad i = 2, 4, \cdots, n. \quad (10)$$

These weighting coefficients of the DD can be calculated by using recursive relations directly deduced from (9) and (10)

$$\frac{a_{i+2}^{(n)}}{a_i^{(n)}} = \frac{i}{i+2}\frac{n-1-i}{n+1+i} \quad (i\,\text{odd}) \quad (11)$$

$$\frac{b_{i+2}^{(n)}}{b_i^{(n)}} = \frac{i}{i+2}\frac{n-i}{n+i+2} \quad (i\,\text{even}). \quad (12)$$

Equations (11) and (12) can be used with increasing values of $i$, by starting with the initial values $a_1^{(n)} = 2^{-2(n-2)}(n-1)!^2/[(n/2-1)!^3(n/2)!]$ and $b_2^{(n)} = 2n/(n+2)$. Note that $a_1^{(n+2)}/a_1^{(n)} = (n+1)^2/[n(n+2)]$ with $a_1^{(2)} = 1$ and $b_2^{(n+2)}/b_2^{(n)} = (n+2)^2/[n(n+4)]$ with $b_2^{(2)} = 1$.

Equations (11) and (12) can also be used with decreasing values of $i$, by starting with the initial values $a_{n-1}^{(n)} = 2^{-2(n-2)}(n-2)!/(n/2-1)!^2$ and $b_n^{(n)} = (n/2-1)!^2/(n-1)!$. Note that $a_{n+1}^{(n+2)}/a_{n-1}^{(n)} = (n-1)/(4n)$ with $a_1^{(2)} = 1$ and $b_{n+2}^{(n+2)}/b_n^{(n)} = n/[4(n+1)]$ with $b_2^{(2)} = 1$.

So relations (9), (10) and (11), (12) are the explicit and recursive formulas giving the weighting coefficients (Table I) for the proposed maximally linear midband frequency digital differentiators, directly derived from the properties of $D_n(\omega)$ and its derivatives.

## III. RECURSIVE RELATION BETWEEN COEFFICIENTS OF TWO SUCCESSIVE DIGITAL DIFFERENTIATORS

Consider two DD $D_{n+2}(\omega)$ and $D_n(\omega)$ of ranks $n+2$ and $n$, respectively ($n$ even), and write (5), where figure powers of the binomial $z + z^{-1}$, for these two ranks. Using filter ranks which differ from 2 and monomials $z$ and $z^{-1}$ the degrees of which also differ from 2 allows us to manipulate separately the odd part $\hat{O}_n^{(1)}(z)$ and the even part $\hat{E}_n^{(1)}(z)$ of $\hat{D}_n^{(1)}(z)$. Coefficients of $z^i$ or $z^{-i}$ in $\hat{O}_n^{(1)}(z)$ and $\hat{E}_n^{(1)}(z)$ are $(\pi/4)ia_i^{(n)}$ and $-\frac{1}{4}ib_i^{(n)}$, respectively. Since $\hat{O}_n^{(1)}(z) = (\pi/4)[\alpha_n/(n-1)](z + z^{-1})^{n-1}$ and $\hat{E}_n^{(1)} = 1 - \frac{1}{4}(\beta_n/n)(z + z^{-1})^n$, and taking the expressions of $\alpha_n$

and $\beta_n$ into account, the following recursive equations can be written

$$\hat{O}_{n+2}^{(1)}(z) = \frac{n+1}{4n}(z+z^{-1})^2 \hat{O}_n^{(1)}(z)$$

and

$$\hat{E}_{n+2}^{(1)}(z) - 1 = \frac{n+2}{4(n+1)}(z+z^{-1})^2[\hat{E}_n^{(1)}(z) - 1].$$

Using simple inverse $Z$ transform techniques, these equations lead to two separate recursive relations allowing a straightforward calculation of $a_i^{(n+2)}$ on one side and $b_i^{(n+2)}$ on the other side

$$\begin{aligned} a_i^{(n+2)} = &\frac{n+1}{4n}\frac{1}{i}[(i+2)\,a_{i+2}^{(n)} \\ &+ (2i+\delta_{i,1})\,a_i^{(n)} + (i-2)\,a_{i-2}^{(n)}] \\ &i = 1, 3, \cdots, n-1 \quad (i\,odd) \end{aligned} \tag{13}$$

$$\begin{aligned} b_i^{(n+2)} = &\frac{n+2}{4(n+1)}\frac{1}{i}\Bigg[(i+2)\,b_{i+2}^{(n)} \\ &+ \left(2i + 2\frac{n+2}{n}\delta_{i,2}\right) b_i^{(n)} + (i-2)\,b_{i-2}^{(n)}\Bigg] \\ &i = 2, 4, \cdots, n \quad (i\,\text{even}) \end{aligned} \tag{14}$$

These relations are valid for $n \geq 2$, with $a_{-1}^{(n)} = 0$, $a_i^{(n)} = 0\,(i > n)$ in (13), and $b_0^{(n)} = 0$, $b_i^{(n)} = 0\,(i > n)$ in (14). So each coefficient of the DD of rank $n+2$ can be easily calculated from three coefficients of the DD of rank $n$.

## IV. Hilbert Transformers with Maximal Flatness at $\omega = \pi/2$

Let us consider, as proposed in [3],

$$H_n(\omega) = \sum_{i=1}^{n} h_i^{(n)} \sin i\omega \tag{15}$$

to approximate $H(\omega)$, with even values of $n$. $H(\omega)$ is symmetrical about $\omega = \pi/2$. So if we also let $H_n(\omega)$ be symmetrical about $\omega = \pi/2$, it can be shown that $h_i^{(n)} = 0$ for even values of $i$ and thus (15) becomes

$$H_n(\omega) = \sum_{i=1,\, i\,\text{odd}}^{n} h_i^{(n)} \sin i\omega. \tag{16}$$

Maximal flatness of $H_n(\omega)$ at $\omega = \pi/2$ demands $dH_n(\omega)|_{\omega=\pi/2} = 1$ and $d^\nu H_n(\omega)/d\omega^\nu|_{\omega=\pi/2} = 0\ (\nu = 1, 2, 3, \cdots, n-2)$. A similar study as for the previous DD gives the simple result [3]

$$h_i = -a_i \quad (i\,\text{odd}). \tag{17}$$

Thus the weighting coefficients (Table I) for the proposed maximally flat midband frequency digital Hilbert transformers can also be readily computed using the explicit and recursive formulas (9), (11) just as well as the recursive formula (13).

## V. Conclusion

A new efficient algorithm for calculating the weighting coefficients of the maximally linear midband frequency digital differentiators has been presented. This algorithm leads to new simple closed-form explicit and recursive formulas of these coefficients in a straightforward manner. Furthermore, a new simple recursive relation has been derived giving the coefficients of a differentiator of given rank from coefficients of the filter of lower rank. Either recursive relation allows a very fast coefficient computation. The same new formulas and relations are valid for calculating the weighting coefficients of the maximally flat midband frequency digital Hilbert transformers.

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# Design of 2-D Perfect Reconstruction Filter Banks Using Transformations of Variables: IIR Case

David B. H. Tay and Nick G. Kingsbury


*Abstract*— **In [1], the authors presented a design technique for 2-channel multidimensional filter banks. The technique is based on the transformation of variables and provides the flexibility of controlling the frequency characteristics of the resulting subband filters with ease. Furthermore, there are several properties of the filters which allow for efficient implementation. The transformation functions used in [1] are FIR and the subband filters are all FIR. In this paper we extend the design technique to yield IIR filters. We emphasize the 2-D diamond subband case (with quincunx sampling) but the technique can be used for the parallelogram and diagonally quadrant subband cases as well. The IIR nature of the filters occurs from the use of IIR transformation functions instead of FIR. Two cases are considered. The first is with zero-phase IIR transformations which yield linear phase subband filters but require noncausal filtering. The second is with nonlinear phase IIR transformations which yield nonlinear phase subband filters that can be implemented in a causal manner.**


## I. Introduction

Multirate digital filter banks are systems that perform analysis/synthesis of digital signals to/from their frequency subband components. They can be used for subband coding which is an increasingly popular technique for the compression of still images and video signals [2]. In this brief, we are concerned with 2-D, 2-channel filter banks where the downsampling is on a quincunx lattice and the ideal subband is diamond in shape. There have been various techniques proposed to such filter banks [3]–[10]. In [1], we presented a technique to design such filter banks using transformations of variables. The technique provides the flexibility of controlling the frequency characteristics of the filters with ease. The filters are FIR, have linear phase and achieve perfect reconstruction and the transformation is equivalent to the generalized McClellan transformation. This brief


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The authors are with Department of Engineering, University of Cambridge Trumpington Street, Cambridge, CB2 1PZ, U.K.



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