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Singularity-free Neural Network Controller with Iterative Training[1]

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***Abstract*—A repetitive control scheme for trajectory tracking of a discrete nonlinear system is presented in this paper, where neural networks are used to approximate the unknown but repeatable nonlinearities. Contrary to the on-line adaptive training of neural networks, the neural networks are trained by tracking a trajectory multiple times so that the tracking performances of the whole trajectory can be improved through repetition. In order to avoid the singularity problem caused by the inverse of approximation of the coupling matrix, this paper modifies the neural network approximations of the coupling matrix and this modification does not cause control instability.**

***Index Terms*--Neural networks, iterative learning control, nonlinear control**

I. INTRODUCTION

Neural network (NN) control was widely investigated in the last decade. Usually, neural networks are used to approximate the uncertainties with only unknown linear weights [1][2] where linear adaptive technologies are used to update the weights on-line and a robust control scheme is used to deal with the residual modeling error. In these approaches, the desired trajectory is supposed to be smooth and the convergence can only be achieved when the time t goes into infinity. For a non-periodic trajectory with a finite time interval, which is widely seen in practical applications such as the robot pick-and-place system, the off-line repetitive training can be considered as an alternative to the on-line adaptive training scheme. The Iterative Learning Control (ILC) [3,8,9] is a special method for such kind of repetitive training.

In the conventional ILC, the learning operator is designed based on the discrete Lyapunov method and the control input u is directly updated in an affine fashion such as P type or D type learning schemes. With the λ-norm technique, a classical stability condition is that the learning gain $L(t)$ should satisfy:

$$\|I - C(t)B(t)L(t)\| < 1$$

where I is the identity matrix, $C(t)$ and $B(t)$ are the output and input coupling matrix of the system, respectively.

For an affine nonlinear system with a linear output equation, it is obvious that the matrix $C(t)B(t)$ is simply the system Jacobian matrix between input and output. Therefore, the above inequality implies that the learning gain $L(t)$ should be an approximation of the inverse of the Jacobian matrix and the relative error of this approximation should be less than 100%. How to obtain this gain is not as simple as this inequality appears because at least the bounds of all entries of the Jacobian matrix along the trajectory have to be known. Unfortunately, sometimes this kind of *a priori* knowledge is not available. This practical difficulty has not been taken serious attention in the field of ILC. One of possible ways to obtain this learning gain $L(t)$ is to estimate the Jacobian matrix along the trajectory then inverse it. However, the singularity problem may appear if the determinant of the estimation is near to zero.

In this paper, an iterative training scheme for the NN approximation is proposed. In our scheme, a series of local NN's are distributed along the desired trajectory to approximate the repeatable uncertainties in a discrete-time nonlinear system and the NN weights are identified by a robust least-square algorithm along the iteration axis by tracking repetitively. With the proposed iterative training scheme for NN approximation, this paper also investigates how to avoid the singularity problem caused by the inversion of the NN approximation of the unknown coupling matrix for control law calculation. The correction vector approach in adaptive control [4][5] is introduced into the iterative training NN control, which modifies the least-square estimate of the coupling matrix by the covariance matrix, and a new weight modification scheme is presented for NN control applications.

II. PROBLEM FORMULATION

Consider the discrete-time representation of a nonlinear sampled-data control system

$$\Delta x\big((k+1)\tau\big)/\tau = f(k\tau, x(k\tau)) + G(k\tau, x(k\tau))u(k\tau) \qquad (1)$$

with state $x(k\tau)\in R^n$, control $u(k\tau)\in R^n$, and unknown smooth nonlinear functions $f(k\tau,x(k\tau))\in R^n$, and $G(k\tau,x(k\tau))\in R^{n\times n}$, where the difference $\Delta x((k+1)\tau)=x((k+1)\tau)-x(k\tau)$, and τ is the sampling period.

Suppose that the system (1) is invertible around a finite time desired trajectory $x_d(k)$, $k=0,\ldots,N$, i.e., $G(k,x(k))$ is invertible nearby. The control goal is to force the state vector $x(k)$ to follow the desired trajectory $x_d(k)$ as exactly as possible. One of the possible ways to improve the tracking performance is to track the desired trajectory repeatedly so as to obtain knowledge about the unknown system iteratively or gradually. Let the repetitive tracking series be described as

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$\Delta x(k+1,i)/\tau = f(k,x(k,i)) + G(k,x(k,i))u(k,i)$, k=0,…,N-1 (2)

where the additional index i expresses the i^{th} iterative tracking and the τ's are omitted from the equation for simplification.

If the nonlinear functions $f(k,x)$ and $G(k,x)$ are known, for feedback linearization, the control law $u(k,i)$ should be:

$$u(k,i) = G(k,x)^{-1}\left(\left(x_d(k+1) - x(k)\right)/\tau - f(k,x) + Ke(k,i)\right)$$

where τK should be stable and the control error is defined as $e(k,i)=x_d(k)-x(k,i)$.

The resulting error dynamics of the original system can behave as a linear system along the time axis:

$$e(k+1,i)+ \tau K\, e(k,i)=0$$

which implies that, for the i^{th} tracking, $e(k,i)\to 0$ as $t=k\tau\to\infty$.

Modifying the linear feedback term $Ke(k,i)$ in the above control law with a one-step-ahead error of the last tracking as $Ke(k+1,i-1)$, we obtain a possible ILC scheme as:

$$u(k,i) = G(k,x)^{-1}\left(\left(x_d(k+1)-x(k)\right)/\tau - f(k,x) + Ke(k+1,i-1)\right) \quad (3)$$

The equivalent error equation becomes

$$e(k+1,i)+ \tau K\, e(k+1,i-1)=0,\ k=0,\ldots,N-1.$$

Clearly, the above error equation is a linear discrete system along the repetitive axis, and the control error of every point along the trajectory, $e(k,i)$, k=1,…,N (except the initial state k=0), will converge to 0 as the number of repetition goes to infinity, i.e., $i\to\infty$.

However, in this paper, we do *not* have any *a priori* knowledge about $f(k,x)$ and $G(k,x)$ except the invertibility assumption about the system along the desired trajectory. Here, we propose to apply a linear parametric neural network to approximate the nonlinear functions $f(k,x)$ and $G(k,x)$ as follows:

$$f(k,x) = W_f^T(k)\varphi(k,x) + \varepsilon_1$$

$$G(k,x) = \begin{bmatrix} W_{11}^T(k)\varphi(k,x)\cdots W_{1n}^T(k)\varphi(k,x) \\ \vdots \\ W_{n1}^T(k)\varphi(k,x)\cdots W_{nn}^T(k)\varphi(k,x)\end{bmatrix} + \varepsilon_2 \quad (4)$$

$$= W_g^T(k)\Phi(k,x) + \varepsilon_2$$

where $\varphi(k,x)\in R^L$ is a vector of basis functions, and

$$\Phi(k,x) = \begin{bmatrix} \varphi(k,x) & 0 & 0\cdots & 0 \\ 0 & \varphi(k,x) & 0\cdots & 0 \\ & \vdots & & \\ 0 & \cdots & 0 & \varphi(k,x)\end{bmatrix} \in R^{(n\times L)\times n}$$

The $W_f^T(k)\in R^{n\times L}$ and $W_g^T(k)\in R^{n\times(nL)}$ are the unknown optimal weights of the neural networks. The ε_1 and ε_2 are the modeling errors of the NN approximation bounded on a compact region Ω around the desired trajectory. The norms of $f(k,x)$ and $G(k,x)$ should be bounded on the region Ω as well. We make the following assumption on the NN approximation accuracy:

A1) *η and μ are known upper bounds of the modeling errors such that* $\|\varepsilon_1\|\le\eta$ and $\|\varepsilon_2\|\le\mu$ *on the region Ω.*

and an assumption about the invertibility of the NN approximation model:

A2) *The coupling matrix of the NN approximation with optimum weights is nonsingular on the region Ω, i.e.*

$$\left|\det\left(W_g^T(k)\Phi(k,x)\right)\right| \ge b > 0$$

The selection of the basis functions depends on what kind of neural networks we wish to use. The following are some typical basis functions:

1) $\varphi_i(k,x) = e^{-\|x-p_i\|^2}/\sigma_i^2$ for a Gaussian RBF neuron, where p_i is the center and the σ_i is the width of the i^{th} neuron [1];

2) $\varphi_i(k,x) = \prod_{j\in I_i} x_j^{d_j(i)}$ for a high-order neuron, where $d_j(i)$ are nonnegative integers [6];

3) $\varphi_i(k,x) = \dfrac{\prod_{j=1}^{n}\mu_{Aj}^i(x_j)}{\sum_{i=1}^{L}\left(\prod_{j=1}^{n}\mu_{Aj}^i(x_j)\right)}$ for a fuzzy neuron, where μ_{Aj}^i is the membership function of a fuzzy set [2].

Note that the proposed neural networks are different from the adaptive neural networks [1][2], where it is assumed that the optimal weights are unknown constants. In equation (4), the optimal weights $W_f(k)$ and $W_g(k)$ can be time-varying. This means that, for every sampling instant $k\tau$ or, more exactly, every point along the desired trajectory $x_d(k)$, k=1,…,N, one can have different optimal weights. So, instead of a unified NN for the whole workspace, equation (4) describes a series of local NN for every point along the desired trajectory. Because every local NN is only concerned with the uncertainties in a neighborhood around a particular point of the desired trajectory, the basis functions can be selected in a very simple form. Then, our task is to train these local and simple NN's having time-varying weights along the iterative axis i instead of the time axis k. More specifically, for different tracking iterations i at the same sampling instant $k\tau$, the optimum weights will keep constant. This is called iterative training NN control in this paper. It can be used to control a system with time-varying but repetitive uncertainties.

III. ITERATIVE IDENTIFICATION OF NEURAL NETWORKS

Substituting (4) into (2), we can get the following linear relation for the i^{th} tracking:

$$\Delta x(k+1,i)/\tau = W^T(k)Y(k,x(k,i)) + v(k,i),\ k=0,\ldots,N-1 \quad (5)$$

where $W^T(k) = \left[W_f^T(k) \quad W_g^T(k)\right]$,

$Y^T(k,x) = \left[\varphi^T(x) \quad u^T(k,i)\Phi^T(k,x)\right]$; $v(k,i) = \varepsilon_1 + \varepsilon_2 u(k,i)$

From this equation, we know that the optimum weights of each local NN at the sample time $k\tau$ are independent of each other and keep invariant for different tracking repetitions. Therefore, we shall identify the optimum weights of every local NN, $k=0,\ldots,N$-1, through repetitive tracking. Obviously, since the system (5) is a linear system with disturbance $v(k,i)$, the least-square algorithm with dead zone [4] can be used to estimate the optimum weights off-line after each tracking repetition. It means that all of the weights are updated together in batch manner after each tracking iteration. This is different to the case in adaptive control where the updating is immediately after each sampling period.

The normalized equation of the system (5) can be written as

$$\Delta\bar{x}(k+1,i)/\tau = W^T(k)\bar{Y}(k,x(k,i)) + \bar{v}(k,i)\ ,\ \ k=0,\ldots,N\text{-}1 \tag{6}$$

with the following normalized variables

$$\Delta\bar{x}(k+1,i) = \Delta x(k+1,i)/\left(1+\left\|Y(k,x)\right\|\right);$$
$$\bar{Y}(k,x) = Y(k,x)/\left(1+\left\|Y(k,x)\right\|\right);\ \bar{v}(k,i) = v(k,i)/\left(1+\left\|Y(k,x)\right\|\right)$$

From A1), the upper bound of $\left\|\bar{v}(k,i)\right\|$ can be expressed as:

$$\begin{aligned}\left\|\bar{v}(k,i)\right\| &\le \left(\eta+\mu\left\|u(k,i)\right\|\right)/\left(1+\left\|Y(k,x)\right\|\right)\\ &\le \eta/\left(1+\left\|Y(k,x)\right\|\right)+\mu\left\|u(k,i)\right\|/\left(1+\left\|\Phi(k,x)u(k,i)\right\|\right)\\ &\le \eta/\left(1+\left\|Y(k,x)\right\|\right)+\\ &\quad \mu\left\|u(k,i)\right\|/\left(1+\sqrt{\lambda_{\min}(\Phi^T(k,x)\Phi(k,x))}\left\|u(k,i)\right\|\right)\end{aligned}$$

According to the invertibility assumption in A2), we can suppose $\lambda_{\min}^{1/2}\left(\Phi^T(k,x)W_g(k)W_g^T(k)\Phi(k,x)\right)\ge b_1>0$, then

$$\lambda_{\max}\left(W_g(k)W_g^T(k)\right)\lambda_{\min}\left(\Phi^T(k,x)\Phi(k,x)\right)\ge$$
$$\lambda_{\min}\left(\Phi^T(k,x)W_g(k)W_g^T(k)\Phi(k,x)\right)\ge b_1>0$$

Because the norm of $W_g(k)$ is upper bounded, we get

$$\lambda_{\min}^{1/2}\left(\Phi^T(k,x)\Phi(k,x)\right)\ge b_1/\left\|W_g(k)\right\|>0$$

The upper bound of $\left\|\bar{v}(k,i)\right\|$ can be further expressed as:

$$\begin{aligned}\left\|\bar{v}(k,i)\right\| &\le \eta/\left(1+\left\|Y(k,x)\right\|\right)+\mu\left\|u(k,i)\right\|/\left(1+b_1/\left\|W_g(k)\right\|\left\|u(k,i)\right\|\right)\\ &\le \eta/\left(1+\left\|Y(k,x)\right\|\right)+\mu' = \delta(k,i)\end{aligned} \tag{7}$$

where the bounded constant $\mu' = \mu\left\|W_g\right\|_{\max}/b_1$ which reflects the modeling error of the NN approximation.

Define a normalized prediction error

$$E(k,i) = \Delta\bar{x}(k+1,i)/\tau - \hat{W}^T(k,i-1)\bar{Y}(k,x(k,i)) \tag{8}$$

and an augmented error term

$$w(k,i) = \left[E^T(k,i)E(k,i)+\bar{Y}^T(k,x(k,i))F^2(k,i-1)\bar{Y}(k,x(k,i))\right]^{1/2} \tag{9}$$

after the i^{th} tracking.

Then the dead zone is defined as

$$\lambda(k,i) = \begin{cases}0 & \text{if}\ \ w^2(k,i)\le\bar{\delta}^2(k,i)\\ 1 & \text{otherwise}\end{cases}$$

where

$$\bar{\delta}^2(k,i) = \alpha\left(\delta^2(k,i)+\varepsilon\delta(k,t)\right),\ \alpha = 1+tr\left(F(k,0)\right),\ \varepsilon>0.$$

Now, the least-square algorithm with dead zone [4] can be used for the weights updating by iterations:

$$F(k,i) = F(k,i-1) - \frac{\lambda(k,i)F(k,i-1)\bar{Y}(k,i)\bar{Y}^T(k,i)F(k,i-1)}{1+\lambda(k,i)\bar{Y}^T(k,i)F(k,i-1)\bar{Y}(k,i)}$$
$$\hat{W}(k,i) = \hat{W}(k,i-1)+\lambda(k,i)F(k,i)\bar{Y}(k,i)E^T(k,i)$$
$$k=0,\ldots,N\text{-}1 \tag{10}$$

with the following properties:

P1) $F(k,i)$ and $\hat{W}(k,i)$ converge;

P2) $\lim\sup_{i\to\infty}\left(w^2(k,i)-\bar{\delta}^2(k,i)\right)\le 0,\quad \forall k=0\cdots N-1$;

P3) The optimum weight can be expressed as $W(k,i)=\hat{W}(k,i)+F(k,i)\beta^*(k,i)$ and $\left\|\beta^*(k,i)\right\|\le h_0(k)$ is upper bounded, where

$$h_0(k) = \left\|F_0^{-1}\tilde{W}(k,0)\right\|$$
$$+\left(\Sigma_{m=1}^n\tilde{W}_m^T(k,0)F^{-1}(k,0)\tilde{W}_m(k,0)+tr(F(k,0))\right)/\varepsilon$$

and the error of the weights $\tilde{W}(k,i)=W(k,i)-\hat{W}(k,i)$, and $\tilde{W}_m(k,i)$ is the m^{th} column vector of $\tilde{W}(k,i)$.

The proof of the above properties can be given following the steps in [4].

Up to now, we have presented the identification law for NN weights. If we directly introduce the identification weights of (10) instead of optimum weights into control law (3), it becomes:

$$u(k,i) = \left(\hat{W}_g^T(k,i-1)\Phi(k,x)\right)^{-1}$$
$$\left(\left(x_d(k+1)-x(k,i)\right)/\tau-\hat{W}_f^T(k,i-1)\varphi(k,x)+Ke(k+1,i-1)\right) \tag{11}$$

In this control law, we use the $(i\text{-}1)^{th}$ identification weights for the i^{th} tracking.

However, this control law may face the singularity problem during the tracking because we can not ensure that the NN approximation of the coupling matrix is invertible in equation (11).

IV. WEIGHTS MODIFICATION FOR SINGULARITY AVOIDANCE

In the last section, P3) points out that the optimum weights can be expressed as the estimate one plus a proper modification of the covariance matrix. Considering the weights of the coupling matrix in $W(k, i)$, the last (nL) rows can be written as:

$$W_g(k,i) = \hat{W}_g(k,i)+F_{nL}(k,i)\beta^*(k,i) \tag{12}$$

where $F_{nL}(k,i)$ is a matrix comprised of the last (nL) rows of the matrix $F(k,i)$.

Then, the optimum NN approximation of the coupling matrix is:

$$W_g^T(k,i)\Phi(k,x) = \hat{W}_g^T(k,i)\Phi(k,x)+\beta^{*T}(k,i)F_{nL}^T(k,i)\Phi(k,x)$$

(13)

Due to A2), the coupling matrix described by equation (13) is invertible. Imitating this form, we expect to find out an appropriate matrix $\beta(k,i)$ such that the estimate coupling matrix can be modified as

$$\overline{W}_g^T(k,i)\Phi(k,x) = \hat{W}_g^T(k,i)\Phi(k,x) + \beta^T(k,i)F_{nL}^T(k,i)\Phi(k,x) \tag{14}$$

and it is invertible as well.

Suppose that the Singular Value Decomposition (SVD) of the estimate of the coupling matrix is

$$\hat{W}_g^T(k,i)\Phi(k,x) = U_1 S_1 V_1^T \tag{15}$$

where the matrices $U_1\in R^{n\times n}$ and $V_1\in R^{n\times n}$ are orthogonal, the $S_1\in R^{n\times n}$ is a real diagonal matrix $S_1=diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq\sigma_2 \geq \ldots\geq \sigma_n\geq 0$ are the singular values of $\hat{W}_g^T(k,i)\Phi(k,x)$. Substituting it into (14) yields:

$$\overline{W}_g^T(k,i)\Phi(k,x) = U_1\left(S_1 + U_1^T\beta^T(k,i)F_{nL}^T(k,i)\Phi(k,x)V_1\right)V_1^T \tag{16}$$

Then, we define a threshold $T>0$ to evaluate the singularity and then let every singular value of the estimate of coupling matrix compare with it. Suppose the singular values from σ_1 to σ_p are greater than the threshold T but the others are less than it. It means that the singular values from σ_1 to σ_p are higher enough even without any modification but σ_{p+1} … σ_n have to be modified. Therefore, at this stage, we first construct the coefficient matrix $\beta(k,i)$ in a form of:

$$\beta(k,i) = c\begin{bmatrix} \overbrace{0}^{\text{p}} & \overbrace{\lambda}^{\text{n-p}} \end{bmatrix} U_1^T \in R^{(n+1)L\times n} \tag{17}$$

where $0<c<1$ is a scalar used to control the intensity of the modification.

Substituting it into equation (16) gives:

$$\begin{aligned}
\overline{W}_g^T(k,i)\Phi(k,x) &= U_1\left(S_1 + \begin{bmatrix} 0 \\ c\lambda^T \end{bmatrix} F_{nL}^T(k,i)\Phi(k,x)V_1\right)V_1^T \\
&= U_1\Bigg(\mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_p, \sigma_{p+1}, \cdots, \sigma_n) + \\
&\qquad \begin{bmatrix} 0 \\ c\lambda^T F_{nL}^T(k,i)\Phi(k,x)V_1 \end{bmatrix}\Bigg)V_1^T \\
&= U_1\begin{bmatrix} diag(\sigma_1, \cdots, \sigma_p) & 0 \\ c\lambda^T F_{nL}^T(k,i)\Phi(k,x)V_{1p} & diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + c\lambda^T F_{nL}^T(k,i)\Phi(k,x)V_{1n} \end{bmatrix} V_1^T
\end{aligned}$$

where $V_{1p}\in R^{n\times p}$ and $V_{1n}\in R^{n\times(n-p)}$ are composed of the $1\ldots p$ and $p+1\ldots n$ column vectors of matrix V_1, respectively.

Now we are going to verify the singularity of this modified coupling matrix by its determinant:

$$\begin{aligned}
\left|\det \overline{W}_g^T(k,i)\Phi(k,x)\right| &= \sigma_1, \cdots, \sigma_p \left|\det U_1\right| \\
&\quad \left|\det\left(diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + c\lambda^T F_{nL}^T(k,i)\Phi(k,x)V_{1n}\right)\right|\left|\det V_1^T\right|
\end{aligned}$$

Because U_1 and V_1 are orthogonal matrices, both absolute values of determinants are equal to 1, thus

$$\begin{aligned}
&\left|\det \overline{W}_g^T(k,i)\Phi(k,x)\right| = \\
&\sigma_1, \cdots, \sigma_p\left|\det\left(diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + c\lambda^T F_{nL}^T(k,i)\Phi(k,x)V_{1n}\right)\right| \\
&\geq T^p\left|\det\left(diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + c\lambda^T F_{nL}^T(k,i)\Phi(k,x)V_{1n}\right)\right|
\end{aligned} \tag{18}$$

Equation (18) points out that the singularity of the modified coupling matrix depends on the singularity of the matrix $B = diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + c\lambda^T F_{nL}^T(k,i)\Phi(k,x)V_{1n}$, where, comparing with the threshold T, the singular values σ_{p+1} … σ_n are near 0. In order to avoid its singularity, we need further construct the coefficient matrix λ. At first, we decompose the matrix $F_{nL}^T(k,i)\Phi(k,x)V_{1n}$ by SVD:

$$F_{nL}^T(k,i)\Phi(k,x)V_{1n} = U_2 S_2 V_2^T \tag{19}$$

where the matrices $U_2\in R^{(n+1)L\times(n+1)L}$ and $V_2\in R^{(n-p)\times(n-p)}$ are orthogonal matrices, the $S_2\in R^{(n+1)L\times(n-p)}$ is a real pseudo-diagonal matrix, i.e.,

$$S_2 = \begin{bmatrix} diag(\sigma'_1, \cdots, \sigma'_{n-p}) \\ 0 \end{bmatrix}$$

where $\sigma'_1 \geq\sigma'_2 \geq \ldots\geq \sigma'_{n-p}\geq 0$ are the singular values of the matrix $F_{nL}^T(k,i)\Phi(k,x)V_{1n}$.

Thus

$$B = diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + c\lambda^T U_2\begin{bmatrix} diag(\sigma'_1, \cdots, \sigma'_{n-p}) \\ 0 \end{bmatrix} V_2^T$$

Now, we propose the coefficient matrix $\lambda\in R^{(n+1)L\times(n-p)}$ as:

$$\lambda = U_2\begin{bmatrix} V_2^T \\ 0 \end{bmatrix} \tag{20}$$

Then $B = diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + cV_2\, diag(\sigma'_1, \cdots, \sigma'_{n-p})V_2^T$ becomes a symmetric nonnegative definite matrix. Substituting it into (18), we get

$$\begin{aligned}
\left|\det \overline{W}_g^T(k,i)\Phi(k,x)\right| &\geq T^p \det\begin{pmatrix} diag(\sigma_{p+1}, \cdots, \sigma_{p+n}) + \\ cV_2 diag(\sigma'_1, \cdots, \sigma'_{n-p})V_2^T \end{pmatrix} \\
&\geq T^p \lambda_{\min}^{n-p}(B) \\
&\geq T^p \lambda_{\min}^{n-p}(cV_2 diag(\sigma'_1, \cdots, \sigma'_{n-p})V_2^T) \\
&= cT^p \sigma'^{\,n-p}_{n-p}
\end{aligned} \tag{21}$$

Therefore, the minimum eigenvalue σ'_{n-p} determines the singularity of the modified coupling matrix. Moreover, we shall derive that σ'_{n-p} is bounded away from 0.

Now let us take account of the optimum NN approximation of the coupling matrix in (13) and substitute (15) into it:

$$\begin{aligned}
\det\left(W_g^T(k,i)\Phi(k,x)\right) &= \det\left(U_1 S_1 V_1^T + \beta^{*T}(k,i)F_{nL}^T(k,i)\Phi(k,x)\right) \\
&= \det\left(S_1 + U_1^T\beta^{*T}(k,i)F_{nL}^T(k,i)\Phi(k,x)V_1\right)
\end{aligned}$$

Partitioning the matrices S_1 and V_1 into two column blocks from column p as treated before, we have

$\det\left(W_g^T(k,i)\Phi(k,x)\right)=\det\left(\left[\begin{bmatrix}\overbrace{B_1}^{p} & \overbrace{B_2}^{n-p}\end{bmatrix}\right]\right)$, where

$$B_1=\begin{bmatrix}diag(\sigma_1,\cdots,\sigma_p)\\0\end{bmatrix}+U_1^T\beta^{*T}(k,i)F_{nL}^T(k,i)\Phi(k,x)V_{1p};$$

$$B_2=\begin{bmatrix}0\\diag(\sigma_{p+1},\cdots,\sigma_n)\end{bmatrix}+U_1^T\beta^{*T}(k,i)F_{nL}^T(k,i)\Phi(k,x)V_{1n}$$

According to the invertible assumption in A2), we know that $|\det([B_1,B_2])|\geq b$. As a result, any column vectors in B_2 are linearly independent and we can further suppose that $\lambda_{\min}\left(B_2^TB_2\right)\geq b_2>0$, namely

$$b_2\leq\lambda_{\min}\left(B_2^TB_2\right)$$

$$\leq 2\lambda_{\min}\begin{pmatrix}diag\left(\sigma_{p+1}^2,\cdots,\sigma_n^2\right)+V_{1n}^T\Phi^T(k,x)F_{nL}(k,i)\cdot\\ \beta^*(k,i)\beta^{*T}(k,i)F_{nL}^T(k,i)\Phi(k,x)V_{1n}\end{pmatrix}$$

(Based on $(A+B)^T(A+B)\leq 2\left(A^TA+B^TB\right)$)

$$\leq 2\left(\sigma_{p+1}^2+\lambda_{\min}\begin{pmatrix}V_{1n}^T\Phi^T(k,x)F_{nL}(k,i)\cdot\\ \beta^*(k,i)\beta^{*T}(k,i)F_{nL}^T(k,i)\Phi(k,x)V_{1n}\end{pmatrix}\right)$$

(Because, for any two Hermitian matrices A and B, $\lambda_{\min}(A+B)\leq\min\{\lambda_{\max}(A)+\lambda_{\min}(B),\lambda_{\min}(A)+\lambda_{\max}(B)\}$)

$$\leq 2\left(\sigma_{p+1}^2+h_0^2(k)\lambda_{\min}\left(V_{1n}^T\Phi^T(k,x)F_{nL}(k,i)F_{nL}^T(k,i)\Phi(k,x)V_{1n}\right)\right)$$

(Because, for any two *Hermitian* matrices A and B, $\lambda_{\min}(A^TBA)\leq\lambda_{\max}(B)\lambda_{\min}(A^TA)$, and the property P3))

Substituting the SVD of (19) into it, we get

$$b_2\leq 2\left(\sigma_{p+1}^2+h_0^2(k)\lambda_{\min}\left(V_2S_2^TS_2V_2^T\right)\right)$$
$$\leq 2\left(\sigma_{p+1}^2+h_0^2(k)\lambda_{\min}\left(V_2\,diag\left(\sigma_1'^2,\cdots,\sigma_{n-p}'^2\right)V_2^T\right)\right).$$

Because all of σ_{p+1} … σ_n are less than the threshold T, we have

$$\lambda_{\min}\left(V_2\,diag\left(\sigma_1'^2,\cdots,\sigma_{n-p}'^2\right)V_2^T\right)\geq\left(b_2/2-T^2\right)/h_0^2(k)$$

If we select the threshold such that $T^2<b_2/2$, i.e., $b_2/2-T^2=q^2$, then

$$\sigma_{n-p}'\geq q/h_0(k)\qquad(22)$$

Substituting (22) into (21), we obtain the following final result:

Theorem 1: If the singular threshold is set to be $T<\sqrt{b_2/2}$ and the coefficient matrix of the modification is determined by equations (17) and (20), the coupling matrix modified by equation (14) will be nonsingular with $\left|\det\overline{W}_g^T(k,i)\Phi(k,x)\right|\geq cT^p\left(q/h_0(k)\right)^{n-p}$.

We have proposed a new modification scheme for NN approximation of an affine nonlinear system. The advantage of this SVD based scheme is that both the lower bound of $\left|\det\overline{W}_g^T(k,i)\Phi(k,x)\right|$ and the upper bound of the norm of the coefficient matrix $\beta(k,i)$ are not related to the NN basis functions, i.e.,

$$\left\|\beta(k,i)\right\|_2=c<1\qquad(23)$$

This is because the matrix $\beta(k,i)$ calculated by equations (17) and (20) consists of orthogonal matrices. This makes the modification uniform for any selected basis functions.

After determining the coefficient matrix $\beta(k,i)$ by (17) and (20), the weights estimated by (10) will be modified as

$$\overline{W}(k,i)=\hat{W}(k,i)+F(k,i)\beta(k,i)\qquad(24)$$

From Theorem 1, with this modified weight matrix $\overline{W}$ instead of the estimated one $\hat{W}$ in the control law (11) as

$$u(k,i)=\left(\overline{W}_g^T(k,i-1)\Phi(k,x)\right)^{-1}$$
$$\left((x_d(k+1)-x(k,i))/\tau-\overline{W}_f^T(k,i-1)\varphi(k,x)+Ke(k+1,i-1)\right)\qquad(25)$$

the singularity problem occurring from the computation of the coupling matrix inverse can be avoided.

IV. STABILITY ANALYSIS OF THE NN CONTROLLER

Now the system (2) will be controlled by the control law (25) to track the desired trajectory repetitively. Using control law (25), we have

$$\Delta x(k+1,i)/\tau=\Delta x(k+1,i)/\tau+$$
$$\overline{W}_g^T(k,i-1)\Phi(k,x)u(k,i)-\overline{W}_g^T(k,i-1)\Phi(k,x)u(k,i)$$
$$=\left(x_d(k+1)-x(k,i)\right)/\tau+Ke(k+1,i-1)+$$
$$\Delta x(k+1,i)/\tau-\overline{W}_f^T(k,i-1)\varphi(k,x)-\overline{W}_g^T(k,i-1)\Phi(k,x)u(k,i)$$

namely,

$$e(k+1,i)=x_d(k+1)-x(k+1,i)$$
$$=-\tau Ke(k+1,i-1)-\tau\left(\Delta x(k+1,i)/\tau-\overline{W}^T(k,i-1)Y(k,x(k,i))\right)$$

Describing the variables as the normalized variables in (6) and using the equations (24) and (8), we get

$$e(k+1,i)=-\tau Ke(k+1,i-1)-$$
$$\tau\left(1+\left\|Y(k,x)\right\|\right)\begin{pmatrix}\Delta\bar{x}(k+1,i)/\tau-\hat{W}^T(k,i-1)\bar{Y}(k,x(k,i))-\\ \beta^T(k,i-1)F(k,i-1)\bar{Y}(k,x(k,i))\end{pmatrix}$$
$$=-\tau Ke(k+1,i-1)-\tau\left(1+\left\|Y(k,x)\right\|\right)\begin{pmatrix}E(k,i)-\beta^T(k,i-1)\cdot\\ F(k,i-1)\bar{Y}(k,x(k,i))\end{pmatrix}$$

Taking norms and using (23) yields

$$\left\|e(k+1,i)\right\|\leq\tau\left|K\right|\left\|e(k+1,i-1)\right\|+\tau\left(1+\left\|Y(k,x)\right\|\right)\cdot$$
$$\left(\left\|E(k,i)\right\|+\left\|\beta^T(k,i-1)\right\|\left\|F(k,i-1)\bar{Y}(k,x(k,i))\right\|\right)$$
$$\leq\tau\left|K\right|\left\|e(k+1,i-1)\right\|+\tau\left(1+\left\|Y(k,x)\right\|\right)\cdot$$
$$\left(\left\|E(k,i)\right\|+\left\|F(k,i-1)\bar{Y}(k,x(k,i))\right\|\right)$$
$$\leq\tau\left|K\right|\left\|e(k+1,i-1)\right\|+\sqrt{2}\tau\left(1+\left\|Y(k,x)\right\|\right)\cdot$$
$$\left(\left\|E(k,i)\right\|^2+\left\|F(k,i-1)\bar{Y}(k,x(k,i))\right\|^2\right)^{1/2}$$

$$=\tau\left|K\right|\left\|e(k+1,i-1)\right\|+\sqrt{2}\tau\left(1+\left\|Y(k,x)\right\|\right)w(k,i)\quad\text{(using (9))}\qquad(26)$$

Note that the effective NN approximation is only within a compact region Ω around the desired trajectory. Within this region, the control law (25) is singularity-free (Theorem 1) and the estimated weights are bounded. As a result, the control (25) will be bounded within Ω, i.e., $\|u(k,i)\| \le \|u\|_{\max}$. Then from the definition of Y in (5), we know that $\|Y(k,x)\| \le C_0 \|\varphi(x(k,i))\|$. If the Taylor series expansion of the basis function exists, we can express it with the first order error around the desired trajectory, then there exist positive constants c_1 and c_2 such that

$$\|\varphi(x(k,i))\| \le c_1 + c_2 \|e(k,i)\|, \text{ for } \forall x(k,i) \in \Omega$$

Furthermore, there exist positive constants C_1,C_2, and C_3 such that

$$\begin{aligned} &1 + \|Y(k,x)\| \le C_3 + C_2 \|e(k,i)\| \\ &\le C_1 + C_2 \|e(k+1,i)\| + C_2 \|\Delta x(k,i)\|, \forall x(k,i) \in \Omega \end{aligned} \tag{27}$$

Because the norms of the unknown $f(k,x)$ and $G(k,x)$ are upper bounded within the region Ω, from equation (2), the maximum change of the state between two samples will be

$$\|\Delta x(k,i)\| \le \tau \left(\|f\|_{\max} + \|G\|_{\max} \|u\|_{\max} \right) = d \tag{28}$$

Substituting (27) and (28) into (26) yields

$$\begin{aligned} &\|e(k+1,i)\| \le \tau |K| \|e(k+1,i-1)\| + \\ &\sqrt{2}\tau w(k,i)(C_1 + C_2 d) + \sqrt{2}\tau w(k,i) C_2 \|e(k+1,i)\| \end{aligned}$$

With the weights updating law in Sec. III, when the system tracks the trajectory repetitively, the augmented error of the estimation $w(k,i) \le \bar{\delta}(k,i)$ ultimately (P2). Since $\delta(k,i) \le \eta + \mu'$ in (7), the $w(k,i)$ can be upper bounded by a function of the modeling errors:

$$w(k,i) \le \bar{\delta}(k,i) \le \sqrt{\alpha\left((\eta + \mu')^2 + \varepsilon(\eta + \mu')\right)} = \sigma(\eta,\mu)$$

Then the above error equation becomes

$$\begin{aligned} &\left(1 - \sqrt{2}\tau C_2 \sigma(\eta,\mu)\right) \|e(k+1,i)\| \le \\ &\tau |K| \|e(k+1,i-1)\| + \sqrt{2}\tau (C_1 + C_2 d) \sigma(\eta,\mu) \end{aligned}$$

If we can select the number of neurons such that the modeling error $\sigma(\eta,\mu)$ satisfies

$$\left(1 - \sqrt{2}\tau C_2 \sigma(\eta,\mu)\right) > 0 \tag{29}$$

and let the control gain K satisfy

$$|K| < \left(1 - \sqrt{2}\tau C_2 \sigma(\eta,\mu)\right) / \tau \tag{30}$$

the sequence of the control error $\|e(k+1,i)\|, k = 0, \cdots, N-1$, will be convergent and bounded with

$$\lim_{i \to \infty} \sup \|e(k,i)\| = \frac{\sqrt{2}\tau (C_1 + C_2 d) \sigma(\eta,\mu)}{1 - \sqrt{2}\tau C_2 \sigma(\eta,\mu) - \tau |K|}, \quad k = 1, \cdots, N \tag{31}$$

However, all of the above results are derived under the assumption that the system always works on the prescribed NN approximation region Ω. This can not be ensured during the iterative training, especially for the first several tracking iterations where the system is controlled with a poorly trained NN and may result in a large deviation getting outside of the region Ω. Recall the difference between the presented iterative training NN in (4) and the adaptive NN, where every point along the desired trajectory has a local NN and they are independent from each other, we can convert a whole trajectory NN training into several segments such that none of the tracking errors may exceed the region Ω. A segmented NN control scheme to keep all trainings within the region Ω was proposed in [7]. During the training, the tracking errors are monitored online. If the tracking errors at any trajectory point exceed the region Ω then the trajectory from this point is divided into two segments. After that, the networks of the first segment are trained repetitively and the next segment can be trained until the tracking of the first one has reached a desired precision Ω_D. Therefore, the training of the whole desired trajectory tracking can be accomplished in a step-by-step or segment-by-segment manner.

V. CONCLUSIONS

In this paper, a new NN controller with iterative training is proposed. It can be used for trajectory tracking control of a discrete-time affine nonlinear system. The main contributions of the paper can be concluded as follows: at first, the presented local NN structure for a particular point of the trajectory is independent of the others. This makes the repetitive segmented training based on Iterative Learning Control possible. It further helps us to retain the system state only within the NN approximation region. Next, we proposed a modification NN approximation scheme for the coupling matrix . This makes the control law a truely singularity-free neural network controller using less *a priori* knowledge compared to other NN control schemes.

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