# Iterative Learning Control of Linear Parameterized Varying Uncertain Systems

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## Abstract

In this paper, we propose a new numerical approach for the iterative learning control of repetitive linear parameterized varying systems with polytope uncertainties. The design problem is formulated as a group of convex optimization problems. Through interpolation, an optimal continuously differentiable switched iterative learning control law is constructed from the solutions of these convex optimization problems. Some sufficient conditions are derived for the convergence of the learning system. Our approach can also be used to consider the iterative learning control of linear time varying systems and nonlinear systems with polytope uncertainties.

**Keywords**: Iterative Learning Control, Convex Optimization Problem, Linear Parameterized Varying Systems, Polytope Uncertainties

## 1 Introduction

Iterative learning control (ILC) of repetitive systems has been studied by many researchers. [1] proposed a first-order iterative control law which has been widely used. [2] studied the iterative learning control with the same non-zero initial condition at each iteration. [3] proposed an initial state learning law to automatically initialize the system at each iteration. Most existing results are focused on the analysis issue of the iterative learning control. However, the convergence conditions found in the literature are typically not sufficient for actual ILC applications. Therefore, it is necessary to consider the design issue of ILC, especailly the optimal design in the case where the system model is not known exactly. [4] have considered iterative learning control design of repetitive linear time invariant (LTI) systems with unknown system matrices by using the estimated knowledge of the system matrices.

In practice, it is very difficult to estimate the system matrices, especially for linear parameterized varying uncertain systems and linear time varying uncertain systems. However, it is reasonable to assume that we do have some rough knowledge of the system matrices. For example, the system matrices may be known to lie within some intervals. Because the system matrices are functions of time variable, it is impossible to use the existing numerical methods such as linear matrix inequalities [5] to calculate the learning gain when there are parametric uncertainties. So far, there does not exist any effective approach for the design of an optimal iterative learning controller for repetitive linear parameterized varying systems and linear time varying systems with parametric uncertainties. However, the system matrices are constant at every fixed time point and they represent an LTI model. Note that the time variable is in a compact set which can be covered by a finite number of subintervals. Inside each subinterval, a fixed time point can be used to obtain an LTI model. Then a method is proposed for the finite LTI systems with polytope uncertainties to find optimal learning gains for these models. However, these piecewise gains cannot be applied directly because the $D$-type iterative learning control [1] is used. Thus, it is necessary to propose a method to interpolate these gains such that a continuously differentiable learning controller can be constructed.

In this note, we present an effective numerical approach for the design of iterative learning control for linear parameterized varying systems with polytope uncertainties by using some rough knowledge of the system matrices. The determination of the piecewise learning gains is formulated as a group of convex optimization problems. This makes the problem computationally tractable by some existing tools [5]. Then a method is proposed to interpolate these gains to form a switched continuously differentiable controller which is applied to the system to achieve the design objectives. Some sufficient conditions are derived for the convergence of the learning system. Our approach extends potential applications of iterative learning control and enlarges the class of systems to include parametric uncertainties. It can also be used to consider the iterative learning control of linear time varying systems and nonlinear systems with polytope uncertainties.

The rest of the note is organized as follows. Section 2 considers the formulation of the problems. Section 3 derives the main results of this note. A numerical example is given in section 4 to illustrate the application of the main result. The concluding remarks are given in section 5.

## 2 Problem Formulation

In this note, we consider the following repetitive linear parameterized varying systems

$$\left\{ \begin{array}{l} \dot{X}_i(t) = A(p(t))X_i(t) + B(p(t))U_i(t) \\ Y_i(t) = C(p(t))X_i(t) \end{array} \right. \tag{1}$$

where $i$ denotes the $i$th repetitive operation of the system; $X_i \in R^n$, $U_i \in R^r$, $Y_i \in R^m$ are the state vector, the input vector and the output vector, respectively; $t \in [t_0, T] \subseteq [0, T]$ is the time variable with $t_0$ and $T$ given, $p(t) \in R^{\hat{l}}$ is continuously differentiable and $A(p(t))$, $B(p(t))$ and $C(p(t))$ are uncertain system matrices defined as follows:

$$A(p(t)) = \sum_{j=1}^{r_3} e_j \hat{A}_j(p(t)) \; ; \; e_j \geq 0; \sum_{j=1}^{r_3} e_j = 1 \tag{2}$$

$$B(p(t)) = \sum_{j=1}^{r_1} f_j \hat{B}_j(p(t)) \; ; \; f_j \geq 0; \sum_{j=1}^{r_1} f_j = 1 \tag{3}$$

$$C(p(t)) = \sum_{j=1}^{r_2} g_j \hat{C}_j(p(t)) \; ; \; g_j \geq 0; \sum_{j=1}^{r_2} g_j = 1 \tag{4}$$

In the above equations, $\hat{A}_j(p(t))(1 \leq j \leq r_3)$, $\hat{B}_j(p(t))(1 \leq j \leq r_1)$ and $\hat{C}_j(p(t))(1 \leq j \leq r_2)$ are known continuously differentiable vertex matrices, $r_3$, $r_1$ and $r_2$ are the numbers of vertices, $e_j$, $f_j$ and $g_j$ are unknown constants.

**Remark 1.** *The class of system (1)-(4) can be used to linearize the following nonlinear system [7]*

$$\left\{ \begin{array}{l} \dot{\tilde{X}}_i(t) = F(\tilde{X}_i(t), \tilde{U}_i(t)) \\ \tilde{Y}_i(t) = H(\tilde{X}_i(t)) \end{array} \right. \tag{5}$$

*where $\tilde{X}_i \in R^n$, $\tilde{U}_i \in R^r$, $\tilde{Y}_i \in R^m$ are the state vector, the input vector and the output vector, respectively; $F$ and $H$ are uncertain system functions defined as follows:*

$$\begin{aligned} F(\tilde{X}_i(t), \tilde{U}_i(t)) &= \sum_{j=1}^{r_4} \hat{f}_j \hat{F}_j(\tilde{X}_i(t), \tilde{U}_i(t)) \; ; \; \hat{f}_j \geq 0 \; ; \; \sum_{j=1}^{r_4} \hat{f}_j = 1 \\ H(\tilde{X}_i(t)) &= \sum_{j=1}^{r_5} \hat{g}_j \hat{H}_j(\tilde{X}_i(t)) \; ; \; \hat{g}_j \geq 0 \; ; \; \sum_{j=1}^{r_5} \hat{g}_j = 1 \end{aligned} \tag{6}$$

*In the above equations, $\hat{F}_j(1 \leq j \leq r_1)$ and $\hat{H}_j(1 \leq j \leq r_2)$ are known smooth vertex functions, $r_4$ and $r_5$ are the numbers of vertices, $\hat{f}_j$ and $\hat{g}_j$ are unknown constants.*

*Suppose that there exists an equilibrium manifold that can be parameterized by a continuously differentiable scheduling variable, $p \in R^{\hat{l}}$. That is, there exist continuous functions, $\tilde{X}_i^o : R^{\hat{l}} \rightarrow R^n$ and $\tilde{U}_i^o : R^{\hat{l}} \rightarrow R^r$ such that [7]*

$$\hat{F}_j(\tilde{X}_i^o(p(t)), \tilde{U}_i^o(p(t))) = 0 \; ; \; 1 \leq j \leq r_4 \tag{7}$$

$$\tilde{Y}_i^o(p(t)) = \hat{H}_j(\tilde{X}_i^o(p(t))) \; ; \; 1 \leq j \leq r_5 \tag{8}$$

*for all $t \in [t_0, T]$. For each $p$, the Jacobian linearization of nonlinear plant (5) about the equilibrium $\tilde{X}_i^o(p(t))$, $\tilde{U}_i^o(p(t))$ is written in the form of (1)-(4) with*

$$\begin{aligned} A(p(t)) &= \frac{\partial F}{\partial \tilde{X}_i}(\tilde{X}_i^o(p(t)), \tilde{U}_i^o(p(t))) \\ B(p(t)) &= \frac{\partial F}{\partial \tilde{U}_i}(\tilde{X}_i^o(p(t)), \tilde{U}_i^o(p(t))) \\ C(p(t)) &= \frac{\partial H}{\partial \tilde{X}_i}(\tilde{X}_i^o(p(t))) \\ X_i(t) &= \tilde{X}_i(t) - \tilde{X}_i^o(p(t)) \\ U_i(t) &= \tilde{U}_i(t) - \tilde{U}_i^o(p(t)) \\ Y_i(t) &= \tilde{Y}_i(t) - \tilde{Y}_i^o(p(t)) \end{aligned}$$

*and*

$$\hat{A}_j(p(t)) = \frac{\partial \hat{F}_j}{\partial \tilde{X}_i}(\tilde{X}_i^o(p(t)), \tilde{U}_i^o(p(t))) \; ; \; 1 \leq j \leq r_4 \tag{9}$$

$$\hat{B}_j(p(t)) = \frac{\partial \hat{F}_j}{\partial \tilde{U}_i}(\tilde{X}_i^o(p(t)), \tilde{U}_i^o(p(t))) \; ; \; 1 \leq j \leq r_4 \tag{10}$$

$$\hat{C}_j(p(t)) = \frac{\partial \hat{H}_j}{\partial \tilde{X}_i}(\tilde{X}_i^o(p(t))) \; ; \; 1 \leq j \leq r_5 \tag{11}$$

$\square$

**Remark 2.** *If $p(t) = t$, system given in (1)-(4) becomes a linear time varying system with polytope uncertainties.* $\square$

We let $Y_d(t)$, $X_d(t_0)$, $X_d(t)$ and $U_d(t)$ denote respectively the desired output, the desired initial state, the desired state and the corresponding input to achieve $Y_d(t)$ and $X_d(t)$. The norms are defined as follows:

$$\begin{aligned} \|b\| &= \sqrt{\sum_{i=1}^{n} b_i^2} \; ; \; b \in R^n \; ; \; \|A\| = \sqrt{\lambda_{max}(A^T A)} \\ \|h(t)\|_\lambda &= \sup_{t \in [0,T]} e^{-\lambda t} \|h(t)\| \; ; \; h : [0, T] \rightarrow R \end{aligned}$$

where $\lambda_{max}(A)$ is the maximum eigenvalue of matrix $A$.

Given a system described (1)-(4), and a desired output trajectory $Y_d(t)$, the tracking error $e_i(t)$ at the

$i$th repetition is given by $e_i(t) = Y_d(t) - Y_i(t)$. Then our iterative learning control problem is formulated as follows. Starting from an arbitrary continuous initial input $U_0(t)$ and initial state $X_0(t_0)$, obtain the next input $U_1(t)$ and the initial state $X_1(t_0)$, and the subsequence $\{U_i(t), X_i(t_0) | i = 2, 3, \cdots\}$ for system (1)-(4) iteratively such that when $i \to \infty$, $Y_i(t) \to Y_d(t)$.

To the best knowledge of the authors, the above tracking problem of linear parameterized varying systems with polytope uncertainties has not been considered yet. To solve the above problem, the following iterative learning law can be used [1].

$$U_{i+1}(t) = U_i(t) + \Gamma(t)\dot{e}_i(t) \qquad (12)$$

where $\Gamma(t)$ is a continuously differentiable function of $t$. The resulting system is convergent if

$$\max_{t \in [t_0, T]} \|I - C(p(t))B(p(t))\Gamma(t)\| \leq \epsilon < 1 \qquad (13)$$

We shall make the following standard assumption about initial conditions.

A1). Repeatability of the initial setting is satisfied within an admissible derivation level, i.e.,

$$\|X_d(t_0) - X_i(t_0)\| \quad \leq \quad b_{x0} \qquad (14)$$

We shall also make the following assumption about the rough knowledge of the system.

A2) . For any fixed $t_l \in [t_0, T]$, suppose that there exists a $\Gamma(t_l)$ which solves the following convex optimization problem

$$\min_{\Gamma}\{\alpha\} \qquad (15)$$

subject to

$$\begin{bmatrix} a_{i,j}I & \Upsilon_{ij}(t_l) \\ \Upsilon_{ij}^T(t_l) & I \end{bmatrix} \geq 0 \qquad (16)$$

$$\Upsilon_{ij}(t_l) = I - \hat{C}_i(p(t_l))\hat{B}_j(p(t_l))\Gamma(t_l) \qquad (17)$$

$$1 > \alpha \geq a_{i,j} \ ; \ 1 \leq i \leq r_2; 1 \leq j \leq r_1 \qquad (18)$$

**Remark 3.** *Suppose that $r_1 = 1$ and $r_2 = 1$. Then Assumption A2 is equivalent to the following standard assumption.*

*FR: $C(p(t))B(p(t))$ is full rank for all $t \in [t_0, T]$.*

*Thus, Assumption A2 is actually the robust version of Assumption FR with polytopic uncertainties.* □

Our design objective is to find a sub-optimal continuously differentiable control gain $\Gamma(t)$ such that $Y_i(t) \to Y_d(t)$ as $i \to \infty$. In the design, the minimization of the following cost function is used as guideline.

$$J = \max_{t \in [t_0, T]} \|I - C(p(t))B(p(t))\Gamma(t)\| \qquad (19)$$

## 3 Main Results

In this section, we shall first present some supporting results.

**Lemma 1.** *Suppose that $\|A(p(t))\| \leq \hat{a}$, then*

$$\|\phi(t, t_0)\| \leq e^{\hat{a}(t - t_0)} \ ; \ \forall t \geq t_0$$

*where*

$$\begin{aligned} \frac{d\phi(t, t_0)}{dt} &= A(p(t))\phi(t, t_0) \\ \phi(t_0, t_0) &= I \end{aligned}$$

**Proof:** The proof is straightforward. □

**Lemma 2.** *The following three conditions are equivalent.*

1. *There exists a $\Gamma$ such that*
$$\|I - CB\Gamma\| < 1$$
2. *There exist an $a < 1$ and $\Gamma$ such that*
$$\begin{bmatrix} aI & I - CB\Gamma \\ (I - CB\Gamma)^T & I \end{bmatrix} \geq 0 \qquad (20)$$
3. *$CB$ is full rank.*

**Proof:** The proof is straightforward. □

We shall now present a method to construct a continuously differentiable $\Gamma(t)$ such that $J$ defined in (19) is less than 1. The design is mainly composed of the following two steps.

**Step 1.** Find finite number of optimal learning gains.

**Step 2.** Through interpolation, construct a continuously differentiable switched learning gain $\Gamma(t)$ from the obtained learning gains such that $J$ defined in (19) is less than 1.

We shall first consider step 1. Within this step, we are required to choose finite number of fixed time points $t_l$ such that the obtained learning gains $\Gamma(t_l)$ satisfy

$$\|I - C(p(t))B(p(t))\Gamma(t_l)\| < 1 \ ; \ \forall t \in \tilde{\Omega}(t, t_l) \subseteq [t_0, T] \qquad (21)$$

where $\tilde{\Omega}(t, t_l)$ is a subinterval containing $t_l$, $1 \leq l \leq N$, $N$ is the number of subintervals or learning gains such that

$$[t_0, T] \subseteq \cup_{l=1}^{N} \tilde{\Omega}(t, t_l) \tag{22}$$

This step is detailed as in the following two lemmas.

**Lemma 3.** *Suppose that Assumption A2 holds. Then, for any $t_l \in [t_0, T]$, we have*

$$\|I - C(p(t_l))B(p(t_l))\Gamma(t_l)\| < 1 \tag{23}$$

**Proof:** The proof is straightforward by using Lemma 2. □

**Lemma 4.** *Suppose that Assumption A2 holds. Then, there exist finite number of subintervals $\tilde{\Omega}(t, t_l)(l = 1, 2, \cdots, N)$ satisfying (22) and for each $\tilde{\Omega}(t, t_l)$, a learning gain $\Gamma(t_l)$ can be found such that (21) holds.*

**Proof:** For any fixed $t_l \in [t_0, T]$, we consider the following functions

$$f_{i,j}(t) = \|I - \hat{C}_i(p(t))\hat{B}_j(p(t))\Gamma(t_l)\| \; ; \; 1 \leq i \leq r_2; 1 \leq j \leq r_1.$$

Note from Lemma 3 that $f_{i,j}(t)$ are continuous functions of $t$ and $f_{i,j}(t_l) < 1$. Thus, there exist open sets $\Omega(t, t_l, i, j)$ including $t_l$ such that $f_{i,j}(t) < 1$ for all $t \in \Omega(t, t_l, i, j)$. Denote

$$\tilde{\Omega}(t, t_l) = \cap_{1 \leq i \leq r_2, 1 \leq j \leq r_1} \Omega(t, t_l, i, j).$$

Obviously,$\tilde{\Omega}(t, t_l)$ is an open set and $t_l$ is in $\tilde{\Omega}(t, t_l)$. Clearly, there exist finite number of $\tilde{\Omega}(t, t_l)(l = 1, 2, \cdots, N)$ such that $\cup_{l=1}^{N} \tilde{\Omega}(t, t_l)$ covers the set $[t_0, T]$ and thus (22) is verified. Also, for each $t \in [t_0, T]$, there exists a $\Gamma(t_l)$ in each subinterval such that

$$\|I - \hat{C}_i(p(t))\hat{B}_j(p(t))\Gamma(t_l)\| < 1 \; ; \; 1 \leq i \leq r_2; 1 \leq j \leq r_1 \tag{24}$$

Similar to the proof of Lemma 3, we know that (21) holds. □

We shall now propose a method to find these appropriate fixed points and subintervals.

**Algorithm 1.**

- **Initialization of $\xi$.**
  - **Step a** Given a small positive $\xi$.
  - **Step b** Let $\tilde{t} = \frac{t_0 + T}{2}$ and calculate $\Gamma(\tilde{t})$ according to (15)–(18). Check if (24) holds for all $t \in [\frac{t_0 + T - \xi}{2}, \frac{t_0 + T + \xi}{2}]$, which is carried out by checking 32 sampling points.
  - **Step c** If yes, then go to Step d. Otherwise check if $\xi < 0.01$. If yes, then exit. Otherwise, $\xi = \xi/2$ and go to Step b.
- **Search fixed points and subintervals.**
  - **Step d** Divide the interval $[t_0, T] = \cup_{i=1}^{m} [t_0, \hat{t}_i]$, where $\hat{t}_i = \hat{t}_{i-1} + \xi (1 \leq i < m)$ and $\hat{t}_m = T$.
  - **Step e** Let $t_l = \frac{\hat{t}_{l-1} + \hat{t}_l}{2}$ and $\tilde{\Omega}(t, t_l) = [\hat{t}_{l-1}, \hat{t}_l]$. Calculate $\Gamma(t_l)$ according to (15)–(18). Check if (24) holds for all $t \in \tilde{\Omega}(t, t_l)$ and all $l$. If yes, then exit. Otherwise, check if $\xi < 0.01$. If yes, then exit. Otherwise, let $\xi = \frac{\xi}{2}$ and go to Step d.

We shall now consider Step 2. Here, we present a new interpolation method. With this method, a continuously differentiable switched controller can be constructed. Note that

$$\tilde{\Omega}(t, t_l) \cap \tilde{\Omega}(t, t_{l+1}) \neq \emptyset \; ; \; l = 1, 2, \cdots, N-1 \tag{25}$$

Without loss of generality, we suppose that [7]

$$[c_l, d_l] \subseteq \tilde{\Omega}(t, t_l) \cap \tilde{\Omega}(t, t_{l+1}) \; ; \; l = 1, 2, \cdots, N-1 \tag{26}$$

Then, $\Gamma(t)$ can be constructed as in the following lemma.

**Lemma 5.** *Let $\Gamma(t)$ be given as follows:*

$$\begin{cases} \Gamma(t_1) \\ \qquad t \in \tilde{\Omega}(t, t_1) - [c_1, d_1] \\ \Gamma(t_l) + \frac{-2t^3 + 3(c_l + d_l)t^2 - 6c_l d_l t + 3c_l^2 d_l - c_l^3}{(d_l - c_l)^3} (\Gamma(t_{l+1}) - \Gamma(t_l)) \\ \qquad t \in [c_l, d_l]; l = 1, 2, \cdots, N-1 \\ \Gamma(t_l) \\ \qquad t \in \tilde{\Omega}(t, t_l) - [c_{l-1}, d_{l-1}] - [c_l, d_l]; l = 2, 3, \cdots, N-1 \\ \Gamma(t_N) \\ \qquad t \in \tilde{\Omega}(t, t_N) - [c_{N-1}, d_{N-1}] \end{cases} \tag{27}$$

*then, $\Gamma(t)$ is continuously differentiable and $J$ defined in (19) is less than 1.*

**Proof:** We shall first prove that $\Gamma(t)$ is continuously differentiable. Let

$$g(t) = \Gamma(t_l) + \frac{-2t^3 + 3(c_l + d_l)t^2 - 6c_l d_l t + 3c_l^2 d_l - c_l^3}{(d_l - c_l)^3} (\Gamma(t_{l+1}) - \Gamma(t_l))$$

It can be shown that

$$\begin{aligned} g(c_l) &= \Gamma(t_l) \\ g(d_l) &= \Gamma(t_{l+1}) \\ g'(t) &= \frac{-6t^2 + 6(c_l + d_l)t - 6c_l d_l}{(d_l - c_l)^3} (\Gamma(t_{l+1}) - \Gamma(t_l)) \\ g'(c_l) &= 0 \\ g'(d_l) &= 0 \end{aligned}$$

It follows that $\Gamma(t)$ is continuously differentiable.

We shall now prove that $J$ defined in (19) is less than 1. Note that when $t \in [c_l, d_l]$, we have

$$0 \leq \frac{-2t^3 + 3(c_l + d_l)t^2 - 6c_l d_l t + 3c_l^2 d_l - c_l^3}{(d_l - c_l)^3} \leq 1$$

Using Lemma 3, (25) and (26), we have

$$\begin{aligned} \|I - C(p(t))B(p(t))\Gamma(t_l)\| < 1 \\ \|I - C(p(t))B(p(t))\Gamma(t_{l+1})\| < 1 \end{aligned}$$

Thus

$$\|I - C(p(t))B(p(t))\Gamma(t)\| < 1 \; ; \; t \in [c_l, d_l] \tag{28}$$

From (28) and Lemma 4, the result is proved. ☐

**Remark 4.** *Note that a continuously differentiable controller can be obtained by using our interpolation method (27) while the controller obtained by the interpolation method in [7] is only continuous.* ☐

The main result of this paper can be stated as follows.

**Theorem 1.** *Consider linear parametrized varying system (1)-(4) satisfying Assumptions A1 and A2. Then, there exists a sub-optimal iterative learning control law (12) such that*

$$\begin{aligned} & \|e_{i+1}(t)\|_\lambda \\ \leq \quad & (\epsilon + \frac{\hat{c}e^{\hat{a}(T-t_0)}((\hat{a}\hat{f}+\hat{e})\hat{b}+\hat{d}\hat{f})}{\lambda})^i \|e_1(t)\|_\lambda + \\ & \frac{b_{x0}\hat{c}(\hat{c}\hat{b}\hat{f}+2)e^{\hat{a}(T-t_0)}((\epsilon + \frac{\hat{c}e^{\hat{a}(T-t_0)}((\hat{a}\hat{f}+\hat{e})\hat{b}+\hat{d}\hat{f})}{\lambda})^i - 1)}{((\epsilon + \frac{\hat{c}e^{\hat{a}(T-t_0)}((\hat{a}\hat{f}+\hat{e})\hat{b}+\hat{d}\hat{f})}{\lambda}) - 1)} \end{aligned} \tag{29}$$

*where*

$$\begin{aligned} \epsilon &\triangleq \max_{t\in[t_0,T]} \|I - C(P(t))B(P(t))\Gamma(t)\| \\ \hat{a} &\triangleq \max_{1\leq i\leq r_3} \max_{t\in[t_0,T]} \{\|\hat{A}_i(p(t))\|\} \\ \hat{b} &\triangleq \max_{1\leq i\leq r_1} \max_{t\in[t_0,T]} \{\|\hat{B}_i(p(t))\|\} \\ \hat{c} &\triangleq \max_{1\leq i\leq r_2} \max_{t\in[t_0,T]} \{\|\hat{C}_i(p(t))\|\} \\ \hat{d} &\triangleq \max_{1\leq i\leq r_1} \max_{t\in[t_0,T]} \{\|\frac{d\hat{B}_i(p(t))}{dp}\frac{dp(t)}{dt}\|\} \\ \hat{e} &\triangleq \max_{t\in[t_0,T]} \|\frac{d\Gamma(t)}{dt}\| \; ; \; \hat{f} \triangleq \max_{t\in[t_0,T]} \|\Gamma(t)\| \end{aligned}$$

*Furthermore, $Y_i(t) \rightarrow Y_d(t)$ when $i \rightarrow \infty$ and $b_{x0} \rightarrow 0$.*

**Proof:** From Lemmas 2–5, we know that $\epsilon < 1$.

From (1), we have

$$\begin{aligned} & e_{i+1}(t) \\ = \; & Y_d(t) - Y_{i+1}(t) \\ = \; & (I - C(p(t))B(p(t))\Gamma(t))e_i(t) \\ & +C(p(t))\phi(t,t_0)B(p(t_0))\Gamma(t_0)e_i(t_0) \\ & -C(p(t))\phi(t,t_0)(X_{i+1}(t_0) - X_i(t_0)) \\ & +C(p(t))\int_{t_0}^{t} \frac{d\phi(t,\tau)B(p(\tau))\Gamma(\tau)}{d\tau} e_i(\tau)d\tau \end{aligned} \tag{30}$$

Taking the norm of equation (30), we have

$$\begin{aligned} e_{i+1}(t) \quad \leq \quad & \epsilon\|e_i(t)\| + \|C(p(t))\phi(t,t_0)B(p(t_0))\Gamma(t_0)\|\|e_i(t_0)\| \\ & +\|C(p(t))\phi(t,t_0)\|\|(X_{i+1}(t_0) - X_i(t_0))\| \\ & +\int_{t_0}^{t} \|C(p(t))\frac{d\phi(t,\tau)B(p(\tau))\Gamma(\tau)}{d\tau}\|\|e_i(\tau)\|d\tau \end{aligned} \tag{31}$$

Note that

$$\begin{aligned} & \frac{d\phi(t,\tau)B(p(\tau))\Gamma(\tau)}{d\tau} \\ = \; & -A(p(\tau))\phi(t,\tau)B(p(\tau))\Gamma(\tau) \\ & +\phi(t,\tau)\frac{dB(p(\tau))}{dp}\frac{dp(\tau)}{d\tau}\Gamma(t) - \phi(t,\tau)B(p(\tau))\frac{d\Gamma(\tau)}{d\tau} \end{aligned}$$

and

$$\begin{aligned} \|A(p(t))\| \quad &\leq \quad \hat{a} \; ; \; \|B(p(t))\| \leq \hat{b} \; ; \; \|C(p(t))\| \leq \hat{c} \\ \|\frac{dB(p(t))}{dp}\frac{dp(t)}{dt}\| \quad &\leq \quad \hat{d} \; ; \; \|\frac{d\Gamma(t)}{dt}\| \leq \hat{e} \; ; \; \|\Gamma(t)\| \leq \hat{f} \end{aligned}$$

Using Lemma 1, we have

$$\|\phi(t,t_0)\| \leq e^{\hat{a}(t-t_0)} \leq e^{\hat{a}(T-t_0)}$$

Thus, we have

$$\begin{aligned} & \|e_{i+1}(t)\| \\ \leq \quad & \epsilon\|e_i(t)\| + \hat{c}(\hat{c}\hat{b}\hat{f}+2)e^{\hat{a}(T-t_0)}b_{x0} \\ & +\int_{t_0}^{t} \hat{c}e^{\hat{a}(T-t_0)}((\hat{a}\hat{f}+\hat{e})\hat{b}+\hat{d}\hat{f})\|e_i(\tau)\|d\tau \end{aligned} \tag{32}$$

Multiplying $e^{-\lambda t}$ on the both side of (32), we have

$$\begin{aligned} \|e_{i+1}(t)\|_\lambda \quad \leq \quad & (\epsilon + \frac{\hat{c}e^{\hat{a}(T-t_0)}((\hat{a}\hat{f}+\hat{e})\hat{b}+\hat{d}\hat{f})}{\lambda})\|e_i(t)\|_\lambda \\ & +\hat{c}(\hat{c}\hat{b}\hat{f}+2)e^{\hat{a}(T-t_0)}b_{x0} \end{aligned}$$

It follows that (29) holds.

Clearly, there exists an $\lambda^*$ such that

$$\epsilon + \frac{\hat{c}e^{\hat{a}(T-t_0)}((\hat{a}+1)\hat{b}\hat{e}+\hat{d}\hat{e})}{\lambda^*} < 1 \tag{33}$$

Then, we know that when $i \rightarrow \infty$ and $b_{x0} \rightarrow 0$, we have $Y_i(t) \rightarrow Y_d(t)$. ☐

**Remark 5.** *A necessary condition for the existence of (15) is that $\hat{C}_i(p(t))\hat{B}_j(p(t))$ is full row rank for all $1 \leq i \leq r_2$, $1 \leq j \leq r_1$ and all $p(t)$.* ☐

**Remark 6.** *The checking of Assumption A2 and the design can be proceeded simultaneously as follows.*

*Check if $\hat{C}_i(p(t))\hat{B}_j(p(t))$ is full row rank for all $1 \leq i \leq r_2$, $1 \leq j \leq r_1$ and all $p(t)$. If yes, then run Algorithm 1 to obtain the answer.* ☐

## 4 An Illustrative Example

To illustrate the theoretical result, we consider the following two-input two-output linear time varying system

$$\begin{aligned} \hat{A}_1(p(t)) &= \begin{bmatrix} -0.5-0.3t & 0 \\ 0 & -1.2-0.2t \end{bmatrix} \\ \hat{A}_2(p(t)) &= \begin{bmatrix} -1.5-0.7t & 0 \\ 0 & -0.8-0.6t \end{bmatrix} \\ \hat{B}_1(p(t)) &= \begin{bmatrix} 16/3-8/3t & -4/3-2/3t \\ -3-1.5t & 1-0.5t \end{bmatrix} \\ \hat{B}_2(p(t)) &= \begin{bmatrix} 1.926+1.5408t & -0.5926-0.47408t \\ -1.1111-0.88888t & 0.4444+0.35552t \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\hat{C}_1(p(t)) &= \begin{bmatrix} 1.5+0.6t & 2+0.8t \\ 3+1.2t & 6+2.4t \end{bmatrix} \\
\hat{C}_2(p(t)) &= \begin{bmatrix} 2.25+0.9t & 3+1.2t \\ 4.5+1.8t & 9+3.6t \end{bmatrix} \\
Y_d(t) &= \begin{bmatrix} Y_{d,1}(t) \\ Y_{d,2}(t) \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}
\end{aligned}$$

It can be know that $N = 5$ and $\xi = 0.2$ by using Lemmas 3 and 4 and Algorithm 1, and

$$\begin{aligned}
\Gamma(0.1) &= \begin{bmatrix} 0.4914 & 0 \\ 0.4914 & 0.4436 \end{bmatrix}; \ \Gamma(0.3) = \begin{bmatrix} 0.3924 & 0 \\ 0.3924 & 0.3542 \end{bmatrix} \\
\Gamma(0.5) &= \begin{bmatrix} 0.3117 & 0 \\ 0.3117 & 0.2814 \end{bmatrix}; \ \Gamma(0.7) = \begin{bmatrix} 0.249 & 0 \\ 0.249 & 0.2248 \end{bmatrix} \\
\Gamma(0.85) &= \begin{bmatrix} 0.212 & 0 \\ 0.212 & 0.1914 \end{bmatrix} \\
c_1 &= 0.16, d_1 = 0.24, c_2 = 0.36, d_2 = 0.44 \\
c_3 &= 0.56, d_3 = 0.64, c_4 = 0.76, d_4 = 0.84
\end{aligned}$$

Thus, from Lemma 5, we can obtain the learning gain as follows.

$$\Gamma(t) = \begin{cases}
\Gamma(0.1) \\
\qquad t \in [0 \quad 0.16]; \\
\Gamma(0.1) + \frac{-2t^3+3(c_1+d_1)t^2-6c_1d_1t+3c_1^2d_1-c_1^3}{(d_1-c_1)^3}(\Gamma(0.3)-\Gamma(0.1)) \\
\qquad t \in [0.16 \quad 0.24]; \\
\Gamma(0.3) \\
\qquad t \in [0.24 \quad 0.36]; \\
\Gamma(0.3) + \frac{-2t^3+3(c_2+d_2)t^2-6c_2d_2t+3c_2^2d_2-c_2^3}{(d_2-c_2)^3}(\Gamma(0.5)-\Gamma(0.3)) \\
\qquad t \in [0.36 \quad 0.44]; \\
\Gamma(0.5) \\
\qquad t \in [0.44 \quad 0.56]; \\
\Gamma(0.5) + \frac{-2t^3+3(c_3+d_3)t^2-6c_3d_3t+3c_3^2d_3-c_3^3}{(d_3-c_3)^3}(\Gamma(0.7)-\Gamma(0.5)) \\
\qquad t \in [0.56 \quad 0.64]; \\
\Gamma(0.7) \\
\qquad t \in [0.64 \quad 0.76]; \\
\Gamma(0.7) + \frac{-2t^3+3(c_4+d_4)t^2-6c_4d_4t+3c_4^2d_4-c_4^3}{(d_4-c_4)^3}(\Gamma(0.85)-\Gamma(0.7)) \\
\qquad t \in [0.76 \quad 0.84]; \\
\Gamma(0.85) \\
\qquad t \in [0.84 \quad 1];
\end{cases} \tag{34}$$

It can be shown that $J$ defined in (19) is less than 1. Using Theorem 1, we know that the iterative learning control laws (12) and (34) ensure that $Y_i(t) \to Y_d(t)$ when $i \to \infty$ and $b_{x0} \to 0$.

## 5 Conclusion

In this note, we have proposed a new numerical approach for the design of iterative learning control of repetitive linear parameterized varying systems with polytope uncertainties. We formulated our approach as a group of convex optimization problems and this makes the problem computationally tractable by using some existing tools. An interpolation method was also presented to construct a continuously differentiable switched controller from the solutions of these convex optimization problems. Some sufficient conditions are derived to ensure the convergence of the learning system. Our approach can also be used to consider the iterative learning control of linear time varying systems and nonlinear systems with polytope uncertainties.

## References

[1] S. Arimoto, S. Kawamura, and F. Miyazaki. Bettering operation of robots by learning. *Journal of Robotic Systems.*, 1:123–140, 1984.

[2] H. K. Lee and Z. Bien. Study on robustness of iterative learning control with non-zero initial error. *International Journal of Control*, 64:345–359, 1996.

[3] Y. Chen, C. Wen, Z. Gong, and M. Sun. An iterative learning controller with initial state learning. *IEEE Transactions on Automatic Control*, 44:371–376, 1998.

[4] Y. Fang and T. W. S. Chow. Iterative learning control of linear discrete–time multivariable systems. *Automatica*, 34:1459–1462, 1998.

[5] S. Boyd, L.EI Ghoui, E. Feron, and V. Balakrishaan. *Linear Matrix Inequalities in system and Control Theory.* SIAM, Boston, 1994.

[6] Z. G. Li, C. Y. Wen, Y. C. Soh, and Y. Chen. Optimal iterative learning control of linear uncertain systems. *IEE Proceedings-D, Control Theory and Applications*, 2000. Submitted.

[7] D. J. Stilwel and W. J. Rugh. Interpolation of observer state feedback controllers for gain scheduling. *IEEE Transactions on Automatic Control*, 44:1225–1229, 1999.