

Simple explicit and recursive formulas for the coefficients of FIR digital differentiators for midband frequencies

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Abstract

An efficient algorithm for calculating the coefficients of FIR digital differentiators for midband frequencies is presented. Simple closed-form explicit and recursive formulas are derived in a straightforward manner. Moreover, a simple recursive relation is obtained, relating the coefficients of two digital filters of adjacent ranks.

I. INTRODUCTION

The interest in suitable filters for digital differentiation has encouraged the development of various techniques. The objective is to design these filters so that they meet given requirements with sufficient accuracy. For instance, good behavior can be expected for low frequencies [1], or for midband frequencies [2].

The frequency response of an ideal digital differentiator (DD) is $D(\omega) = j\omega$ ($-\pi \leq \omega \leq \pi$). Various approximations of $D(\omega)$ can be found in the literature. Some designs are convenient for wideband frequency ranges but less appropriate for narrow bands of frequencies

around $\omega = \pi/2$. Efficient solutions for this latter situation have been proposed in [2], by using maximal linearity as a criterion of optimality for the differentiators. Mathematical formulas have been derived for computation of the respective weighting coefficients required in the design. However, these coefficients are given as weighted sums of numerous other coefficients, and moreover the length of these sums grows with the rank of the filter.

In this work, we present a fast attractive method for obtaining the coefficients of the DD. New closed-form explicit formulas are given, as well as immediate recursive relations allowing a straightforward calculation of the coefficients of any filter of a given rank. In addition, we show a new recursive relation for deriving the general coefficient of a filter from three coefficients of the filter of previous rank.

II. DIGITAL DIFFERENTIATORS

Consider as in [2] the following form of the DD approximation: $D_n(\omega) = \sum_{i=1}^n c_i^{(n)} \sin i\omega$, where $2n$ is the filter's order. Let n denote the filter's rank in the studied filter set. $D_n(\omega)$

presents the property of being antisymmetric, in particular $D_n(0) = 0$.

Considering as in [2] even values of n , which yields more efficient approximations than odd values of n , we can express $D_n(\omega)$ as follows, without loss of generality

$$D_n(\omega) = \frac{\pi}{2} \sum_{i=1, i \text{ odd}}^{n-1} a_i^{(n)} \sin i\omega - \frac{1}{2} \sum_{i=2, i \text{ even}}^n b_i^{(n)} \sin i\omega \quad (1)$$

Maximal linearity of $D_n(\omega)$ at $\omega = \pi/2$ requires $D_n(\omega)|_{\omega=\pi/2} = \pi/2$, $dD_n(\omega)/d\omega|_{\omega=\pi/2} = 1$ and $d^\nu D_n(\omega)/d\omega^\nu|_{\omega=\pi/2} = 0$ ($\nu = 2, 3, \dots, n-1$).

III. EXPLICIT AND RECURSIVE FORMULAS FOR DD COEFFICIENTS

Denote $\hat{D}_n(z) = D_n(\omega)|_{e^{j\omega}=z}$, $\hat{D}_n^{(1)}(z) = dD_n(\omega)/d\omega|_{e^{j\omega}=z}$ and $\hat{D}_n^{(2)}(z) = d^2D_n(\omega)/d\omega^2|_{e^{j\omega}=z}$. $\hat{D}_n(z)$ can be written: $\hat{D}_n(z) = (2j)^{-1} \sum_{i=1}^n c_i^{(n)} (z^i - z^{-i})$. In this study, we take directly into account the properties of $D_n(\omega)$ and its derivatives, and the conditions of maximal linearity. We show that the coefficients can be advantageously derived by interpretation of these conditions with regard to the form of $\hat{D}_n(z)$, $\hat{D}_n^{(1)}(z)$ and $\hat{D}_n^{(2)}(z)$. By using simple Z transform techniques, we then derive the following simple explicit formulas for the DD coefficients

$$a_i^{(n)} = \frac{1}{2^{2(n-1)}} \frac{n}{i} \binom{n-1}{n-1-i} \binom{n}{2} \quad (2)$$

$$i = 1, 3, \dots, n-1$$

$$b_i^{(n)} = 4 \frac{1}{i} \binom{n}{n-i} / \binom{n}{2} \quad (3)$$

$$i = 2, 4, \dots, n$$

From eqns 2 and 3, immediate recursive relations can be deduced for calculating the DD coefficients: $a_{i+2}^{(n)}/a_i^{(n)} =$

$[i/(i+2)] [(n-i-1)/(n+i+1)]$ (i odd),
 $b_{i+2}^{(n)}/b_i^{(n)} = [i/(i+2)] (n-i)/(n+i+2)$ (i even). The first coefficients are given in Table I.

Table I

Coefficients for maximally linear midband frequency digital differentiators

n	NF1	$a_1^{(n)}$	$a_3^{(n)}$	$a_5^{(n)}$	$a_7^{(n)}$
2	1	1			
4	8	9	1		
6	128	150	25	3	
8	1024	1225	245	49	5

NF1: normalizing factor for $a_i^{(n)}$, i odd

n	NF2	$b_2^{(n)}$	$b_4^{(n)}$	$b_6^{(n)}$	$b_8^{(n)}$
2	1	1			
4	6	8	1		
6	30	45	9	1	
8	420	672	168	32	3

NF2: normalizing factor for $b_i^{(n)}$, i even

IV. RECURSIVE RELATION BETWEEN COEFFICIENTS OF TWO SUCCESSIVE DD

Consider two DD $D_{n+2}(\omega)$ and $D_n(\omega)$ of ranks $n+2$ and n , respectively (n even). Properties of $\hat{D}_n^{(1)}(z)$ allow to manipulate separately the odd part $\hat{O}_n^{(1)}(z)$ and the even part $\hat{E}_n^{(1)}(z)$ of $\hat{D}_n^{(1)}(z)$, and to deduce two very simple recursive equations satisfied by $\hat{O}_n^{(1)}(z)$ and $\hat{E}_n^{(1)}(z)$ respectively. Then using simple Z transform techniques lead to two separate recursive relations allowing a straightforward calculation of each coefficient of the DD of rank $n+2$ from three coefficients of the DD of rank n

$$a_i^{(n+2)} = \frac{n+1}{4n} \frac{1}{i} [(i+2)a_{i+2}^{(n)} + (2i + \delta_{i,1})a_i^{(n)} + (i-2)a_{i-2}^{(n)}] \quad (4)$$

(i odd)

$$b_i^{(n+2)} = \frac{n+2}{4(n+1)} \frac{1}{i} \left[(i+2)b_{i+2}^{(n)} + (2i + 2\frac{n+2}{n}\delta_{i,2})b_i^{(n)} + (i-2)b_{i-2}^{(n)} \right]$$

(i even) (5)

for $n \geq 2$, with $a_{-1}^{(n)} = 0$, $a_i^{(n)} = 0$ ($i > n$), and $b_0^{(n)} = 0$, $b_i^{(n)} = 0$ ($i > n$).

V. CONCLUSION

An efficient algorithm for fast computation of the coefficients of the maximally linear FIR digital differentiators has been presented. Simple closed-form explicit and recursive formulas have been derived for these coefficients. Moreover, a simple recursive relation has been obtained giving the coefficients of a differentiator of given rank from coefficients of the filter of lower rank.

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