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Analytical Stability Bound for a Class of Delayed Fractional-Order Dynamic Systems ⋆

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Abstract. Delayed Linear Time-Invariant (LTI) fractional-order dynamic systems are considered. The analytical stability bound is obtained by using Lambert function. Two examples are presented to illustrate the obtained analytical results.

Keywords: Delay, fractional-order dynamic systems, fractional-order integrator, fractional-order differentiator, stability bound, analytical solutions, Lambert function.

1. Introduction

In real world systems, delay is everywhere. Generally speaking, delays in systems can be classified into two groups: lumped delays and distributed delays. Examples for systems with lumped delays are transportation processes such as control of coal flow rate on belt, economic systems, computer controlled systems, remote control, etc. Heat exchangers and electric transmission lines are examples for systems with distributed delays. For more and detailed examples of systems with time delays, refer to [1, 2].

Delay Differential Equations (DDE), initially introduced in the eighteenth century by Laplace and Condorcet [1], are used to describe dynamic systems with time delays, also known as dead-time or transport-lag which were also studied in terms of difference differential equations in [3]. In general, the time-delay is believed to have a negative impact on the control system performance. To compensate its effect, Smith predictor scheme works fine for slow processes [4, 5]. There are many variants in using Smith predictor scheme, see e.g. [1, 6] and the references therein. However, it should be pointed out that the time delay can be utilized to achieve a better control performance. For example, in Iterative Learning Control (ILC) [7, 8], the learning updating law uses the delayed error signals and control signals (previous trials) to construct the control signal for the present iteration.

The analysis of DDE stability has been a long history efforts for applied mathematicians as well as control engineers. Although Bellman and Cooke tried to get a closed form solution for the first order DDE with some tedious infinite series, a concise analytical solution for the first

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order DDE with constant coefficients has not been possible until the rediscovery of Lambert[1] W function [9]. Lambert function $W(x)$ satisfies

$$W(x)\,\mathrm{e}^{W(x)} = x.$$

Actually, Euler noticed the importance of Lambert function $W(x)$ in [10, 11], according to the introductory paper on Lambert function $W(x)$ [12]. Lambert function got its name during the implementation of Maple [13, 14], a symbolic computation system. Among many interesting applications for Lambert function [12, 15], getting an analytical stability bound for the first order DDE has been an exciting one [16]. Actually, for a simple DDE

$$\dot{y}(t) = ay(t-1),$$

with its characteristic equation

$$s = a\,\mathrm{e}^{-s},$$

one may immediately get

$$s = W_k(a),$$

where k is the index for branches [17]. For a with a real value, only the principal branch $W_0(a)$, or simply $W(a)$, is to be considered. In symbolic software packages such as Maple, MATLAB Symbolic Toolbox and etc. $W(x)$ is a standard function now. Recently, a new extension for the above result on DDE stability bound was presented in [18] where the following formula is used:

$$\text{Given:}\quad (a+bx)\,\mathrm{e}^{(cx)} + d = 0;$$

$$\text{One gets:}\quad x = \frac{1}{c}W\left(-\frac{cd}{b}\mathrm{e}^{(ac)/b}\right) - \frac{a}{b}.$$

In [19], the results were extended to the case of high-order systems with repeating poles. For some special cases, using other techniques, the DDE stability can be analyzed symbolically, e.g. the analytical expressions for linear DDE stability of cutting process [20]. However, so far, there is no effort in the literature in finding an analytical stability bound for fractional-order DDE with constant coefficients using Lambert function. Fractional-order means that the DDE order is non-integer.

Using the notion of fractional-order, it might be a step closer to the real world situation because the real processes are generally or most likely *fractional* [21]. However, for many of them, the fractionality might be very low. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy resistance-capacitance line or the diffusion of heat in a semi-infinite solid, where the heat flow $q(t)$ is in nature equal to the semi-derivative of temperature $T(t)$ [22]

$$\frac{\mathrm{d}^{0.5}T(t)}{\mathrm{d}t^{0.5}} = q(t).$$

[1] Johann Heinrich Lambert was born on 26 August 1728 in Mülhausen, Alsace; died 25 September 1777 in Berlin, Prussia. Lambert was a colleague of Euler and Lagrange at the Berlin Academy of Sciences. One of his achievements was to first provide a rigorous proof that π is irrational.

Clearly, using an integer order ODE description for the above system may differ significantly to the actual situation. However, the fact that the integer-order dynamic models are more welcome is probably due to the absence of solution methods for Fractional-Order Differential Equations (FODEs). The early attempts to apply fractional-order calculus to dynamic systems control include [23–25]. Recently, some progresses in analysis of dynamic systems modeled by FODEs have been made in [24, 26–32]. It is now accepted that the extra degree of freedom from using of fractional-order integrator and differentiator can further improve the performance of traditional controllers, e.g. [25, 26, 33–36]. For issues related to practical applications, we refer to [37, 38].

In this paper, based on [39], we shall use Lambert W function to get an analytical stability bound for a special class of DFODEs with some extended discussions. The problem formulation is given in Section 2 with a brief introduction of the fractional-order notion. Detailed derivation for the stability bound is given in Section 3. An illustrative example is given in Section 4. Section 5 concludes this paper with some concluding remarks for further investigations.

2. Problem Formulation

2.1. Fractional-Order Differential Equation

The fractional calculus is a generalization of integration and derivation to non-integer order operators [28, 40, 41, 30]. The idea of fractional calculus has been known since the development of the normal calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695. A fundamental operator ${}_a\mathrm{D}_t^r$, a generalization of differential and integral operators, is introduced as follows:

$$
{}_a\mathrm{D}_t^r = \begin{cases} \frac{\mathrm{d}^r}{\mathrm{d}t^r}, & \Re(r) > 0, \\ 1, & \Re(r) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-r}, & \Re(r) < 0, \end{cases}
$$

where r is the fractional order and a is related to the initial condition. There are two commonly used definitions for the general fractional differentiation and integral, i.e., the Grünwald definition and the Riemann–Liouville definition [28, 30, 41]. The Grünwald definition is that

$$
{}_a\mathrm{D}_t^r f(t) = \lim_{h\to 0} \frac{1}{h^r} \sum_{j=0}^{[t-a/h]} (-1)^j \binom{r}{j} f(t-jh), \tag{1}
$$

where $[\cdot]$ is a flooring-operator while the Riemann–Liouville definition is given by

$$
{}_a\mathrm{D}_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} \mathrm{d}\tau, \tag{2}
$$

for $(n-1 < r < n)$ and where $\Gamma(x)$ is the well-known Euler's `gamma` function.

The Laplace transform method is used for solving engineering problems. The formula for the Laplace transform of the Riemann-Liouville fractional derivative (2) has the form [29]:

$$
\int_0^\infty \mathrm{e}^{-pt} \, {}_0\mathrm{D}_t^r f(t) \, \mathrm{d}t = p^r F(p) - \sum_{k=0}^{n-1} p^k \, {}_0\mathrm{D}_t^{r-k-1} f(t)\big|_{t=0}, \tag{3}
$$

for $(n-1 < r \leq n)$.

For numerical calculation of fractional-order derivation we can use the relation (4) derived from the Grünwald definition (1). This relation has the following form:

$$_{(t-L)}\mathrm{D}_t^r f(t) \approx h^{-r} \sum_{j=0}^{N(t)} b_j f(t-jh), \tag{4}$$

where L is the 'memory length', h is the step size of the calculation,

$$N(t) = \min\left\{\left[\frac{t}{h}\right], \left[\frac{L}{h}\right]\right\}, \tag{5}$$

where b_j is the binomial coefficient given by the following recursive formula.

$$b_0 = 1, \quad b_j = \left(1 - \frac{1+r}{j}\right) b_{j-1}. \tag{6}$$

To solve the FODEs, the Laplacian transformation of the Mittag–Leffler function in two parameters was proposed as an effective means [29]. A two-parameter function of the Mittag–Leffler type is defined by the series expansion:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha, \beta > 0). \tag{7}$$

In fact, it is shown [29] that the Mittag-Leffler function is a generalization of exponential function e^z, i.e.,

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \mathrm{e}^z.$$

In general, LTI fractional-order controlled system can be described by

$$\begin{aligned} &a_n \mathrm{D}_t^{\beta_n} y(t) + \cdots + a_1 \mathrm{D}_t^{\beta_1} y(t) + a_0 \mathrm{D}_t^{\beta_0} y(t) \\ &\quad = b_m \mathrm{D}_t^{\alpha_m} u(t) + \cdots + b_1 \mathrm{D}_t^{\alpha_1} u(t) + b_0 \mathrm{D}_t^{\alpha_0} u(t), \end{aligned} \tag{8}$$

where $u(t)$ and $y(t)$ are control and controlled signals respectively; β_k, α_k $(k = 0, 1, 2, \ldots)$ are generally real numbers, $\beta_n > \ldots > \beta_1 > \beta_0$, $\alpha_m > \ldots > \alpha_1 > \alpha_0$ and a_k, b_k $(k = 0, 1, \ldots)$ are arbitrary constants. Note that here $({}_0\mathrm{D}_t^{\mu} \equiv \mathrm{D}_t^{\mu})$.

Using Laplacian transformation method [29, 42, 43], the transfer function of (8) can be written as

$$G_s(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{\sum_{k=0}^{m} b_k (j\omega)^{\alpha_k}}{\sum_{k=0}^{n} a_k (j\omega)^{\beta_k}}. \tag{9}$$

2.2. Stability Problem

In this paper, we consider a simple class of FODE with a constant time delay as follows

$$\frac{\mathrm{d}^r y(t)}{\mathrm{d}t^r} = K_p y(t-\tau), \tag{10}$$

where r and K_p are real numbers; delay τ is a positive constant. We suppose all the initial values are zeros. We are interested in telling whether the system (10) is stable or not for a given set of combination of the three parameters: r, K_p and τ.

Remark 2.1. In a control system using network, the delay τ may be the sum of τ_c, the communication delay for certain protocol and τ_p, the plant delay due to physical transportation delay intrinsic to the system under control. We cannot change τ_p but may be able to adjust τ_c. Therefore, τ may become another design parameter, in addition to K_p, to shape the stability bound.

3. Analytical Stability Bound Using Lambert Function *W*

To study the stability issue for DFODE (10), it is desirable to discuss in the frequency domain.

The Laplacian transformation of DFODE (10) gives the characteristic equation in terms of s ($s = j\omega = \mathrm{d}/\mathrm{d}t$)

$$s^r - K_p\,\mathrm{e}^{-\tau s} = 0. \tag{11}$$

To obtain the stability bound, if possible, directly solving the transcendental equation (11) is the simplest way. This is possible by using the $W(\cdot)$ function discussed above. Multiplying $\mathrm{e}^{\tau s}$ to both sides of (11) gives

$$s^r\,\mathrm{e}^{\tau s} = K_p \tag{12}$$

and putting the r-th root for the both sides of the above yields

$$(s)\,\mathrm{e}^{(\tau/r)s} = \sqrt[r]{K_p} = (K_p)^{1/r}. \tag{13}$$

To use Lambert function, let

$$s_1 = \frac{\tau}{r}s$$

and (13) becomes

$$s_1\,\mathrm{e}^{s_1} = \frac{\tau}{r}(K_p)^{1/r}. \tag{14}$$

Immediately,

$$s_1 = W\left(\frac{\tau}{r}(K_p)^{1/r}\right) \tag{15}$$

and in turn

$$s = \frac{r}{\tau}W\left(\frac{\tau}{r}(K_p)^{1/r}\right) \tag{16}$$

which is what is pursued: the 'analytical stability bound' for DFODE (10). Obviously, the stability condition is that for all possible τ, r and K_p,

$$\frac{r}{\tau}W\left(\frac{\tau}{r}(K_p)^{1/r}\right) \le 0. \tag{17}$$

Remark 3.1. A more general result. We can extend the analytical result to a class of delayed infinite dimensional systems described by DFODEs with the characteristic equation given by

$$(s+\alpha)^r - K_p\,\mathrm{e}^{-\tau s} = 0. \tag{18}$$

Clearly, DFODE (10) is a special case when $\alpha = 0$. To get the analytical stability bound, similarly, let

$$s_2 = s + \alpha$$

and (18) becomes

$$s_2\, \mathrm{e}^{\tau(s_2-\alpha)/r} = (K_p)^{1/r}. \tag{19}$$

Simplifying (19), one gets

$$\left(\frac{\tau}{r}s_2\right)\mathrm{e}^{((\tau/r)s_2)} = \frac{\tau}{r}\mathrm{e}^{((\tau/r)\alpha)}(K_p)^{1/r}. \tag{20}$$

Again, immediately, by using the Lambert W function,

$$s_2 = \frac{r}{\tau}W\left(\frac{\tau}{r}\mathrm{e}^{((\tau/r)\alpha)}(K_p)^{1/r}\right), \tag{21}$$

i.e.,

$$s = \frac{r}{\tau}W\left(\frac{\tau}{r}\mathrm{e}^{(\tau/r)\alpha)}(K_p)^{1/r}\right) - \alpha, \tag{22}$$

and obviously, the stability condition is that for all possible r, τ, α and K_p,

$$\frac{r}{\tau}W\left(\frac{\tau}{r}\mathrm{e}^{((\tau/r)\alpha)}(K_p)^{1/r}\right) - \alpha \le 0. \tag{23}$$

Remark 3.2. A special case. Let us examine the case when $\tau = 0$, i.e., there is no delay. From (23),

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{r}{\tau}W\left(\frac{\tau}{r}\,\mathrm{e}^{((\tau/r)\alpha)}(K_p)^{1/r}\right)\\
&\quad = \lim_{\tau\to 0} 2W'\left(\frac{\tau}{r}\mathrm{e}^{((\tau/r)\alpha)}(K_p)^{1/r}\right)\left(\frac{(K_p)^{1/r}}{r}\right)\mathrm{e}^{((\tau/r)\alpha)}\left(1+\frac{\alpha}{r}\tau\right)\\
&\quad = W'(0)((K_p)^{1/r}) = (K_p)^{1/r},
\end{aligned} \tag{24}$$

where

$$W'(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{W(x)}{x(1+W(x))}, \quad W(0)=0 \quad \text{and} \quad W'(0)=1.$$

Therefore, the stability bound (23) regenerates to

$$(K_p)^{1/r} - \alpha \le 0. \tag{25}$$

Is easy to check that when $r = 1$, the above condition is in accordance with the classical stability result.

4. An Illustrative Example

We can get a 3D plot for $s(\tau, K_p)$ for any given α and r. Here we present an example to illustrate the obtained analytical stability bound results.

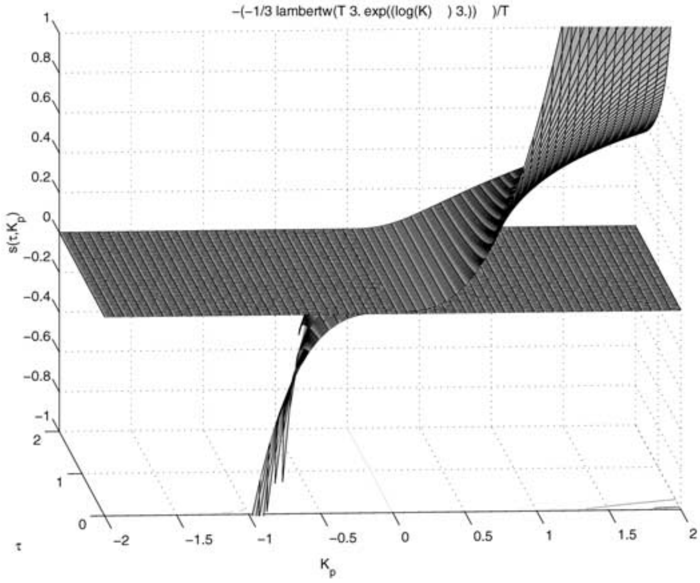

Figure 1. The stability bound when $\alpha = 0$ and $r = 1/3$.

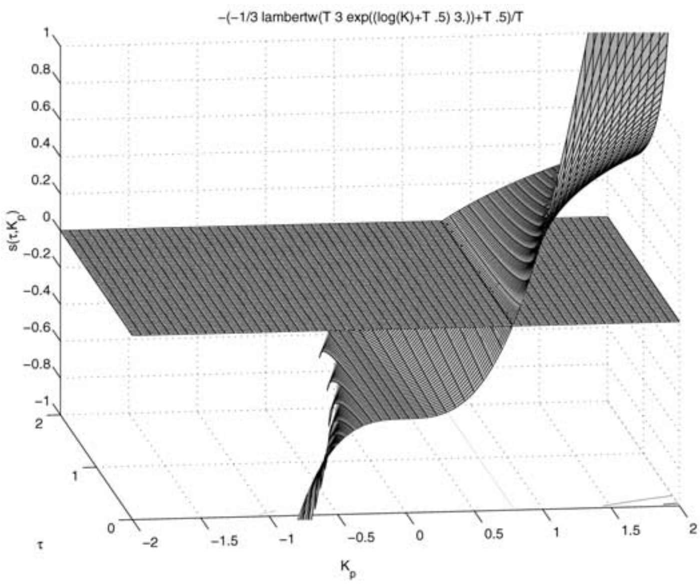

Figure 2. The stability bound when $\alpha = 0.5$ and $r = 1/3$.

Consider $r = 1/3$. In Figure 1, where $\alpha = 0$ and $r = 0.5$, the stability bound can be clearly seen from the intersection of the surface $s(\tau, K_p)$ and surface $s = 0$. Similarly, the stability bound is obtained in Figure 2 for $\alpha = 0.5$ and $r = 0.5$. Comparing Figures 1 and 2, we can understand that that time-delay in FODE can destabilize the system. It is also interesting to find that for K_p less than 0, the DFODE is always stable. For $\alpha \neq 0$, the stability bound is enlarged when comparing Figures 1 and 2.

In practice, we are more interested in determining a suitable range for K_p, given an α and a range for possible delays $[\tau_{\min}, \tau_{\max}]$. Figure 3 shows a sample run using the obtained

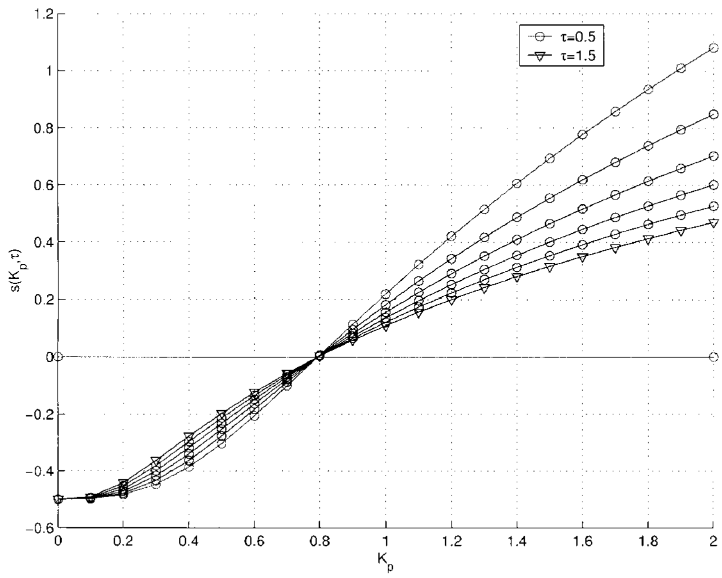

Figure 3. The stabilizing range for K_p when $r = 1/3$, $\alpha = 0.5$ and $\tau \in [0.5, 1.5]$.

analytical stability bound. In Figure 3, black line with triangles corresponds to the case for $\tau_{\max} = 1.5$ while red line with circles for $\tau_{\min} = 0.5$ Note that the *minimum* (with surprise) K_p is restricted by $\tau_{\max}$. This is true when a stability margin is to be considered.

5. Concluding Remarks

In this paper, by using Lambert function W, an analytical stability bound has been obtained for a class of delayed fractional-order differential equations with constant coefficients. An example is given for illustration.

For further investigations, first of all, the branches of Lambert function should be included into the analytical bound. More complex forms of DFODE's should be explored when applying Lambert function technique. Moreover, mixed numerical and analytical schemes can be considered.

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