

# Correspondence

## Analytic Closed-Form Matrix for Designing Higher Order Digital Differentiators Using Eigen-Approach

Soo-Chang Pei and Jong-Jy Shyu

**Abstract**—The eigenfilter method has been recently proposed for designing higher order differentiators very effectively. The design method is based on the computation of an eigenvector of an appropriate real, symmetric, and positive-definite matrix. The elements of this matrix are usually evaluated by very time-consuming numerical integration. In this correspondence, we present simple analytic closed-form formulas to compute these matrix-elements very efficiently. Hence, the eigenfilter approach for differentiators becomes much easier and more accurate than before and design time is reduced greatly for designing long filters.

### I. INTRODUCTION

Higher order digital differentiators are very useful for the calculation of geometric moments [1] and for biological signal processing [2] as well as for the analysis of seismic and geological data [3], [4]. Recently, Rahenkamp and Kumar [5] have modified the well-known McClellan-Parks program [6] for designing higher order differentiators. However, this modified McClellan-Parks algorithm often leads to very large deviation or fails to converge, especially for designing full-band higher order differentiators. An analytic closed-form method was also proposed for the higher order differentiator design by Shah *et al.* [7]. But the method is only suitable for the design of higher order differentiators with low relative error around spot frequencies  $\omega = \pi/(\text{any integer})$ . Pei and Shyu have extended Vaidyanathan and Nguyen's eigenfilter approach [8] to the design of FIR Hilbert transformers and differentiators [9], higher order digital differentiators [10], and 2-D FIR filters [11]. In this work, we present a new closed-form matrix for designing wide-band or full-band higher order differentiators, and we find that the proposed general recursive formula can be derived for arbitrary higher order digital differentiators. Hence, the eigenfilter approach becomes much easier and more accurate than before and design time is reduced greatly.

### II. CLOSED-FORM MATRIX FOR DESIGNING EVEN-ORDER DIFFERENTIATORS

The eigenfilter method is based on minimizing a quadratic measure of the error in the frequency band. Let  $D(\omega)$  and  $H(\omega)$  denote the desired and designed frequency responses of the differentiator; the total error function can be formulated as

$$E = \frac{1}{\pi} \int_0^{\omega_p} [D(\omega) - H(\omega)]^2 d\omega \quad (1)$$

where  $\omega_p$  is the highest radian frequency for which the differentiating action is required. The objective is to express the above error in the

Manuscript received August 16, 1991; revised August 29, 1995. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Robert A. Gabel.

S.-C. Pei is with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, R.O.C. (e-mail: pei@cc.ee.ntu.edu.tw).

J.-J. Shyu is with the Department of Computer Science and Engineering, Tatung Institute of Technology, Taipei, Taiwan, R.O.C.

Publisher Item Identifier S 1053-587X(96)02409-7.

following quadratic eigenformulation:

$$E = \mathbf{A}^t \mathbf{Q} \mathbf{A} \quad (2)$$

where  $t$  is the vector transpose operation,  $\mathbf{Q}$  is a real, symmetric and positive-definite matrix, and  $\mathbf{A}$  is a real vector related to the filter coefficients in some manner. By the Rayleigh principle [12], the eigenvector  $\mathbf{A}$  associated with the smallest eigenvalue of matrix  $\mathbf{Q}$  minimizes the total error  $E$ .

Assume the filter length is  $N$  and it has been shown in [10] that the elements of matrix  $\mathbf{Q}$  for even-order differentiators are given by

$$\begin{aligned} q(n, m) &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^k}{\omega_0^k} \cos(n\omega_0) - \cos(n\omega) \right] \\ &\quad \times \left[ \frac{\omega^k}{\omega_0^k} \cos(m\omega_0) - \cos(m\omega) \right] d\omega \\ &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^{2k}}{\omega_0^{2k}} \cos(n\omega_0) \cos(m\omega_0) - \frac{\omega^k}{\omega_0^k} \cos(n\omega_0) \cos(m\omega) \right. \\ &\quad \left. - \frac{\omega^k}{\omega_0^k} \cos(n\omega) \cos(m\omega_0) + \cos(n\omega) \cos(m\omega) \right] d\omega \\ &\quad N \text{ odd, } 0 \leq n, m \leq \frac{N-1}{2} \quad (3) \end{aligned}$$

and

$$\begin{aligned} q(n, m) &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^k}{\omega_0^k} \cos\left(\left(n - \frac{1}{2}\right)\omega_0\right) - \cos\left(\left(n - \frac{1}{2}\right)\omega\right) \right] \\ &\quad \times \left[ \frac{\omega^k}{\omega_0^k} \cos\left(\left(m - \frac{1}{2}\right)\omega_0\right) - \cos\left(\left(m - \frac{1}{2}\right)\omega\right) \right] d\omega \\ &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^{2k}}{\omega_0^{2k}} \cos\left(\left(n - \frac{1}{2}\right)\omega_0\right) \cos\left(\left(m - \frac{1}{2}\right)\omega_0\right) \right. \\ &\quad \left. - \frac{\omega^k}{\omega_0^k} \cos\left(\left(n - \frac{1}{2}\right)\omega_0\right) \cos\left(\left(m - \frac{1}{2}\right)\omega\right) \right. \\ &\quad \left. - \frac{\omega^k}{\omega_0^k} \cos\left(\left(n - \frac{1}{2}\right)\omega\right) \cos\left(\left(m - \frac{1}{2}\right)\omega_0\right) \right. \\ &\quad \left. + \cos\left(\left(n - \frac{1}{2}\right)\omega\right) \cos\left(\left(m - \frac{1}{2}\right)\omega\right) \right] d\omega \\ &\quad N \text{ even, } 1 \leq n, m \leq \frac{N}{2} \quad (4) \end{aligned}$$

where  $k$  is even and denotes the order of the differentiator, and  $\omega_0$  is the reference frequency point that is generally chosen at the center of the band in our eigenfilter approach [10]. It is noted that in the first term and the last term of (3) and (4), it is easy to obtain closed-form expressions for the integrals, but it is difficult with the other terms, especially for higher order  $k$ . Hence, numerical integration for evaluating (3) and (4) is usually taken by Simpson's rule, which not only results in larger numerical error but also takes much more computation time to design. In the following description, we will derive the closed-form formulas for (3) and (4).

Let

$$q(n, m) = q_1(n, m) - q_2(n, m) - q_3(n, m) + q_4(n, m) \quad (5)$$

where  $q_i(n, m)$ ,  $i = 1, 2, 3, 4$  represents the respective result of the four terms in (3) or (4). First suppose  $N$  is odd, then it is easy to

obtain

$$q_1(n, m) = \frac{\cos(n\omega_0) \cos(m\omega_0)}{(2k+1)\pi\omega_0^{2k}} \omega_p^{2k+1} \quad (6)$$

and

$$q_4(n, m) = \frac{1}{2\pi} \left[ \frac{\sin((n-m)\omega_p)}{n-m} + \frac{\sin((n+m)\omega_p)}{n+m} \right]. \quad (7)$$

As for evaluating  $q_2(n, m)$  and  $q_3(n, m)$ , the following equations [13] are used:

$$\int x^n \sin(ax) dx = -\frac{x^n}{a} \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx, \quad (8)$$

$$\int x^n \cos(ax) dx = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx. \quad (9)$$

Let

$$q_2(n, m) = \frac{\cos(n\omega_0)}{\pi\omega_0^k} \hat{q}_c(m) \quad (10)$$

and by (8) and (9), we can get

$$\begin{aligned} \hat{q}_c(m) &= \int_0^{\omega_p} \omega^k \cos(m\omega) d\omega \\ &= \frac{\omega_p^k \sin(m\omega_p)}{m} \\ &+ \sum_{l=1}^{k/2} \left[ (-1)^{l+1} \frac{k(k-1)\cdots(k-2l+2)\omega_p^{k-2l+1} \cos(m\omega_p)}{m^{2l}} \right. \\ &\quad \left. + (-1)^{l+2} \frac{k(k-1)\cdots(k-2l+1)\omega_p^{k-2l} \sin(m\omega_p)}{m^{2l+1}} \right], \\ & \quad m \neq 0. \quad (11) \end{aligned}$$

Notice that  $q_3(n, m)$  is the same as  $q_2(n, m)$  in form by interchanging the variables  $m$  and  $n$ ; we get

$$q_3(n, m) = q_2(m, n) = \frac{\cos(m\omega_0)}{\pi\omega_0^k} \hat{q}_c(n). \quad (12)$$

For  $N$  even, the expressions for  $q_i(n, m), i = 1, 2, 3, 4$  are very similar to those above for  $N$  odd if we replace the variables  $n, m, n+m$  by  $(n-\frac{1}{2}), (m-\frac{1}{2}),$  and  $(n+m-1)$ , respectively, as follows:

$$q_1(n, m) = \frac{\cos((n-\frac{1}{2})\omega_0) \cos((m-\frac{1}{2})\omega_0)}{(2k+1)\pi\omega_0^{2k}} \omega_p^{2k+1}, \quad (13)$$

$$q_2(n, m) = \frac{\cos((n-\frac{1}{2})\omega_0)}{\pi\omega_0^k} \hat{q}_c\left(m-\frac{1}{2}\right), \quad (14)$$

$$q_3(n, m) = \frac{\cos((m-\frac{1}{2})\omega_0)}{\pi\omega_0^k} \hat{q}_c\left(n-\frac{1}{2}\right) \quad (15)$$

and

$$q_4(n, m) = \frac{1}{2\pi} \left[ \frac{\sin((n-m)\omega_p)}{n-m} + \frac{\sin((n+m-1)\omega_p)}{n+m-1} \right]. \quad (16)$$

### III. CLOSED-FORM MATRIX FOR DESIGNING ODD-ORDER DIFFERENTIATORS

The elements of matrix  $\mathbf{Q}$  for odd-order differentiator design are given by [10]

$$\begin{aligned} q(n, m) &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^k}{\omega_0^k} \sin(n\omega_0) - \sin(n\omega) \right] \\ &\quad \times \left[ \frac{\omega^k}{\omega_0^k} \sin(m\omega_0) - \sin(m\omega) \right] d\omega \\ &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^{2k}}{\omega_0^{2k}} \sin(n\omega_0) \sin(m\omega_0) - \frac{\omega^k}{\omega_0^k} \sin(n\omega_0) \sin(m\omega) \right. \\ &\quad \left. - \frac{\omega^k}{\omega_0^k} \sin(n\omega) \sin(m\omega_0) + \sin(n\omega) \sin(m\omega) \right] d\omega \\ & \quad N \text{ odd, } 0 \leq n, m \leq \frac{N-1}{2} \quad (17) \end{aligned}$$

and

$$\begin{aligned} q(n, m) &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^k}{\omega_0^k} \sin\left(\left(n-\frac{1}{2}\right)\omega_0\right) - \sin\left(\left(n-\frac{1}{2}\right)\omega\right) \right] \\ &\quad \times \left[ \frac{\omega^k}{\omega_0^k} \sin\left(\left(m-\frac{1}{2}\right)\omega_0\right) - \sin\left(\left(m-\frac{1}{2}\right)\omega\right) \right] d\omega \\ &= \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\omega^{2k}}{\omega_0^{2k}} \sin\left(\left(n-\frac{1}{2}\right)\omega_0\right) \sin\left(\left(m-\frac{1}{2}\right)\omega_0\right) \right. \\ &\quad \left. - \frac{\omega^k}{\omega_0^k} \sin\left(\left(n-\frac{1}{2}\right)\omega_0\right) \sin\left(\left(m-\frac{1}{2}\right)\omega\right) \right. \\ &\quad \left. - \frac{\omega^k}{\omega_0^k} \sin\left(\left(n-\frac{1}{2}\right)\omega\right) \sin\left(\left(m-\frac{1}{2}\right)\omega_0\right) \right. \\ &\quad \left. + \sin\left(\left(n-\frac{1}{2}\right)\omega\right) \sin\left(\left(m-\frac{1}{2}\right)\omega\right) \right] d\omega \\ & \quad N \text{ even, } 1 \leq n, m \leq \frac{N}{2}. \quad (18) \end{aligned}$$

Again, let

$$q(n, m) = q_1(n, m) - q_2(n, m) - q_3(n, m) + q_4(n, m). \quad (19)$$

For  $N$  odd

$$q_1(n, m) = \frac{\sin(n\omega_0) \sin(m\omega_0)}{(2k+1)\pi\omega_0^{2k}} \omega_p^{2k+1} \quad (20)$$

and

$$q_4(n, m) = \frac{1}{2\pi} \left[ \frac{\sin((n-m)\omega_p)}{n-m} - \frac{\sin((n+m)\omega_p)}{n+m} \right]. \quad (21)$$

Also let

$$q_2(n, m) = \frac{\sin(n\omega_0)}{\pi\omega_0^k} \hat{q}_s(m) \quad (22)$$

and by (8) and (9), we can get

$$\begin{aligned} \hat{q}_s(m) &= \int_0^{\omega_p} \omega^k \sin(m\omega) d\omega \\ &= \sum_{l=1}^{\frac{k+1}{2}} \left[ (-1)^l \frac{r\omega_p^{k-2l+2} \cos(m\omega_p)}{m^{2l-1}} \right. \\ &\quad \left. + (-1)^{l+1} \frac{k(k-1)\cdots(k-2l+2)\omega_p^{k-2l+1} \sin(m\omega_p)}{m^{2l}} \right], \\ & \quad m \neq 0 \quad (23) \end{aligned}$$

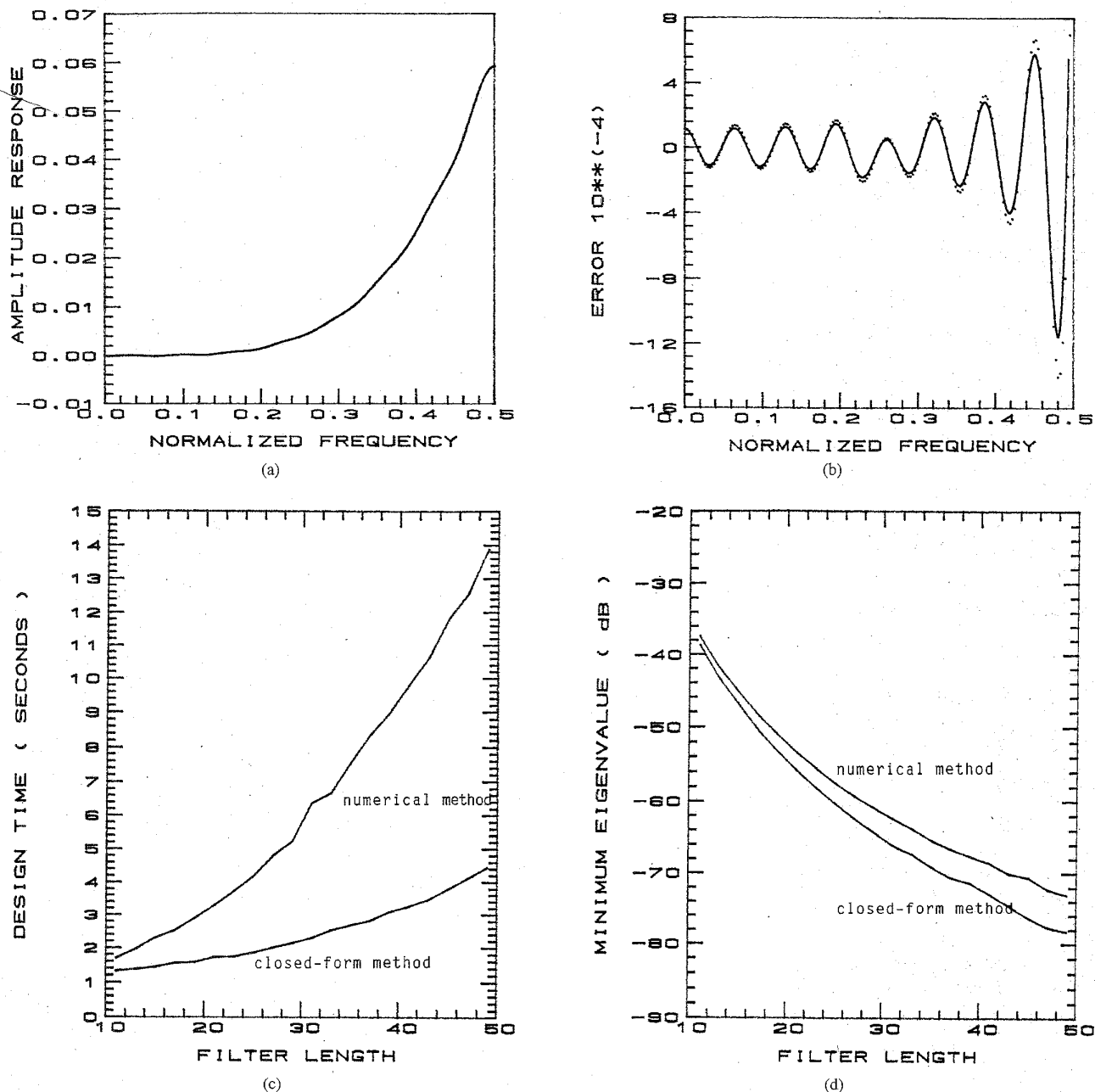


Fig. 1. Comparison of fourth-order, full-band differentiator design with closed-form method and numerical method: (a) magnitude response of differentiator with length  $N = 31$ ; (b) error curves for closed-form (solid line) and numerical method (dotted line); (c) design time; (d) minimum eigenvalues.

where

$$r = \begin{cases} 1, & l = 1, \\ k(k-1)(k-2)\cdots(k-2l+3), & l = 2, 3, \dots, \frac{k+1}{2}. \end{cases} \quad (24)$$

Similarly, by (22),

$$q_3(n, m) = q_2(m, n) = \frac{\sin(m\omega_0)}{\pi\omega_0^k} \hat{q}_s(n). \quad (25)$$

For  $N$  even, as above, if we replace the variables  $n, m, n+m$  with  $(n - \frac{1}{2}), (m - \frac{1}{2}),$  and  $(n + m - 1)$ , respectively, we can get

$$q_1(n, m) = \frac{\sin((n - \frac{1}{2})\omega_0) \sin((m - \frac{1}{2})\omega_0)}{(2k+1)\pi\omega_0^{2k}} \omega_p^{2k+1}, \quad (26)$$

$$q_2(n, m) = \frac{\sin((n - \frac{1}{2})\omega_0)}{\pi\omega_0^k} \hat{q}_s\left(m - \frac{1}{2}\right), \quad (27)$$

$$q_3(n, m) = \frac{\sin((m - \frac{1}{2})\omega_0)}{\pi\omega_0^k} \hat{q}_s\left(n - \frac{1}{2}\right) \quad (28)$$

and

$$q_4(n, m) = \frac{1}{2\pi} \left[ \frac{\sin((n-m)\omega_p)}{n-m} - \frac{\sin((n+m-1)\omega_p)}{n+m-1} \right]. \quad (29)$$

#### IV. EXAMPLES AND COMPARISONS

In this section, two higher order differentiators are designed by closed-form expressions and numerical integration, respectively. We

have taken the general rectangular Simpson's rule to evaluate the numerical integration with 100 grid points, and all simulations are done on VAX 11/780 computer in FORTRAN. In our work, we use the IMSL software package [14].

For the fourth-order full-band differentiator design, Fig. 1(a) and (b) show the magnitude response and error curves of a full-band, fourth-order differentiator with length  $N = 31$ , respectively. Notice that the error by the closed-form method (solid line) is smaller than that of the numerical integration method (dotted line) in all of the bands, especially near the band edge. Fig. 1(c) illustrates design time for various length fourth-order full-band differentiator design from length 11 to 49, in which the amount of time saved by the closed-form method is greatly increased with the larger filter length. For example, with length  $N = 49$ , the closed-form method is about four times faster than the numerical method. As for its accuracy and total error measure in the frequency band, their corresponding minimum eigenvalues for two methods are illustrated in Fig. 1(d). Obviously, the closed-form method is much faster and its error is smaller than that of the numerical method.

Another example of full-band third-order differentiators with different lengths from 10 to 50 are also designed by both the proposed closed-form method and the numerical method; results similar to the above example are obtained.

## V. CONCLUSION

We have presented a closed-form method for designing arbitrary higher order differentiators by the eigenfilter approach. We have shown that the closed-form method is much faster and performs better than the numerical method. A design example has been used to illustrate the effectiveness of this approach.

## REFERENCES

- [1] B. V. K. Vijaya Kumar and C. A. Rahenkamp, "Calculation of geometric moments from Fourier plane intensities," *Appl. Optics*, vol. 25, pp. 997-1007, Mar. 1986.
- [2] S. Usui and I. Amidror, "Digital low-pass differentiation for biological signal processing," *IEEE Trans. Biomed. Eng.*, vol. BME-29, pp. 686-693, Oct. 1982.
- [3] J. Claerbout, *Fundamentals of Geophysical Data Processing*. New York: McGraw-Hill, 1976.
- [4] A. Mesko, *Digital Filtering: Application in Geophysical Exploration for Oil*. London: Pitman, 1984.
- [5] C. A. Rahenkamp and B. V. K. Vijaya Kumar, "Modifications to the McClellan, Parks and Rabiner computer program for designing higher order differentiating FIR filters," *IEEE Trans. Acoust., Speech, Signal Proc.*, vol. ASSP-34, pp. 1671-1674, Dec. 1986.
- [6] J. H. McClellan, T. W. Parks, and L. R. Rabiner, "A computer program for designing optimal FIR linear phase digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 506-526, Dec. 1973.
- [7] H. Shah, S. C. D. Roy, and B. Kumar, "On the design of efficient second and higher degree FIR digital differentiators at the frequency  $\pi/(any\ integer)$ ," *Signal Processing*, vol. 27, pp. 117-124, May 1992.
- [8] P. P. Vaidyanathan and T. Q. Nguyen, "Eigenfilter: A new approach to least-squares FIR filter design and applications including Nyquist filters," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 11-23, Jan. 1987.
- [9] S. C. Pei and J. J. Shyu, "Design of FIR Hilbert transformers and differentiators by eigenfilters," *IEEE Trans. Circuits Syst.*, vol. 35, pp. 1457-1461, Nov. 1988.
- [10] ———, "Eigenfilter design of higher order digital differentiators," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 505-511, Apr. 1989.
- [11] ———, "2-D FIR eigenfilters: A least-square approach," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 24-34, Jan. 1990.
- [12] B. Nobel and J. W. Daniel, *Applied Linear Algebra*. Englewood Cliffs, NJ: Prentice-Hall, 1977.
- [13] L. J. Giacoletto, *Electronics Designer's Handbook*. New York: McGraw-Hill, 1977.
- [14] *IMSL Library Reference Manual*, International Mathematical and Statistical Libraries, Inc., Houston, TX, 1982.

## Prefiltering Approach for Optimal Polynomial Prediction

Timo I. Laakso and Seppo J. Ovaska

**Abstract**—A prefiltering approach for optimal prediction of polynomial signals is proposed. The new scheme enables the use of an arbitrary stable prefilter for which an optimal FIR postfilter is designed such that polynomial signals of given order are predicted unchanged. Additional degrees of freedom are used for noise suppression. The advantages of the approach are demonstrated with examples employing a first-order recursive prefilter.

## I. INTRODUCTION

Numerous real-world data sets are comprised of samples from slowly varying analog signals, and they can be modeled well as segments of low-order polynomials [1]. This assumption is valuable when it is desired to predict future samples of such signals. Predictive filtering (i.e., filtering the signal without delaying the primary component) is important, e.g., in automatic control applications where the delay in a feedback loop must be kept as small as possible to ensure fast controllability of the system. Several types of predictors have been proposed for extrapolation of polynomial signals, e.g., optimal Heinonen-Neuvo predictors [2] and computationally efficient smoothed Newton predictors [3].

Heinonen-Neuvo predictors are nonlinear-phase FIR filters that are constructed by requiring an  $L$ th-order polynomial to pass the filter unaltered and, in addition, by minimizing the wideband noise gain of the filter. In this sense, the Heinonen-Neuvo predictors are optimal, and they are in fact a modification of the so-called Savitzky-Golay filters for nonpredictive filtering of polynomial signals discussed in [4] and [5]. However, as polynomial signals are very narrowband lowpass signals, a high FIR filter order is needed to get low noise gain. Instead, as is the case with conventional (nonpredictive) filtering, recursive structures are, in general, more efficient for narrowband signal processing. Furthermore, the fullband linear-phase property of conventional FIR filters (usually the main reason for using FIR filters instead of IIR filters) cannot be utilized in prediction applications since the phase can be made approximately linear only in a narrow band.

Manuscript received September 18, 1992; revised August 29, 1995. The associate editor coordinating the review of this paper and approving it for publication was Dr. Robert A. Gabel.

T. I. Laakso is currently with the School of Electronic and Manufacturing Systems Engineering, University of Westminster, London, UK, on leave from the NOKIA Research Center, Helsinki, Finland.

S. J. Ovaska is currently with the Laboratory of Telecommunications Technology, Helsinki University of Technology, Espoo, Finland, on leave from Lappeenranta University of Technology, Lappeenranta, Finland.

Publisher Item Identifier S 1053-587X(96)02403-6.