

OPTIMAL CONTROL FOR FRACTIONALLY DAMPED FLEXIBLE SYSTEMS

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Abstract

This paper is concerned with the linear-quadratic optimal control of a rod whose damping mechanism is described in terms of fractional derivatives. A state-space representation of the original system is obtained by a suitable transformation. This transformation is based on the description of the heredity of the system through an additional state variable and thereby allows the solution of the quadratic optimal control problem to be simply determined. Some numerical simulations are provided also with the aim of demonstrating the legitimacy of the approach in the experimental field.

1. Introduction

Over the last decades, there have been several researches on the modeling, analysis and control of beams spurred on by the increasing use of flexible structures in space technology. In particular, the significance of properly modeling internal damping in the control of structures has been repeatedly stressed. As a result, various damping models have been devised in order to take into account the phenomenon of dissipation in structures. Among them are the well-known internal viscous damping mechanisms, the thermoelastic damping mechanisms [3] and the relatively more recent shear-diffusion damping model [12].

The purpose of the present paper is to treat the optimal linear quadratic control of the axial vibrations of an elastic rod in the presence of *fractional* damping mechanism. The idea of utilizing fractional calculus in the modeling of constitutive relations in viscoelastically damped structures is not new. In fact, it dates as far back to 1960 [2]. Though originally suggested by mechanical engineers, on the basis of experiments, the utilization of fractional calculus for the modeling of viscoelastically materials now enjoys a firm theoretical basis.

Unfortunately, the control (unlike the modeling) of fractionally damped structures has not received as much attention. This may be partly attributed to the fact that, when it comes to the question of optimal quadratic control problem,

fractionally damped systems (being part of the integrodifferential theory) do not benefit from a theoretical setting as convenient as the one provided by the semigroup theory. Indeed, to the authors' knowledge, there is no direct approach to solving the quadratic cost control problem for general integro-differential systems. Recently, there have been a number of important attempts to deal with this problem (see [4] and the references therein) but the assumptions made there are such that these works do not cover our model.

By economy of space some of the proofs of the results and some lengthier explanations may be found in a forthcoming paper [5].

2. Fractional calculus

let $f : [0, +\infty) \rightarrow (-\infty, +\infty)$ be a real-valued function. The fractional integral and derivative operators of order α of f are defined as :

$$I_t^\alpha f(t) \triangleq \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \quad 0 < \alpha < 1, \quad (1)$$

and

$$\partial_t^\alpha f(t) \triangleq \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{dt} f(\tau) d\tau, \quad 0 < \alpha < 1, \quad (2)$$

respectively.

From the above definitions, note that:

$$\partial_t^\alpha = I_t^{1-\alpha} \frac{d}{dt}. \quad (3)$$

Also, by fairly easy calculations one shows that:

$$I_t^\alpha \partial_t^\alpha f(t) = f(t) - f(0). \quad (4)$$

Remark: The operators I_t^α and ∂_t^α defined above are *hereditary* in the sense that they do take into account the entire history of the function $f(t)$ from 0 up to time t .

3. The model

The governing equations for the axial vibration of an elastic thin rod (clamped at both ends) in the presence of fractional damping are:

$$m\partial_t^\alpha \theta(x, t) - EA\partial_x^2 \theta(x, t) - CA\partial_t^\alpha \partial_x^2 \theta(x, t) = b(x)w(t), \quad (5)$$

with the boundary conditions:

$$\theta(0, t) = \theta(1, t) = 0, \quad (6)$$

and the initial conditions:

$$\theta(x, 0) = \theta_0(x), \quad \partial_t \theta(x, 0) = \theta_1(x), \quad (7)$$

in which, the damping mechanism is described by the fractional derivative operator (2).

In the above, $t \in [0, \infty)$ is the time variable and $x \in (0, 1)$ is the space coordinate along the rod of unit length. The function $\theta(x, t)$ denotes the displacement of the point of abscissa x at time t . The coefficients m , A , C and E are the mass per unit length, the cross-sectional area, the modulus of viscosity and the Young's modulus of elasticity of the rod, respectively. The function $w(t)$ is the external force amplitude applied to the rod, and $b(x)$ is the spatial force distribution assumed to be in $L^2(0, 1)$.

From the point of view of strength of materials, equations (1-4) are derived in the same way as in the case of a purely elastic rod except that here, the constitutive relation between the stress $\sigma(x, t)$ and strain $\varepsilon(x, t)$ functions within the material is assumed to be of the form [2,13]:

$$\sigma(x, t) = E\varepsilon(x, t) + C\partial_t^\alpha \varepsilon(x, t), \quad 0 < \alpha < 1. \quad (8)$$

Therefore, equations (1-4) may be regarded as interpolating between the undamped rod equations for which $\sigma(x, t) = E\varepsilon(x, t)$ and the internally damped rod model for which the constitutive relationship is of Kelvin-Voigt type: $\sigma(x, t) = E\varepsilon(x, t) + C\partial_t \varepsilon(x, t)$

Our objective is to study the linear quadratic control problem:

minimize

$$J(w) = r\mathcal{E}(T) + q \int_0^T \mathcal{E}(t)dt + \int_0^T w^2(t)dt, \quad (9)$$

$$\mathcal{E}(t, \alpha) = \frac{1}{2} \int_0^1 \{m(\partial_t \theta)^2 + EA(\partial_x \theta)^2\} dx, \quad (10)$$

subjected to the equations of evolution (5-8).

From here on, we will assume, without loss of generality, that the system's parameters m , A , C and E are unity.

4. A non hereditary formulation

As the semigroup theory offers a natural setting for linear quadratic control problems, our first step is therefore to put equations (5)-(8) in a form which is especially suited for the application of the theory of semigroup in Hilbert spaces.

Introducing the following function:

$$\phi(x, \xi, t) := \int_0^t \xi^\beta e^{-\xi^2(t-\tau)} \partial_\tau \partial_x \theta(x, \tau) d\tau, \quad \xi > 0 \quad (11)$$

where $-\frac{1}{2} < \beta = \alpha - \frac{1}{2} < \frac{1}{2}$, we show that:

proposition 1: If $\mu = 2\pi^{-1} \sin \pi(\frac{1}{2} - \beta)$, then

$$\mu \int_0^\infty \xi^\beta \phi(x, \xi, t) d\xi = \partial_t^\alpha \partial_x \theta(x, t). \quad (12)$$

proof: Making use of the definition of equation (11), a suitable change of variable, and the property $\Gamma(\frac{1}{2} - \beta)\Gamma(\frac{1}{2} + \beta) = \pi[\sin(\frac{1}{2} + \beta)\pi]^{-1}$, we have:

$$\int_0^\infty \xi^\beta \phi(x, \xi, t) d\xi = \int_0^t \left[\int_0^\infty \mu \xi^{2\beta} e^{-\xi^2(t-\tau)} d\xi \right] \partial_\tau \partial_x \theta(x, \tau) d\tau \quad (13)$$

$$= \int_0^t \frac{\partial_\tau \partial_x \theta(x, \tau)}{\Gamma(1/2 - \beta)(t - \tau)^{\beta+1/2}} d\tau \quad (14)$$

$$= \partial_t^{\beta+1/2} \partial_x \theta(x, t). \quad \blacksquare \quad (15)$$

Now note that relation (11) is equivalent to the following differential system defined on $\xi \in \mathbf{R}^+$:

$$\partial_t \phi(x, \xi, t) = -\xi^2 \phi(x, \xi, t) + \xi^\beta \partial_x \partial_t \theta(x, t), \quad (16)$$

$$\phi(x, \xi, 0) = 0. \quad (17)$$

Therefore, utilizing the above proposition, it is easy to see that system (5-8) is equivalent to:

$$\partial_t^2 \theta(x, t) - \partial_x^2 \theta(x, t) - \mu \int_0^\infty \xi^\beta \partial_x \phi(x, \xi, t) d\xi = b(x)w(t), \quad (18)$$

$$\partial_t \phi(x, \xi, t) + \xi^2 \phi(x, \xi, t) - \xi^\beta \partial_x \partial_t \theta(x, t) = 0, \quad (19)$$

with the boundary conditions:

$$\theta(0, t) = \theta(1, t) = 0 \quad (20)$$

and the initial conditions:

$$\theta(x, 0) = \theta_0(x), \quad \partial_t \theta(x, 0) = \theta_1(x), \quad \phi(x, \xi, 0) = 0 \quad (21)$$

We define the following state space:

$$\mathbf{H} := H_0^1((0, 1)) \times L^2((0, 1)) \times L^2((0, 1) \times (0, \infty)), \quad (22)$$

with scalar product:

$$\left\langle \begin{pmatrix} u \\ v \\ \phi \end{pmatrix}, \begin{pmatrix} f \\ g \\ \psi \end{pmatrix} \right\rangle = \langle u, f \rangle_{H_0^1((0,1))} + \langle v, g \rangle_{L^2((0,1))} + \mu \langle \phi, \psi \rangle_{L^2((0,1) \times \mathbb{R})}. \quad (23)$$

Now, setting:

$$z = \begin{pmatrix} \theta \\ \partial_t \theta \\ \phi \end{pmatrix} \in \mathbf{H}, \quad (24)$$

system (5-7) can be put in the operator form:

$$\frac{d}{dt} z(t) = Az(t) + Bw(t), \quad z_0 = \begin{pmatrix} \theta_0 \\ \theta_1 \\ 0 \end{pmatrix}, \quad (25)$$

where the operator A defined by:

$$A \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{bmatrix} \partial_x^2 u + \mu \int_0^\infty \xi^\beta \partial_x \phi dx \\ \xi^\beta \partial_x v - \xi^2 \phi \end{bmatrix}. \quad (26)$$

is an unbounded operator in the Hilbert space \mathbf{H} with domain

$$D(A) = \{(u, v, \phi)^T \in \mathbf{H} : \partial_x^2 u + \mu \int_0^\infty \xi^\beta \phi_x dx \in L^2((0,1)), v \in H_0^1((0,1)), \xi^\beta \partial_x v - \xi^2 \phi \in L^2((0,1) \times \mathbb{R})\} \quad (27)$$

and

$$B = \begin{bmatrix} 0 \\ b(x) \\ 0 \end{bmatrix}. \quad (28)$$

a bounded operator from \mathbf{R} into \mathbf{H} .

Theorem 1:

2. The operator A generates a dissipative C_0 -semigroup $S(t)$.

Proof:

By making use of (23) and (26), one easily proves the dissipativity, that is:

$$\left\langle A \begin{pmatrix} u \\ v \\ \phi \end{pmatrix}, \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} \right\rangle = -2\mu \|\xi \phi\|_{L^2((0,1) \times \mathbb{R})}^2; \quad (29)$$

the proof of the existence of a C_0 -semigroup is based on the Lumer-Phillip theorem [11]. For further details see [5]. ■

Theorem 2: In the absence of control, the solution of system (23) converges asymptotically towards zero.

Proof: The proof is based on the LaSalle's invariance principle which requires the following two conditions:

1. the maximal invariance set is $\{0\}$,
2. the system's trajectory is precompact.

Here is the demonstration of part 1. Let $(\theta, \partial_t \theta, \phi)$ be in the maximal invariance set; then from (29):

$$\|\xi \phi(t)\|_{L^2((0,1) \times \mathbb{R})} = 0, \quad \forall t \geq 0, \quad (30)$$

hence:

$$\phi(t) = 0, \quad \forall t \geq 0; \quad (31)$$

now combining (31), (12) and (4), we see that:

$$\partial_x \theta(x, t) = \partial_x \theta(x, 0), \quad \forall t \geq 0; \quad (32)$$

also utilizing (31), equations (18) and (20) become:

$$\partial_t^2 \theta(x, t) = \partial_x^2 \theta(x, t), \quad (33)$$

$$\theta(0, t) = \theta(1, t) = 0; \quad (34)$$

now from (32) and (33-34) we deduce that $\theta(x, t) \equiv 0$.

The proof of part 2. is technically more elaborate; see [5]. ■

5. Quadratic control on finite horizon

We now consider system (25) with control. For every $w \in L^2(0, T)$, the mild solution [4] of (5-7) is determined by the first two components of z_0 , that is:

$$z(t) = S(t)[u_0, v_0, 0]^T + \int_0^t S(t-\tau)Bw(\tau)d\tau. \quad (35)$$

The cost functional (9-10) may be put in the form:

$$J(w) = \langle Rz(T), z(T) \rangle + \int_0^T \langle Qz, z \rangle dt + \int_0^T w^2 dt, \quad (36)$$

with $Q, R \in \mathcal{L}(\mathbf{H})$ defined by:

$$Q = \begin{bmatrix} \frac{q}{2}I & 0 & 0 \\ 0 & \frac{1}{2}I & 0 \\ 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} \frac{r}{2}I & 0 & 0 \\ 0 & \frac{1}{2}I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad r, q > 0, \quad (37)$$

I being the identity operator in the corresponding spaces.

The problem (35-36) is a standard optimal quadratic control problem [4], in the space state form. Its solution is given by:

$$w^{opt}(t) = -B^* P(t)z(t) \quad (38)$$

where $P(t)$ is the unique self-adjoint non-negative solution of the Riccati differential equation on $[0, T]$:

$$\frac{dP}{dt} + A^* P + P A + Q^* Q - P B^* B P = 0, \quad P(T) = R. \quad (39)$$

The minimal cost corresponding to the problem (35-36) is therefore given by:

$$\min_{w \in L^2(0,1)} J(w) = \langle P(0)z_0, z_0 \rangle; \quad (40)$$

setting:

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \quad (41)$$

and recapitulating, we arrive at the following result:

Theorem3: *The quadratic control problem (5-7), (9-10) has a unique solution. The optimal control and the corresponding minimal cost are respectively given by:*

$$w^{opt}(t) = -\langle b(x), P_{21}(t)\theta(x, t) + P_{22}(t)\partial_t\theta(x, t) \rangle_{L^2(0,1)} \quad (42)$$

and

$$J(w^{opt}) = \left\langle \begin{bmatrix} P_{11}(0) & P_{12}(0) \\ P_{21}(0) & P_{22}(0) \end{bmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix}, \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} \right\rangle_{H_0^1 \times L^2} \quad (43)$$

6. Numerical Simulations

In this section we present some numerical results concerning system (18-21) in the absence of control ($w \equiv 0$). The aim is to illustrate the convenience of the non-hereditary formulation from the numerical point of view and also to exhibit some differences between the classical (viscous) and the fractional damping mechanisms. Note that, due to the increasing memory required for evaluating the integral term, the classical integral formulation of (5-7) does not permit numerical approximations with a reasonable volume of computation. Besides, the presence of a singularity in the kernel and the fact that $t^{-\alpha}$ decays slowly considerably complicates the problem of the numerical resolution of (5-7).

The classical viscous model which is compared to (5-7) is:

$$\partial_t^2 \varphi - \partial_x^2 \varphi - \lambda \partial_t \partial_x^2 \varphi = 0 \quad (44)$$

with suitable initial and boundary conditions. Coefficient $\lambda > 0$ plays a similar role as μ in (5-7). Remember that the mechanical energy of (44) is a decreasing function of the time variable.

Finite difference schemes have been used to approximate the solutions of (18-21) and (44). A high precision approximation of $\phi(x, \xi, t) : \tilde{\phi}(x_i, \xi_j, t_k)$ may be obtained with only a few discretization points in the ξ -direction, provided correct meshes are used [7]. The stability and consistency of the scheme remain simple thanks to formulation (18-19).

Figures 1 and 2 show the energy of the rod both for the cases of viscous and fractional damping (φ and θ) respectively. The initial states are taken as:

$$\theta_0(x) = \varphi_0(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2-2x & 1/2 \leq x \leq 1, \end{cases} \quad (45)$$

$$\theta_1(x) = \varphi_1(x) = 0 \quad (46)$$

While the mechanical energy in the fractional damping case is globally decreasing (recall that system (16-17), and therefore (5-7), converges towards zero), it is sometimes increasing. This is clearly due to the presence of the hereditary term. However, from (29) the "total" energy defined by the sum of the mechanical energy of the rod and the L^2 -norm of ϕ is always decreasing. In figure 2, the rates of decay of the mechanical energies can be compared. The fractional damping seems to generate slow asymptotic behavior; this is presently under study.

Figures 3 and 4 are 3-D representations of an evolution of θ and φ respectively. They show significant differences between the two types of damping mechanisms and thereby stress the need for accurate models for the analysis and control of fractionally damped systems.

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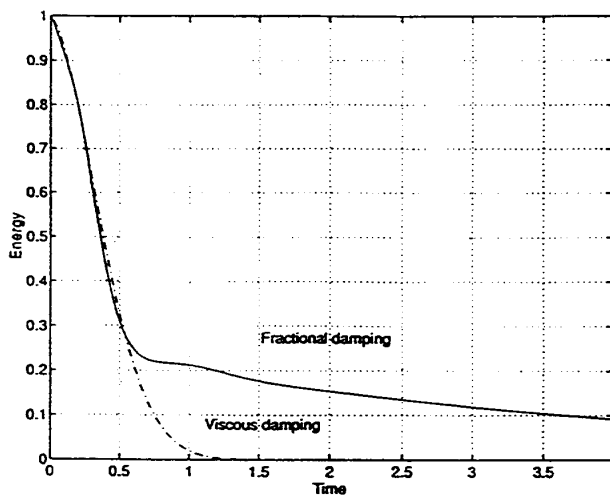


Fig 1: energy of the rod ($\lambda = \mu = 0.5$).

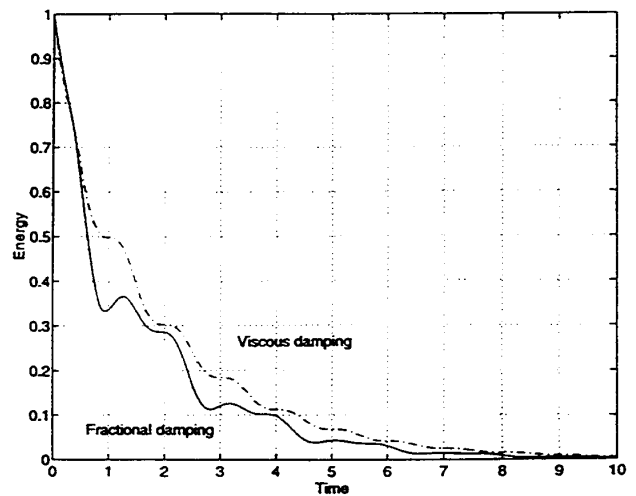


Fig 2: energy of the rod ($\lambda = \mu = 0.05$).

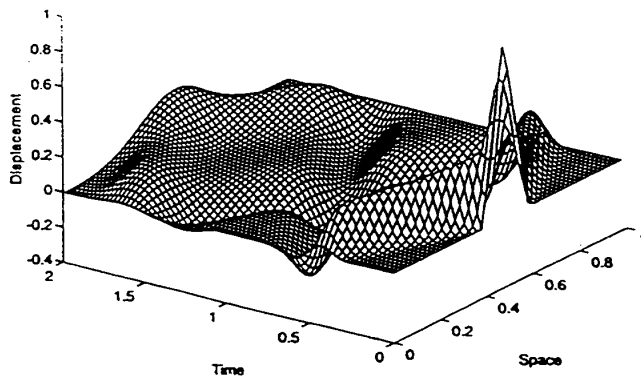


Fig 3: evolution of φ ($\lambda = 0.01$).

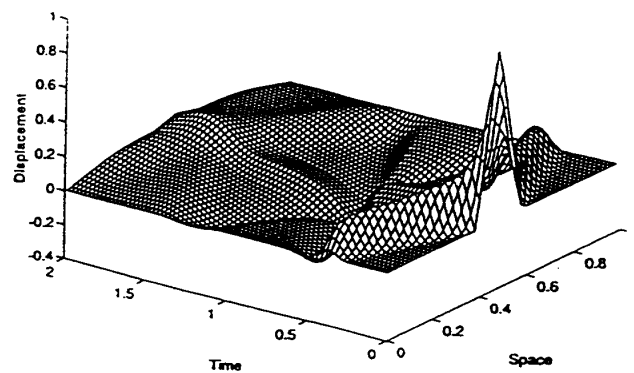


Fig 4: evolution of θ ($\mu = 0.04$).