

On Fractional Derivatives, Fractional-Order Dynamic Systems and $PI^\lambda D^\mu$ -controllers

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Abstract

These notes are devoted to some methods used in the fractional calculus (theory of integration and differentiation of an arbitrary order) and to application of the fractional calculus to modelling and control of dynamical systems.

1 On two definitions of fractional derivatives

In this section two different definitions of the fractional derivatives (the Letnikov-Riemann-Liouville definition and the Caputo definition) are discussed from the viewpoint of the use for the formulation and solution of applied problems leading to the differential equations of the fractional (non-integer) order.

There are various ways of interpolating the definition of the integer-order derivative to non-integer (and even not necessarily real) order (e.g., [10, 8, 16]). However, we will consider the relationship only between two of them, which we call from here on the Letnikov-Riemann-Liouville definition (LRL definition) and the Caputo definition. The main question of our interest is the comparison of these definitions from the viewpoint of their application in physics and engineering.

Keeping in mind this aim, let us start with recalling some facts of the historical background since the second half of the XIXth century.

In 1867, A. K. Grünwald [4] considered the definition of the fractional derivative, which can be written using the modern notation in terms of the fractional-order differences as

$${}_a^G D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{{}_a \Delta_h^\alpha f(t)}{h^\alpha}, \quad (1)$$

$${}_a \Delta_h^\alpha f(t) = \sum_{j=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^j \binom{\alpha}{j} f(t-jh),$$

where $[x]$ means the integer part of x .

In 1868, A. V. Letnikov, who did not know about Grünwald's results, suggested a similar approach. Letnikov proved the following relationship

$${}_a^G D_t^\alpha f(t) = \sum_{k=0}^m \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha+1)} \int_a^t (t-\tau)^{m-\alpha} f^{(m+1)}(\tau) d\tau \quad (2)$$

Note that m in the formula (2) can be any positive integer number. The choice of m is limited only by the condition that the integral on the right-hand side must exist. A different choice of m leads to different function classes upon which the definition of the fractional derivative serves as an extension of the definition of integer-order derivatives. In particular, it seems to be convenient to take, for example

$$m < \text{Re}(\alpha) < m + 1.$$

We see that Letnikov obtained his formula (2) under the assumption that $f(t)$ is sufficiently differentiable and that $f^{(k)}(a) < \infty$, $k = 0, 1, \dots, m$.

However, the sum on the right-hand side causes certain inconvenience during the direct manipulation with the fractional derivatives defined by (2). This explains why Letnikov in his work [6] introduced a new version of the definition of the fractional derivative, which can be written as

$${}_a^L D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(\tau) d\tau}{(t-\tau)^{\alpha-n+1}}, \quad (3)$$

$$(n-1 < \alpha < n).$$

Letnikov required the function $f(t)$ to be continuous in the interval $t > a$ and allowed it to be infinite of order r at $t = a$, i.e. $\lim_{t \rightarrow a} (t-a)^r f(t) = A (\neq 0)$. This is a weaker condition than the existence and finiteness of a number of integer-order derivatives of the function $f(t)$. The definition (3) is a very natural generalization

of the right-hand side of the relationship (2). Indeed, assuming that $f(t)$ has a necessary number of continuous derivatives having finite values at $t = a$ and integrating (3) by parts, we obtain the right-hand side of (2). This also shows that there is a function class upon which the Grünwald-Letnikov definition (1) and the Letnikov definition (3) are equivalent.

The demands of the modern technology require a certain revision of the well-established pure mathematical approach. There appeared a number of works, especially in the theory of viscoelasticity and in hereditary solid mechanics, where the fractional derivatives are used for the better description of the material properties. Mathematical modelling based on enhanced rheological models naturally leads to differential equations of the fractional order — and to the necessity of the formulation of initial conditions to such equations. Applied problems require the definitions of the fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a)$, $f'(a)$ etc.

Unfortunately, the LRL approach leads to initial conditions containing the limit values of the LRL fractional derivatives at $t = a$, for example of the type

$$\lim_{t \rightarrow a} {}_a^L D_t^{\alpha-k} f(t) = b_k,$$

where b_k , $k = 1, 2, \dots, n$ are given constants.

In spite of the fact that initial value problems with such initial conditions can be successfully solved mathematically (see, for example, solutions given in [16] and [12]), their solutions are useless practically, because there is no physical interpretation known for such type of initial conditions.

A certain solution to this conflict between the well-established and polished mathematical theory and the practical needs was proposed by Caputo first in his paper [1] and two years later in his book [2]. Caputo's definition can be written as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t - \tau)^{\alpha+1-n}}, \quad (4)$$

$$(n - 1 < \alpha < n).$$

The main advantage of Caputo's approach is that the initial conditions for the fractional differential equation with the Caputo derivatives take on the same form as for integer-order differential equations.

To underline the difference in the form of the initial conditions which must accompany fractional differential equations in terms of the LRL and the Caputo derivatives, let us recall the corresponding Laplace transform formulas for the case $a = 0$.

The classical formula for the Laplace transform of the LRL fractional derivative, is (e.g., [10, p.134] or [8, p.123]):

$$\int_0^\infty e^{-pt} \{ {}_0^L D_t^\alpha f(t) \} dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^k {}_0^L D_t^{\alpha-k-1} f(t) \Big|_{t=0},$$

$$(n - 1 < \alpha \leq n).$$

whereas Caputo's formula obtained in [1] for the Laplace transform of the Caputo derivative is

$$\int_0^\infty e^{-pt} \{ {}_0^C D_t^\alpha f(t) \} dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0),$$

$$(n - 1 < \alpha \leq n).$$

The Laplace transform method is frequently used for solving applied problems. To choose the appropriate Laplace transform formula, it is very important to understand which type of the definition of the fractional derivative (in other words, which type of initial conditions) must be used.

Another difference between the LRL definition and the Caputo definition is that the Caputo derivative of a constant is 0, whereas in the cases of a finite value of the lower integration limit a the LRL fractional derivative of a constant C is not equal to 0, but

$${}_0^L D_t^\alpha C = \frac{C t^{-\alpha}}{\Gamma(1 - \alpha)}.$$

Integrating (3) by parts gives

$${}_a^L D_t^\alpha f(t) = {}_a^C D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{(t - a)^{k-\alpha}}{\Gamma(k - \alpha + 1)} f^{(k)}(a),$$

which shows that the LRL definition (3) and the Caputo definition (4) coincides when $f^{(k)}(a) = 0$, ($k = 0, 1, \dots, n - 1$). This has been used by Podlubny for the time-domain analysis of the fractional-order control systems [13].

The further comparison of the two definitions of fractional derivative (the details can be found in [14]) led us to the conclusion that *the Letnikov-Riemann-Liouville definition (3) serves as a generalization of the notion of the generalized (in the sense of distributions) derivative, while the Caputo derivative (4) is a generalization of differentiation in the classical sense.*

2 The Laplace transform method

The Laplace transform method for the solution of linear differential equations with constant coefficients, described in detail in [12], is based on the following formula of the Laplace transform of the k -th order derivative of the Mittag-Leffler function in two parameters:

$$\int_0^{\infty} e^{-pt} t^{\alpha k + \beta - 1} E_{\alpha, \beta}^{(k)}(\pm at^{\alpha}) dt = \frac{k! p^{\alpha - \beta}}{(p^{\alpha} \mp a)^{k+1}},$$

$$(Re(p) > |a|^{1/\alpha}),$$

where the Mittag-Leffler function is defined by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0)$$

The Laplace transform can be used for the solution of fractional differential equations with the LRL derivatives, the Caputo derivatives, and with so-called sequential Miller-Ross derivatives, as well as for solving partial fractional differential equations with constant coefficients. Examples of the use of the Laplace transform method can be found in [12] and [13].

3 Fractional Green's function

For the solution of fractional differential equations for an arbitrary right-hand side, Podlubny [13] suggested a generalization of the definition of Green's function given in [9].

The most important property of the fractional Green's function is that the solution of an initial-value problem with inhomogeneous initial conditions and an arbitrary right-hand side of the fractional differential equation (i.e., a fractional-order model of a dynamical system for an arbitrary input signal) can be expressed in terms of the known fractional Green's function (see [13]).

4 Fractional difference method

If the use of the Laplace transform method is impossible or time-consuming, we use the fractional difference method for the numerical solution of fractional differential equations [15].

Omitting the sign of limit in the Grünwald-Letnikov definition (1), we obtain the first order approximation of the derivative of order α . It is also possible to use higher order approximation formulas [7]. On this way, we arrive at explicit algorithms, which allows step-by-step evaluation of the values of solution of the fractional-order differential equations.

The main difficulty is the growing number of terms in the sums in the recurrence formulas used for computations. Due to this, for large intervals of change of

an independent variable t evaluation becomes, in fact, impossible. To overcome this difficulty, the "short-memory" principle was suggested [15], which can be written most concisely as

$${}_a D_t^{\alpha} y(t) \approx {}_{t-L} D_t^{\alpha} y(t),$$

where L is the "memory" length, the choice of which depends on the required precision of the output.

The short-memory principle provides a certain maximal number of terms in computed sums and therefore leads to faster algorithms.

To our opinion, the most important open problem in the field of numerical solution of fractional differential equations is obtaining of formulas for approximation of initial conditions of the form ${}_a D_t^{\alpha} y(t) \Big|_{t=a} = b, (b \neq 0)$.

5 On identification of parameters of fractional-order models

In this section we describe an approach to identification of parameters of fractional-order models of real dynamical systems. The method is illustrated on the following examples.

5.1 Heating furnace.

A set of measured values y_i^* ($i = 0, M$) was obtained for the transfer function of a real experimental heating furnace. Then three models were developed for this object.

The first model was obtained using integer-order derivatives. Assuming that the system can be described by the second-order differential equation

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = u(t), \quad (5)$$

we obtained the following values

$$a_2 = -73043.360, \quad a_1 = 4892.733, \quad a_0 = 1.928,$$

which minimize the criterion Q

$$Q = \frac{1}{M+1} \sum_{i=0}^M (y_i^* - y_i),$$

where y_i is the output of the model at the point of i -th measurement. In this case, the minimal value of Q is $1.02 \cdot 10^{-3}$.

Another model was obtained assuming that the system can be described by the three-term fractional differential equation

$$a_2 y^{(\alpha)}(t) + a_1 y^{\beta}(t) + a_0 y(t) = u(t).$$

In this case, the following values for orders α and β and for the coefficients a_0 , a_1 and a_2 were obtained:

$$\alpha = 1.31, \quad \beta = 0.97,$$

$$a_2 = -14994.3, \quad a_1 = 6009.52, \quad a_0 = 1.69.$$

giving the value $Q = 2.7 \cdot 10^{-4}$ for the criterion. The result of fitting of the unit-impulse response of the furnace is presented in Fig.1.

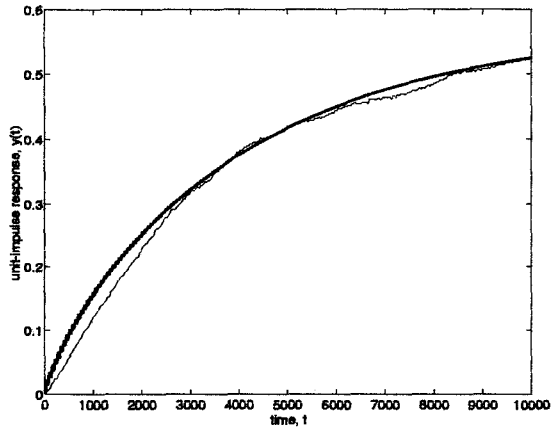


Figure 1: Fractional-order model of the unit-impulse response of a heating furnace.

Moreover, the object can also be modelled by a two-term fractional differential equation. In such a case we must put $a_2 = 0$ in the equation (5), so the term with α -th derivative disappears. The remaining parameters of the two-term fractional model take on the values

$$\beta = 0.73, \quad a_1 = 788.35, \quad a_0 = 1.39,$$

and the corresponding value of criterion is $Q = 6.3 \cdot 10^{-4}$.

5.2 Change of properties of transport wires.

As we will show below, fractional derivatives can also be used as an instrument for modelling the impact of hereditary effects in steel wires on the change of their mechanical properties without the detailed study of the causes of the changes.

Until now, the polynomial regression is usually used for the solution of the problem. To determine basic disadvantages of the polynomial regression models which are frequently applied for estimation of reliability of steel wires, used in mining transport machines, it is necessary to recall the main features of the process of the change of properties of such a wire:

- during a certain period after installation of a wire an enhancement of its properties is observed;
- then properties of a wire become worse and worse, until it breaks down;

- the period of enhancement is shorter than the period of decrease, and the general shape of the process curve is not symmetric.

The linear regression can give a rough estimate for the second phase (decrease of the performance of a wire), but it cannot describe the period of the initial enhancement of wire properties. The parabolic regression gives a symmetric shape of a fitting curve, which does not correspond to the physical background of the considered process. Higher order polynomial regression models can give better interpolation within the time interval for which measurements are available, but they give wrong picture if used for prediction.

In industrial practice, the parabolic regression is preferred in most cases even inspite of its physical inadequacy. As a consequence, this leads to underestimation of the strength of a wire and to its premature replacement.

One may try to use another regression model, for example, exponential, logarithmic, combined etc. However, all those types of regression curves, in fact, dictate a certain shape of a fitting curve, and the whole responsibility for the selected shape lies upon a researcher/engineer. All such approaches miss much of necessary flexibility.

An alternative approach, which we introduce in this paper, is based on the use of fractional integral for the description of hereditary changes of mechanical properties of steel wires.

A set of experimental measurements y_1, y_2, \dots, y_n is fitted with the help of the function $y(t)$ satisfying the following integral equation:

$$y(t) = \sum_{k=0}^{m-1} a_k t^k - a_m {}_0D_t^{-\alpha} y(t), \quad (0 < \alpha \leq m), \quad (6)$$

and the constant α , a_k , ($k = 0, \dots, m$) must be determined. For the determination of these parameters we used the least squares method, although any other criterion can be used as well. Regarding to the parameter m it is worth mentioning that m is the smallest integer number which is not less than α , so once one knows α , m is also known.

The parameters in the equation (6) allow obvious physical interpretation. Namely, a_k , ($k = 0, \dots, m-1$) are initial values of the fitting function $y(t)$ and its first $(m-1)$ derivatives. The fractional-order integral in the right-hand side represents the cumulative impact of previous history of loading on the present state of a wire, and the order of integration, α , determines the shape of the memory function of the wire material. By omitting the fractional-order integral we obtain the classical general polynomial regression model.

The problem can be reduced to the initial-value problem for the auxiliary unknown function $z(t)$, where $z(t) = y(t) - \sum_{k=0}^{m-1} a_k t^k$:

$${}_0 D_t^\alpha z(t) + a_m z(t) = -a_m \sum_{k=0}^{m-1} a_k t^k, \quad (7)$$

$$z^{(k)}(0) = 0, \quad (k = 0, \dots, m-1).$$

The fact that initial conditions are zero allow application of fractional difference method [15] for the numerical solution of the problem (7) for any fixed combination of parameters a_k , ($k = 0, \dots, m$). After the solution $z(t)$ is computed, we can use the backward substitution and evaluate the value of the least-squares criterion for the function $y(t)$. The optimal set of parameters a_k can be determined by using known optimization methods. In particular, in the presented work we used the standard MATLAB functions for unconstrained optimization.

The developed approach was applied to the set of 14 measurements made each 6 months during 7 years at the mining transport equipment at the Nizna Slana Mines.

In Fig. 2 a comparison of the parabolic regression model ($y(t) = -0.0330t^2 + 0.5619t + 10.7236$) and the described fractional differential equation approach is given. The following parameters of the equation (6) had been computed: $\alpha = 1.32$, $m = 2$, $a_1 = 10.1955$, $a_2 = 1.2760$, $a_3 = 0.0457$. There is no surprize in the fact that the fractional-order model gives the less value of the least squares criterion. More important is the fact that all significant characteristics of the shape of the process curve, which we mentioned above, are preserved.

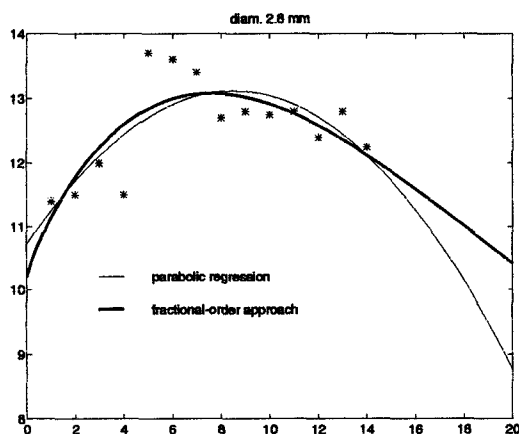


Figure 2: Fractional order model versus parabolic regression

We would like to mention that our experiments with the identification of parameters of fractional-order

models showed that, in general, fractional-order models can better describe reality than integer-order models. However, we discovered that in the case of using fractional derivatives there may be many combinations of orders of parameters, which give very close values for the chosen criterion of fitting. It seem to us that models involving only fractional integrals do not have this property. The problem of the definite selection of orders of derivatives and integrals, appearing in the fractional-order models, must be studied with attention.

6 $PI^\lambda D^\mu$ -controllers

The idea of using fractional-order controllers for the control of dynamic systems belongs to A. Oustaloup, who developed the so-called CRONE controller (CRONE is an abbreviation of *Comande Robuste d'Ordre Non Entier*), which is described in his book [11] along with examples of applications in various fields (see also other references in [11]). A. Oustaloup demonstrated the advantage of the CRONE controller in comparison with the PID -controller.

In 1994, Podlubny [13] proposed a generalization of the PID -controller, which is called the $PI^\lambda D^\mu$ -controller because of involving an integrator of order λ and differentiator of order μ , and which also shows better performance when used for the control of fractional-order systems than the classical PID -controller.

The transfer function of such the $PI^\lambda D^\mu$ -controller has the form:

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\lambda} + K_D s^\mu, \quad (8)$$

$$(\lambda, \mu > 0)$$

The equation for the $PI^\lambda D^\mu$ -controller's output in the time domain is:

$$u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^\mu e(t) \quad (9)$$

Taking $\lambda = 1$ and $\mu = 1$, we obtain a classic PID -controller. $\lambda = 1$ and $\mu = 0$ give a PI -controller. $\lambda = 0$ and $\mu = 1$ give a PD -controller. $\lambda = 0$ and $\mu = 0$ give an gain (see the $P - I - D$ plane, Fig. 3).

All these classical types of PID -controllers are the particular cases of the fractional $PI^\lambda D^\mu$ -controller (8). However, the $PI^\lambda D^\mu$ -controller is more flexible and gives an opportunity to better adjust the dynamical properties of a fractional-order control system. In terms of the $P - I - D$ plane, this means that instead of "jumping" between four fixed points in the plane it is possible to move in this plane continuously.

The advantages of the $PI^\lambda D^\mu$ -controllers in comparison with classical PID -controllers were discussed in

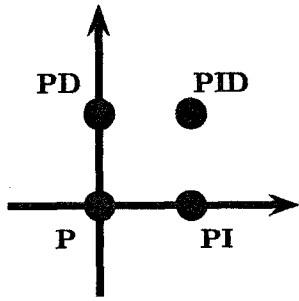


Figure 3: The $P - I - D$ plane

[13], [17], and [18]. One of the most important advantages is the better control of dynamical systems, which are described by fractional-order mathematical models. The second one is the fact that the $PI^\lambda D^\mu$ -controllers are less sensitive to changes of parameters of a controlled system and to changes of parameters of the controller itself. The same property is typical also for the CRONE controllers [11].

7 Conclusion

The benefits of using of fractional-order models of real dynamical objects and processes become more and more obvious, and examples of applications appear in various fields of science and technology. On this way, the most important problems, in our opinion, are the following:

- methods for identification of the *structure* of a fractional-order mathematical model of a real object,
- methods for identification of the *model parameters*,
- *technical realization* of fractional-order controllers,
- further development of *numerical methods and fast computational algorithms* for solution of ordinary and partial differential equations with fractional-order derivatives, and for computation of special functions of the fractional calculus,
- establishment of a strong *link between "fractional" and "fractal"*,
- development of approaches to obtaining fractional-order models of real objects and processes, and to physical interpretation of fractional-order derivatives (LRL, Caputo, Miller-Ross, etc.) and integrals,

and the last one is introducing the fractional calculus approach, as a convenient and natural part of the mathematical analysis, to a wide community of its potential users — applied scientists, technologists and engineers.

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