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A Comparative Introduction of Four Fractional Order Controllers [1]

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Abstract

Using the differentiation and integration of fractional order or non-integer order in systems control is gaining more and more interests from the systems control community. In this paper, four representative fractional-order controllers in the literature are briefly introduced, namely, TID (Tilted Proportional and Integral) controller, CRONE controller (*Contrôle Robuste d'Ordre Non Entier*), $\mathrm{PI}^\lambda\mathrm{D}^\mu$ controller and fractional lead-lag compensator. The basic ideas and technical formulations are presented with some comparative comments. The major purpose of this paper is to draw attention to the non-conventional way of robust control based on the fractional order calculus.

Key words: Fractional order calculus, fractional order control, Bode's ideal loop transfer function, robust controller design.

1 Introduction

Using the notion of fractional-order, it may be a step closer to the real world life because the real processes are generally or most likely *fractional* [1]. However, for many of them, the fractionality may be very small. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy RC line or diffusion of the heat into a semi-infinite solid, where the heat flow $q(t)$ in nature is equal to the semi-derivative of the temperature $T(t)$ [2]

$$\frac{\mathrm{d}^{0.5}T(t)}{\mathrm{d}t^{0.5}} = q(t).$$

Clearly, using an integer order ordinary differential equation (ODE) description for the above system may differ significantly to the actual situation. However, the fact that the integer-order dynamic models are more welcome is probably due to the absence of solution methods for fractional-order differential equations (FODEs). Recently, some progresses in analysis of dynamic systems modeled by FODEs have been made in [3, 4, 5, 6, 7, 8, 9, 10]. For example, PID controllers, which have been dominating industrial controllers, have been modified using the notion of fractional-order integrator and differentiator. It is shown that extra degree of freedom from the use of fractional-order integrator and differentiator made it possible to further improve the performance of traditional PID controllers.

In theory, the control systems can include both the fractional order dynamic system or plant to be controlled and the fractional-order controller. However, in control practice, more common is to consider the fractional-order controller. This is due to the fact that the plant model may have already been obtained as an integer order model in classical sense. In most cases, our objective is to apply the fractional order control (FOC) to enhance the system control performance. Therefore, in this paper, we will concentrate on this scenario - controller being fractional-order.

This paper serves as a comparative introduction to four existing schemes: TID controller [11], CRONE controller [12, 13], $\mathrm{PI}^\lambda\mathrm{D}^\mu$ controller [14] and fractional lead-lag compensator [15]. The basic ideas and technical formulations are presented with some comparative comments. The major purpose of this paper is to draw attention to the non-conventional way of robust control based on fractional order calculus.

This paper is organized as follows. In Sec. 2, we briefly introduce the definition of fractional order operator and FODE. Four FOC schemes are introduced in Sec. 3 with some of our comparative comments in Sec. 4. Sec. 5 concludes this paper with some additional remarks on the possible future developments in FOC.

2 Fractional-order Calculus, Fractional-order Differential Equations and Their Laplace Transformation

Fractional order calculus is as old as the calculus of differentiation. The theory of fractional-order derivative was developed mainly in the 19-th century. For more references, see [6, 16, 17, 8].

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2.1 Definitions of Fractional-order Differentiator

A fundamental operator ${}_a\mathrm{D}_t^\alpha$, a generalization of differential and integral operators, is introduced as follows:

$$ {}_a\mathrm{D}_t^\alpha = \begin{cases} \mathrm{d}^\alpha/\mathrm{d}t^\alpha, & \Re(\alpha) > 0, \\ 1, & \Re(\alpha) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-\alpha}, & \Re(\alpha) < 0, \end{cases} \quad (1) $$

where α can be a complex number but it is assumed to a real number in this paper, a is a real number related to initial value which can be usually taken as 0. There are two commonly used definitions for the general fractional differentiation and integral, i.e., the Grünwald-Letnikov definition and the Riemann-Liouville definition [6, 17, 8]. The Grünwald-Letnikov definition is that

$$ {}_a\mathrm{D}_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} \quad (2) $$

where $[\cdot]$ is a flooring-operator while the Riemann-Liouville definition is given by

$$ {}_a\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \quad (3) $$

for $(n-1 < \alpha < n)$ where $\Gamma(x)$ is the well known Euler's `gamma` function. One can observe that by introducing notion of the fractional-order operator ${}_a\mathrm{D}_t^\alpha$, the differentiator and integrator can be unified. Therefore, in this paper, we shall use the term "fractional-order differentiator" or "fractional derivative" alone which should be understood to imply both differentiator and integrator as shown in (1).

2.2 Some Properties of Fractional-order Differentiator

Here we introduce two general properties of fractional derivative. The first is the composition of fractional with integer-order derivative and the second is the property of linearity. For more properties of fractional derivative, refer to [17, 8].

The fractional-order derivative commutes with integer-order derivation [8],

$$ \frac{\mathrm{d}^n}{\mathrm{d}t^n}({}_a\mathrm{D}_t^p f(t)) = {}_a\mathrm{D}_t^p \left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n} \right) = {}_a\mathrm{D}_t^{p+n} f(t), \quad (4) $$

under the condition $t = a$ one gets $f^{(k)}(a) = 0$, $(k = 0, 1, 2, \ldots, n-1)$. The relationship (4) says the operators $\mathrm{d}^n/\mathrm{d}t^n$ and ${}_a\mathrm{D}_t^p$ commute.

Similar to the integer-order differentiation, ${}_a\mathrm{D}_t^\alpha$ is a linear operator

$$ {}_a\mathrm{D}_t^p \left(\lambda f(t) + \mu g(t)\right) = \lambda\, {}_a\mathrm{D}_t^p f(t) + \mu\, {}_a\mathrm{D}_t^p g(t). \quad (5) $$

2.3 Linear Fractional-order Differential Equations (FODEs)

A typical n-term linear FODE in time domain is give by

$$ a_n \mathrm{D}_t^{\beta_n} y(t) + \cdots + a_1 \mathrm{D}_t^{\beta_1} y(t) + a_0 \mathrm{D}_t^{\beta_0} y(t) = 0 \quad (6) $$

where $a_k (k = 0, 1, \cdots, n)$ are constant coefficients of the FODE; $\beta_k, (k = 0, 1, 2, \cdots, n)$ are real numbers. Without loss of generality, assume that $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 \geq 0$.

It is possible to solve the above FODE analytically by using Mittag-Leffler function in two parameters which is a generalization of exponential function e^z. The Mittag-Leffler function in two parameters is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta > 0). \quad (7) $$

Clearly, e^z is a particular case of the Mittag-Leffler function [7]

$$ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. $$

The analytical solution of the n-term FODE (6) is given in general form in [7].

2.4 Laplace Transformation Method

The Laplace transform formula for the Riemann-Liouville fractional derivative (3) has the form [7]:

$$ \int_0^\infty \mathrm{e}^{-st}\, {}_0\mathrm{D}_t^\alpha f(t)\, \mathrm{d}t = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k\ {}_0\mathrm{D}_t^{\alpha-k-1} f(t)\Big|_{t=0}, \quad (8) $$

for $(n-1 < \alpha \leq n)$ where $F(s) = \mathcal{L}[f(t)]$ is the normal Laplace transformation.

Consider a control function which acts on the FODE system (6) as follows:

$$ a_n \mathrm{D}_t^{\beta_n} y(t) + \cdots + a_1 \mathrm{D}_t^{\beta_1} y(t) + a_0 \mathrm{D}_t^{\beta_0} y(t) = u(t). \quad (9) $$

By Laplace transform, we can get a fractional transfer function [18]:

$$ G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_n s^{\beta_n} + \cdots + a_1 s^{\beta_1} + a_0 s^{\beta_0}}. \quad (10) $$

In general, a fractional-order dynamic system can be represented by [7, 19, 20]

$$ \begin{aligned} \frac{Y(\mathrm{j}\omega)}{U(\mathrm{j}\omega)} &= \frac{b_m(\mathrm{j}\omega)^{\alpha_m} + \cdots + b_1(\mathrm{j}\omega)^{\alpha_1} + b_0(\mathrm{j}\omega)^{\alpha_0}}{a_n(\mathrm{j}\omega)^{\beta_n} + \cdots + a_1(\mathrm{j}\omega)^{\beta_1} + a_0(\mathrm{j}\omega)^{\beta_0}} \\ &= \frac{\sum_{k=0}^m b_k(\mathrm{j}\omega)^{\alpha_k}}{\sum_{k=0}^n a_k(\mathrm{j}\omega)^{\beta_k}} \end{aligned} \quad (11) $$

in frequency domain where $s = \mathrm{j}\omega = \mathrm{j}(2\pi f)$ and f is the frequency in Hertz. It should be pointed out that, for fractional-order control, in the literature, discussions in the frequency-domain dominate.

3 Four Fractional Order Controllers

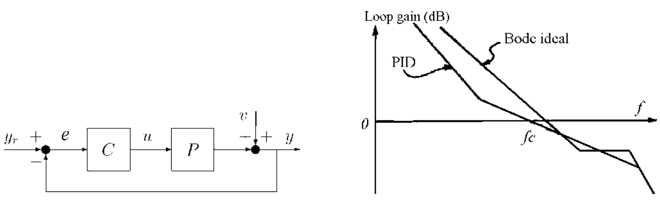
The early attempts to apply fractional-order derivative to systems control can be found in [21, 3, 22]. In this section, four representative fractional-order controllers in the literature will be briefly introduced, namely, TID controller, CRONE controller, $\mathrm{PI}^\lambda\mathrm{D}^\mu$ controller and fractional lead-lag compensator.

3.1 TID Controller

In [11], a feedback control system compensator of the PID type is provided, wherein the proportional component of the compensator is replaced with a **t**ilted component having a transfer function $s^{-\frac{1}{n}}$. The resulting transfer function of the entire compensator more closely approximates an optimal transfer function, thereby achieving improved feedback controller. Further, as compared to conventional PID compensators, the TID compensator allows for simpler tuning, better disturbance rejection ratio, and smaller effects of plant parameter variations on closed loop response.

3.1.1 Basic Motivations:

The motivation for TID (Tilted Proportional and Integral) control is from the consideration of the so-called theoretically optimal loop response due to Bode. Consider the conventional feedback control system block diagram described by Fig. 1(a) where C is the feedback controller, y_r is the reference input signal, e is the control error signal , u and y are input and output signals respectively. In Fig. 1(a), the additive disturbance is denoted by v. The major goals for the feedback control system are to minimize the effect of disturbances at the output of the system, and to minimize sensitivity of the closed loop response to plant parameter variations. To satisfy these requirements, the feedback of the system, properly weighted in frequency, must be maximized. These constraints uniquely define the optimal transfer function for the feedback loop. The purpose for the compensator of the feedback system is to implement a loop response reasonably close to the optimal one. A commonly-used compensator employed in feedback control systems is a proportional-integral-derivative (PID) compensator. In fact, a PID controller provides varying degrees of gain and phase shift of the signal according to the frequency contents. The conventional PID compensator transfer function typically has two real zeros. Typically, the P-term dominates near f_c, the D-term dominates at frequencies over $4f_c$, and the I-term dominates at frequencies up to f_c /4, where f_c is the crossover frequency at which loop gain is 0 dB as shown in Fig. 1(b).



(a) Block diagram of the classic feedback control system with disturbance

(b) Bode plots for PID controlled plant and the ideal loop response

Figure 1: Classic control system and its ideal Bode plot.

Referring to Fig. 1(b), a theoretically optimal loop response has been determined by Bode. For the purpose of industrial control, a simplified suboptimal Bode loop response can be employed. The suboptimal response is illustrated in Fig. 1(b) by a solid line. The slope of this suboptimal gain response is about -10 dB/octave. The transcendental loop transfer function which characterizes the suboptimal response can be closely approximated by a rational function. As can be seen from Fig. 1(b), rather sharp corners occur at the sides of the Bode step. Any smoothing of the corners, especially the left one, caused by an improper or inaccurate rational function approximation, reduces the available feedback, resulting in reduced performance. A typical loop gain Bode diagram of the system with a PID compensator is also shown in Fig. 1(b). When provided with the same stability margin and the same average loop gain as an optimal Bode controller, the crossover frequency f_c of the PID controller is about one-half that of the optimal Bode loop response. The feedback at frequency $f_c/4$ is about 10 dB lower than that of a simplified Bode controller. The conventional PID controllers in common use when applied to a great variety of plants, are easy to tune to provide robust and fairly good performance. However, the performance is not optimal as explained above.

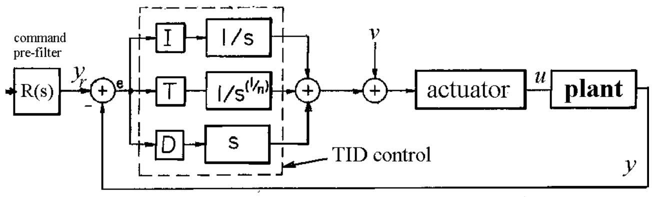


Figure 2: Block diagram of TID control scheme

The object of TID is to provide an improved feedback loop compensator having the advantages of the conventional PID compensator, but providing a response which is closer to the theoretically optimal response.

3.1.2 Brief Introduction to TID Control Scheme:

Similar to PID control, TID scheme is shown in Fig. 2. where the the proportional compensating unit is replaced with a compensator having a transfer function characterized by $1/s^{\frac{1}{n}}$ or $s^{-1/n}$. This compensator is herein referred to as a "Tilt" compensator, as it provides a feedback gain as a function of frequency which is tilted or shaped with respect to the gain/frequency of a conventional or positional compensation unit. The entire compensator is herein referred to as a Tilt-Integral-Derivative (TID) compensator. For the Tilt compensator, n is a *nonzero real number*, preferably between 2 and 3. Thus, unlike the conventional PID controller, wherein exponent coefficients of the transfer functions of the elements of the compensator are either 0, -1, or +1, TID scheme exploits an exponent coefficient of $-1/n$. By replacing the conventional proportional compensator with the tilt compensator of the invention, an overall response is achieved which is closer to the theoretical optimal response determined by Bode as illustrated in Fig. 1(b).

In Fig. 2, $R(s)$ is a prefilter provided for proper command signal prefiltering which is commonly seen in practice. A preferred transfer function for the prefilter is

$$R(s) = \frac{s^2 + 2\omega_c s + \omega_c^2}{s^2 + 5.25\omega_c s + \omega_c^2}$$

Since the T-term eliminates static error, the coefficient of the I-term can be set to zero for many problems, thus simplifying controller tuning. A suggested tuning procedure for the TID compensator is:

(a) set K_I=0, K_D=0, and set the coefficient K_T for the loop gain to be 0 dB at a desired crossover frequency f_c;

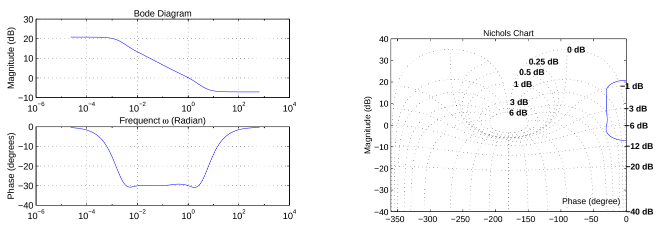
(b) set K_D such that the phase stability margin at the crossover frequency is about *5 degrees larger than desired*; and

(c) set K_I=$0.25 K_T f_c^{(1-1/n)}$.

Taking $n = 1/3$ as an example, the transfer function $1/s^{1/3}$ can be approximated by a transfer function having alternating real poles and zeros in a complex plane representation. Three poles and three zeros per decade generally suffice to achieve the phase error of less than 1 degree and the amplitude error of less than 0.1 db which is given by

$$T_{6/6}(s)=\frac{\begin{array}{c}.442s^6+2.23s^5+1.86s^4+0.428s^3+.0295s^2\\ +.000568s+2.18\times10^{-6}\end{array}}{\begin{array}{c}s^6+2.42s^5+1.304s^4+.201s^3+.0092s^2\\ +.0001098s+1.98\times10^{-7}\end{array}}.$$

Enter the coefficients for the above approximated transfer function $T_{6/6}(s)$ for $1/s^{1/3}$ into `CtrlLAB`© [23], three mouse clicks give the Bode plot, Nichols chart and root locus as shown in Fig. 3.



(a) Bode plot

(b) Nichols chart

Figure 3: Frequency responses of transfer function $T_{6/6}(s)$

From the Bode plot, we can see that $T_{6/6}(s)$ is a good approximate for $1/s^{1/3}$ in both magnitude and in particular the phase (constant phase angle). The vertical line in Nichols chart Fig. 3(b), is a desired robustness property for controller design.

In TID patent [11], an analog circuit using op-amps plus capacitors and resistors is introduced with a detailed component list which is useful in some cases where the computing power to implementing $T_3(s)$ digitally is not possible. An example is given in [11] to illustrate the benefits from TID over conventional PID in both time and frequency domain.

3.2 CRONE Controller

The CRONE control was proposed by Oustaloup in pursuing *fractal robustness* [12, 13]. CRONE is a French abbreviation for "*Contrôle Robuste d'Ordre Non Entier*" (which means non-integer order robust control). In this section, we shall follow the basic concept of *fractal robustness*, which motivated the CRONE control, and then mainly focus on the second generation CRONE control scheme and its synthesis based on the desired frequency template which leads to fractional transmittance [24, 25].

3.2.1 Fractal Robustness: In [26], "fractal robustness" is used to describe the following two characteristics: the isodamping and the vertical sliding form of frequency template in the Nichols chart. This desired robustness motivated the use of fractional-order controller in classical control systems to enhance their performance.

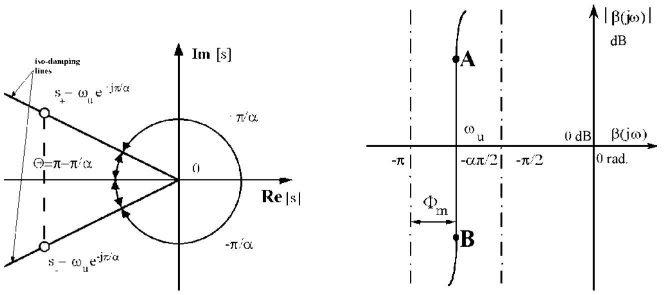
1. **Isodamping lines**. Consider the characteristic equation
$$1+(\tau s)^{\alpha}=0 \qquad (12)$$
where τ is a constant. The two poles are given by
$$s=\frac{1}{\tau}e^{\pm j\pi/\alpha}; \qquad (13)$$
for $1 < \alpha < 2$. The poles are complex and conjugated and form a center angle 2Θ with $\Theta = (\pi - \pi/\alpha)$ as shown in Fig. 4(a). Clearly, the poles move at a constant angle (fixed by the order α) when τ varies. The robustness in plane s is then illustrated by two half-straight lines which form the same angle Θ in relation to the real axis and are called *isodamping half-straight lines*.

   The *natural frequency* and the *damping ratio* are directly deducible from the poles, through their modulus $1/\tau$ and the half-center angle Θ as follows:
$$\omega_p=\frac{1}{\tau}\sin\Theta=\frac{1}{\tau}\sin\left(\pi-\frac{\pi}{\alpha}\right)=\frac{1}{\tau}\sin\left(\frac{\pi}{\alpha}\right) \qquad (14)$$
and
$$\zeta(\alpha)=\cos\Theta=\cos\left(\pi-\frac{\pi}{\alpha}\right)=-\cos\left(\frac{\pi}{\alpha}\right). \qquad (15)$$
It can be clearly seen that *the damping ratio ζ is exclusively a function of fractionality order α*, thus allowing the introduction of the notion of *robust oscillatory mode*.

2. **Frequency template**.

   With a unit negative feedback, the forward path transfer function, or open-loop transmittance, for the characteristic equation (12) is
$$\beta(s)=\left(\frac{1}{\tau s}\right)^{\alpha}=\left(\frac{\omega_u}{s}\right)^{\alpha}, \qquad (16)$$
which is the transmittance of a *non integer integrator* in which $\omega_u = 1/\tau$ denotes the unit gain (or transitional) frequency.

   As $\arg\beta(j\omega) = -\alpha\pi/2$ with $1 < \alpha < 2$, the Nichols chart of $\beta(j\omega)$ is a *vertical straight line* between $-\pi/2$ and $-\pi$. This is illustrated in Fig. 4(b). When τ, the system parameter, changes, the vertical straight line shown in



(a) Isodamping half-straight lines

(b) Robustness in Nichols chart

Figure 4: Illustrations of *fractal robustness.*

Fig. 4(b) slides. Such a vertical displacement ensures a constant phase margin Φ_m, and thus correspondingly a constant damping ratio in the time domain.

In controller design, the objective is to achieve such a similar frequency behavior, in a medium frequency range around ω_u, knowing that the closed loop dynamic behavior is exclusively linked to the open loop behavior around ω_u. Therefore, the ideal controller design comprises

- an open loop Nichols locus which forms a vertical straight line segment around ω_u for the nominal parametric state of the plant, called open loop frequency template (or more simply template) (Fig. 4(b));
- and a sliding of the template on itself when there exist parameter changes in the plant (assume that the parameter change will lead to gain variations around ω_u).

Synthesizing such a template defines the non-integer approach that the second generation CRONE control uses.

3.2.2 The Second Generation CRONE Control - Basic Concept: For typical disturbed feedback control system as shown in Fig. 1(a), its control performance is fully characterized by the sensitivity function $\mathbb{S}(s)$, also known as *the transmittance in regulation*, or the complementary sensitivity function $\mathbb{T}(s)$, also known as *the transmittances in tracking* and we know that $\mathbb{S}(s) + \mathbb{T}(s) = 1$. It is practically true that given the open loop behavior around the unit gain frequency, one can determine the dynamic behavior in closed loop. Therefore, we use the transmittance frequency template , $\beta(s)$, as shown in Fig. 4, to define the desired behavior of $\mathbb{T}(s)$ or $\mathbb{S}(s)$. Let's choose a template such that

$$\beta(s) = \beta(s) \qquad \forall\omega \in [\omega_A, \omega_B], \tag{17}$$

where

$$\beta(s) = \left(\frac{\omega_u}{s}\right)^{\alpha}, \qquad \alpha \in [1, 2]. \tag{18}$$

Referring to Fig. 1(a), the desired or ideal $\mathbb{T}(s)$ and $\mathbb{S}(s)$ are to set as follows:

$$\mathbb{T}(s) = \left[\frac{Y(s)}{Y_r(s)}\right]_{V(s)=0} = \frac{\beta(s)}{1+\beta(s)} = \frac{1}{1+(s/\omega_u)^{\alpha}} \tag{19}$$

and

$$\mathbb{S}(s) = \left[\frac{Y(s)}{V(s)}\right]_{Y_r(s)=0} = \frac{1}{1+\beta(s)} = \frac{(s/\omega_u)^{\alpha}}{1+(s/\omega_u)^{\alpha}}. \tag{20}$$

In tracking, gain reaches a maximum for resonance frequency

$$\omega_t = \left(-\cos\alpha\frac{\pi}{2}\right)^{1/\alpha}\omega_u, \tag{21}$$

and in regulation, gain reaches a maximum for resonance frequency

$$\omega_r = \left(-\cos\alpha\frac{\pi}{2}\right)^{-1/\alpha}\omega_u. \tag{22}$$

This result reveals the existence of a resonance when $\cos(\alpha\pi/2) < 0$, namely for $1 < \alpha < 3$ and therefore for the CRONE control since $1 < \alpha < 2$. The *resonance ratio in tracking* is

$$Q_t(\alpha) = \frac{|\mathbb{T}(j\omega_t)|}{|\mathbb{T}(j0)|} = \frac{1}{\sin\alpha\frac{\pi}{2}}, \tag{23}$$

while the *resonance ratio in regulation* is

$$Q_r(\alpha) = \frac{|\mathbb{S}(j\omega_r)|}{|\mathbb{S}(j\infty)|} = \frac{1}{\sin\alpha\frac{\pi}{2}}. \tag{24}$$

These results show that *the resonance ratio depends exclusively on control order* α, thus allowing the introduction of the notion of *robust resonance*. By observation, it can be found that

$$(\omega_t\omega_r)^{1/2} = \omega_u \tag{25}$$

and

$$Q_t(\alpha) = Q_r(\alpha). \tag{26}$$

From (25)-(26), *the resonance frequencies in tracking and in regulation are symmetrically distributed with regard to the open loop unit gain frequency* while *the resonance ratios in tracking and in regulation are identical.*

3.2.3 The Second Generation CRONE Control - Design Steps: Usually, descriptive specifications of the open loop behavior (for the nominal plant) will be given such as

- *the accuracy specifications at low frequencies* ;
- *the vertical template around unit gain frequency* ω_u ;
- *the input sensitivity specifications at high frequencies.*

For a stable minimum phase plant, it turns out that the behavior thus defined can be described by a *transmittance based on the frequency-limited real non integer differentiator*, i.e.,

$$\beta(s) = \left[K_b\left(\frac{\omega_b}{1}+1\right)\right]^{n_b}\left(\sqrt{\frac{1+(\omega_u/\omega_b)^2}{1+(\omega_u/\omega_h)^2}}\,\frac{1+s/\omega_h}{1+s/\omega_b}\right)^{\alpha}\left(\frac{K_h}{1+s/\omega_h}\right)^{n_h} \tag{27}$$

with

$$K_b = \left(1 + (\omega_b/\omega_u)^2\right)^{-1/2} \text{ and } K_h = \left(1 + (\omega_u/\omega_h)^2\right)^{1/2}. \quad (28)$$

In the particular case where transitional frequencies ω_b and ω_h are sufficiently distant from frequency ω_u, around this frequency (i.e. $\omega_b \ll \omega \ll \omega_h$), $\beta(s)$ can be reduced to transmittance

$$\beta(s) = (\omega_u/s)^\alpha, \quad (29)$$

which is the same as that described by the template (relation (18)).

The order α transmittance of relation (27) describes the frequency truncation of the template defined by the transitional frequencies ω_b and ω_h. This transmittance results from the substitution of the part raised at power α for the transmittance ω_b/p which is used in the description of the template between frequencies ω_A and ω_B, as shown in Fig. 4(b).

Finally, referring to Fig. 1(a), the controller $C(s)$ in cascade with the plant is synthesized from its frequency response according to

$$C(\mathrm{j}\omega) = \frac{\beta(\mathrm{j}\omega)}{G_0(\mathrm{j}\omega)}, \quad (30)$$

where $G_0(\mathrm{j}\omega)$ denotes the frequency response of the nominal plant.

There are a number of real life applications of CRONE controller such as the car suspension control [27, 13], flexible transmission [12], hydraulic actuator [28] etc. CRONE control has been evolved to a powerful non-conventional control design tool with a dedicate MATLAB toolbox for it [29]. For an extensive overview, refer to [30] and the references therein.

3.3 PI$^\lambda$D$^\mu$ Controller

3.3.1 Basic Formulae:

PI$^\lambda$D$^\mu$ controller, also known as PI$^\lambda$D$^\delta$ controller, was studied in time domain in [14] and in frequency domain in [31]. In general form, referring to Fig. 1(a), the transfer function of PI$^\lambda$D$^\delta$ is given by

$$C(s) = \frac{U(s)}{E(s)} = K_p + T_i s^{-\lambda} + T_d s^{\delta}, \quad (31)$$

where λ and δ are positive real numbers; K_p is the proportional gain, T_i the integration constant and T_d the differentiation constant. Clearly, taking $\lambda = 1$ and $\delta = 1$, we obtain a classical PID controller. If $\lambda = 0$ ($T_i = 0$) we obtain a PD$^\delta$ controller, etc. All these types of controllers are particular cases of the PI$^\lambda$D$^\delta$ controller. The time domain formula is that

$$u(t) = K_p e(t) + T_i \mathrm{D}_t^{-\lambda} e(t) + T_d \mathrm{D}_t^{\delta} e(t). \qquad (\mathrm{D}_t^{(*)} \equiv {}_0\mathrm{D}_t^{(*)}). \quad (32)$$

The digital realization techniques for the above FOC will be introduced in detail in the next section.

It can be expected that PI$^\lambda$D$^\delta$ controller (32) may enhance the systems control performance due to more tuning knobs introduced. Actually, in theory, PI$^\lambda$D$^\delta$ itself is an infinite dimensional linear filter due to the fractional order in differentiator or integrator.

3.3.2 A Simple Controller Synthesis Scheme:

Unlike conventional PID controller, there is no systematic and yet rigor design or tuning method existing for PI$^\lambda$D$^\delta$ controller. Here, a simple scheme based on the dominant root principle to design PI$^\lambda$D$^\delta$ controller is briefly introduced. The pole distribution of the characteristic equation of the controlled system in the complex plane should be located at the desired dominant roots which are designed based on the control performance requirement. Assume that the desired dominant roots are a pair of complex conjugate root as follows:

$$p_{1,2} = -r \pm \mathrm{j}\omega. \quad (33)$$

It is clear that the above dominant root defines the stability measure S_t and damping measure T_l. In this simplified situation, the parameters design of the PI$^\lambda$D$^\delta$ controller can be divided into two steps, i.e.,

1. **The design of K_p.** Proportional gain K_p is related to the static error E_t [%], settling time T_r [sec.], and overshoot P_r [%]. In general, the larger the K, the smaller the control time T_r [s] as well as the static error E_t [%]. Therefore, K_p can be simply set via

$$K_p \geq (100/E_t).$$

2. **The design of T_d, δ, T_i, λ.** From the complex conjugate roots (33), the (required) stability measure $S_t = r$ and damping measure $T_l = r/\omega$ can be computed. Given S_t and T_l, using classical root locus method, we can numerically solve T_d, δ, T_i, λ from the characteristic equation with fractional-order controller $C(s)$ which is given by

$$C(s)P_0(s) + 1 = 0 \quad (34)$$

where $P_0(s)$ is a nominal model of $P(s)$ as shown in Fig. 1(a). More specifically, for simple plant models, this can now be done by solving

$$\min_{T_d,\delta,T_i,\lambda} \|C(s)P_0(s) + 1\|_{s=-r\pm\mathrm{j}\omega}.$$

3.4 Fractional Lead-Lag Compensator

In the previous subsections, fractional controllers are directly related to the use of fractional-order differentiator or integrator. It is possible to extend the classical lead-lag compensator to the fractional-order case which was studied in [15]. The fractional lead-lag compensator is given by

$$C_r(s) = C_0 \left(\frac{1 + s/\omega_b}{1 + s/\omega_h}\right)^r \quad (35)$$

where $0 < \omega_b < \omega_h$, $C_0 > 0$ and $r \in (0, 1)$.

Consider the feedback control loop in Fig.1. A robust control problem of interest is to find C guaranteeing robust Q-factor (amplitude magnification factor at the resonance frequency) for the transfer functions from y_y to y and from v to y for all plants in the form $P = \alpha P_0$, with P_0 the nominal plant model and $\alpha \in [\alpha_m, \alpha_M]$.

An ideal solution to this problem is to make the nominal loop transfer function $L_0(s) = C(s)P_0(s) = (s/\omega_0)^n$, where $1 < n < 2$. For $L(s) = C(s)P(s)$, its Nyquist plot is similar to $L_0(s)$. Therefore, the phase margin and Q-factors for the closed-loop transfer functions from y_r to y and from v to y will be independent of α and uniquely determined by n. Furthermore, since the transfer function from y_r to y can be approximated by a second-order stable system, the maximum closed-loop step response overshoot is almost independent of α. Thus, a change of α will result in a slower or faster but equally damped response, which is a desirable robustness property in some applications, such as the car suspension design problem [15].

3.4.1 Design Concept: The ideal loop transfer function can be approximated by shaping $L_0(j\omega)$ close to $(j\omega/\omega_0)^n$ around the cross-over frequency ω_0 at which $\|L_0(j\omega_0)\| = 1$. To achieve this, when the plant $P(s) = Ms^m$, it is required that $\arg(C(j\omega))$ is close to $m - 2 + \phi/\pi$, where ϕ is the desired phase margin, in a given frequency range around ω_0. One could obviously seek to achieve this by using the classical rational lead-lag filter corresponding to (35) with $r = 1$ due to the fact that $\arg[C_1(j\omega)]$ is close to $\arg[C_1(j\omega_0)]$ around the frequency $\omega_0 = \sqrt{\omega_b\omega_h}$. Clearly, when $r = 1$, the width of the frequency range over which the condition $\arg[C_1(j\omega)] \approx \arg[C_1(j\omega_0)] = m - 2 + \phi/\pi$ holds is entirely determined by the choice of ϕ, and thus cannot be adjusted to meet robustness requirements.

When a non-integer value of r in (35) is used, it is possible to guarantee robust phase margin, closed-loop resonance and overshoot for any given range of variation of α.

To apply this control strategy to more general linear systems, the control $C(s)$ in Fig. 1 should take the form $C(s) = C_r(s)G(s)$, where $G(s)$ is to be chosen such that

$$G(s) \approx Ms^m/P_0(s). \tag{36}$$

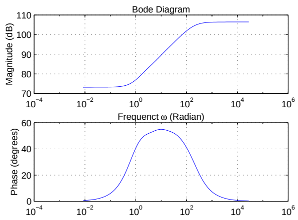
3.4.2 Realization of Fractional Lead-Lag Compensator: A state-space representation of the fractional lead-lag compensator is proposed in [15] together with an error bound estimate. However, the stable minimum-phase frequency-domain fitting is an easier and more effective method.

In the suspension controller design example of [15], which is cited from [13], the plant is $P(s) = 1/(Ms^2)$. The range of change in M is from 100 kg to 900 kg. The parameters [15] of the designed fractional lead-lag compensator are: $\omega_b = 0.5$, $\omega_h = 200$, $r = 0.65$. Nominal frequency $\omega_0 = 10$ rad./sec. at the nominal mass $M = M_0 = 300$kg. λ is set to 20 so that $\omega_b = \omega_0/\lambda$ and $\omega_h = \lambda\omega_0$. C_0 is determined from $\|P(j\omega)C_r(j\omega)\| = 1$ which gives $C_0 = M_0\omega_0^2\lambda^{-r}$ such that ω_0 is the gain crossover frequency for the nominal case when $M = M_0$.

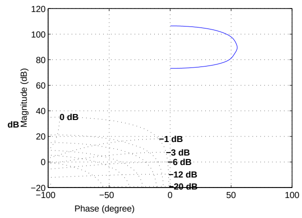
Here we give out the fitting result for $C_{0.65}(s)$ using the stable frequency fitting method introduced in [32]. The 4/4 fitting result is that

$$C_{0.65}(s) = 4280.1\left(\frac{1+2s}{1+0.005s}\right)^{0.65}$$
$$\approx \frac{9.457\times10^{-11}s^4+1.218\times10^{-8}s^3+3.07\times10^{-7}s^2+1.476\times10^{-6}s+9.794\times10^{-7}}{4.5\times10^{-16}s^4+1.161\times10^{-13}s^3+6.99\times10^{-12}s^2+9.516\times10^{-11}s+2.14\times10^{-10}}$$

with its Bode plot and Nichols chart drawn in CtrlLAB in Fig. 5. We can see that Fig. 5 is quite similar to the characteristic of a frequency-band fractional differentiator.



(a) Bode plot



(b) Nichols chart

Figure 5: Stable minimum phase 4/4 frequency domain fitting of $C_{0.65}(s)$

4 Comparative Comments

The four representative fractional-order controllers briefly introduced in Sec. 3 have their respective advantages. We offer our comparative comments as follows.

- In terms of readiness for real applications, CRONE method is the best choice since it has a clear design interpretation with connections to the familiar conventional controller design methods based on Bode plot and Nichols chart. In addition, CRONE has many industrial applications especially for the car suspension systems. The recently developed MATLAB Toolbox for CRONE design [29] makes it ready for commercial applications.
- Since PID control is popular in many industry sections, $PI^\lambda D^\mu$ controller should provide additional potentials to achieve better performance. Compared to many well-proven PID parameter setting techniques, development of setting or autotuning techniques for the 5 parameters in $PI^\lambda D^\mu$ is strongly desired.
- Although TID can be regarded as a special type of $PI^\lambda D^\mu$ controller, it is observed that a systematic parameter setting method has been proposed and tested. Therefore, TID should find its wide applications in process control industry.
- Lead-lag compensator is also a popular control system design method. Fractional order lead-lag

compensator should have its equal value compared to CRONE or $PI^{\lambda}D^{\mu}$ controller. However, more intuitive systematic design and parameter tuning method are needed.

In summary, the introduction of fractional order calculus idea to conventional controller design extends the opportunity of added performance improvement. For example, one may extend the another popular controller design method - "pole-placement" to "fractional pole-placement". The success depends on the systematic FOC tuning/design method.

5 Concluding Remarks

We have presented an introduction of four representative fractional-order controllers in the literature, namely, TID (Tilted Proportional and Integral) controller, CRONE controller (*Contrôle Robuste d'Ordre Non Entier*), $PI^{\lambda}D^{\mu}$ controller and fractional lead-lag compensator. The basic ideas and technical formulations are presented with some comparative comments. The major purpose of this paper is to draw attention to the non-conventional way of robust control based on fractional order calculus.

We believe that fractional order control (FOC) can benefit control engineering practitioners in a number of ways. A significant benefit may be due to the reduced number of tuning knobs compared to other robust control design methods. Further research efforts will be on the efficient digital implementation methods (discretization method) and the investigation the FOC performance for systems with actuator nonlinearities. Another possibility for future work is the use of FOC with real or complex time varying order.

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