Analytic and Numerical Computation of Stability Bound for A Class of Linear Delay Differential Equations Using Lambert Function

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Outline of the presentation

- **Motivation**
- Lambert Function
- Analytical Solution
- Numerical Solution
- Concluding Remarks

1. Motivation

Consider the dynamical control system

$$\frac{d^2y(t)}{dt^2} + a_m\frac{dy(t)}{dt} + b_m y(t) = u(t) \tag{1}$$

where $u(t)$ is the control signal while the output is $y(t)$. This may be a physical one-mass system under exogenous acting force. The position $y(t)$ may not be available immediately. There is a (lumped) time delay τ. This is the case when a communication network is used in the control system such as the web-based manipulation of the mass position. Suppose the control law used is a simple proportional control, i.e.,

$$u(t) = K_p y(t - \tau), \tag{2}$$

where K_p is the proportional controller gain

Then, the system (1) under controller (2) is actually governed by a second-order delayed differential equation (DDE)

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a_m \frac{\mathrm{d}y(t)}{\mathrm{d}t} + b_m y(t) - K_p y(t-\tau) = 0. \tag{3}$$

Question:

How parameters a_m, b_m, K_p and τ affect the overall stability property of DDE (3). More specifically, for a given system parameters a_m, b_m, and τ, how to determine the stability bound on controller gain K_p?

Remark 1.1 *Here the delay τ may be the sum of τ_c, the communication delay for certain protocol and τ_p, the plant delay due to physical transportation delay intrinsic to the system under control. We cannot change τ_p but may be able to adjust τ_c. Therefore, τ may become a design parameter to shape the stability bound.*

Under zero initial conditions, the Laplacian representation of (3) is

$$s^2 + a_m s + b_m - K_p e^{-\tau s} = 0 \tag{4}$$

(4) is a transcendental algebraic equation on s. No symbolic solution exists to our best knowledge. However, in this paper, we are able to obtain an analytical solution for a special case of (4) by assuming that the poles in system (1) are repeating. That is, when assuming $b_m = a_m^2/4 \triangleq \alpha^2$, (4) becomes

$$(s + \alpha)^2 - K_p e^{-\tau s} = 0. \tag{5}$$

From (5), in the next section, as a special case, an analytical solution for s in terms of symbolic α, K_p and τ can be obtained by using Lambert function W.

In this paper, we are interested in obtaining an analytical stability bound for higher order DDE with multiple repeating poles, i.e., for

$$(s + \alpha)^n - K_p e^{-\tau s} = 0. \tag{6}$$

Remark 1.2 *In many process control systems such as mixing systems of noninteracting series structure , the open-loop transfer function can be given by*

$$G(s) = \frac{K e^{-\tau s}}{(s + \alpha)^n}. \tag{7}$$

Its closed-loop characteristic equation under proportional control with gain K_p/K is the same as (6). Therefore, the stability bound analytically given in this paper will be practically useful in some control applications.

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2. Lambert Function

Johann Heinrich Lambert was born on 1728 Aug 26 in Mlhausen, Alsace; died 1777 Sep 25 in Berlin, Prussia. Lambert was a colleague of Euler and Lagrange at the Berlin Academy of Sciences. One of his achievements was to first provide a rigorous proof that π is irrational.

The Lambert function $W(x)$ satisfies

$$W(x)e^{W(x)} = x.$$

- Euler noticed the importance of Lambert function $W(x)$ in 1777.
- The Lambert function got its name during the implementation of Maple in 1990s, a symbolic computation system.
- Many interesting applications. For example, a simple DDE (delayed differential equation)

$$\dot{y}(t) = ay(t-1), \quad s = ae^{-s},$$

immediately

$$s = W_k(a)$$

where k is the index for branches. For a with a real value, only the principal branch $W_0(a)$, or simply $W(a)$, is to be considered.

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To obtain the stability bound, if possible, directly solving the transcendental equation (5) is the simplest way. This is possible by using the $W(\cdot)$ function discussed in the previous section. Multiplying $e^{\tau s}$ to both side of (5) gives

$$(s+\alpha)^n e^{\tau s} = K_p \tag{8}$$

and putting the n-th root for the both sides of the above yields

$$(s+\alpha)e^{\frac{\tau}{n}s} = \sqrt[n]{K_p}. \tag{9}$$

Now, let

$$s_1 = s + \alpha \tag{10}$$

and (9) becomes

$$s_1 e^{\frac{\tau}{n}(s_1-\alpha)} = \sqrt[n]{K_p}. \tag{11}$$

Simplifying (11), one gets

$$(\frac{\tau}{n}s_1)e^{(\frac{\tau}{n}s_1)} = \frac{\tau}{n}e^{(\frac{\tau}{n}\alpha)}(\sqrt[n]{K_p}). \tag{12}$$

Immediately, by using the Lambert W function,

$$s_1 = \frac{n}{\tau} W(\frac{\tau}{n} e^{(\frac{\tau}{n}\alpha)} (\sqrt[n]{K_p})), \tag{13}$$

i.e.,

$$s = \frac{n}{\tau} W(\frac{\tau}{n} e^{(\frac{\tau}{n}\alpha)} (\sqrt[n]{K_p})) - \alpha, \tag{14}$$

which is what we called the "analytical stability bound" for DDE (6). Obviously, the stability condition is that for all possible τ, α and K_p,

$$\frac{n}{\tau} W(\frac{\tau}{n} e^{(\frac{\tau}{n}\alpha)} (\sqrt[n]{K_p})) - \alpha \leq 0. \tag{15}$$

When $\tau = 0$, i.e., there is no delay, we have

$$\begin{aligned} \lim_{\tau\to 0} \frac{n}{\tau} W(\frac{\tau}{n} e^{(\frac{\tau}{n}\alpha)} (\sqrt[n]{K_p})) \;&=\; \\ &\lim_{\tau\to 0} nW'(\frac{\tau}{n} e^{(\frac{\tau}{n}\alpha)} (\sqrt[n]{K_p}))(\frac{\sqrt[n]{K_p}}{n}) e^{(\frac{\tau}{n}\alpha)} (1 + \frac{\alpha}{n}\tau) \\ &= W'(0)(\sqrt[n]{K_p}) = \sqrt[n]{K_p}, \end{aligned} \tag{16}$$

where $W'(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{W(x)}{x(1+W(x))}$, $W(0) = 0$ and $W'(0) = 1$. Therefore, the stability bound (15) for DDE with zero delay is that

$$\sqrt[n]{K_p} - \alpha \leq 0, \tag{17}$$

which is in accordance with the classical stability result.

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3. Numerical Schemes for The Case of Distinct Poles

3.1. Simple Cases With No Analytic Solution Using Lambert Function

In this section, we focus on the characteristic equation (4). In general, we can re-write (4) in the following form

$$(s+\alpha)(s+\beta) - K_p e^{-\tau s} = 0, \tag{18}$$

where $\beta \neq \alpha$, $a_m = \alpha + \beta$, and $b_m = \alpha\beta$. It is an open problem to obtain a similar analytical solution as discussed in the previous section for (18) by using the Lambert function.

We will show that, even under simplified conditions, the analytical solutions may not be possible. For example, by setting $\beta = 0$, (18) becomes

$$s(s+\alpha)e^{\tau s} = K_p. \tag{19}$$

This is the form of a typical motion control system with its position loop closed via a delayed proportional feedback. In this case, the output is the position and the plant transfer function is $\frac{1}{s(s+\alpha)}$. We comment that although it looks easy, the analytical solution for (19) by using Lambert function is not possible. Another simplified example is to consider the velocity as the output signal with the plant transfer function $\frac{1}{(s+\alpha)}$.

If we consider the P-controller (delayed velocity feedback), the characteristic equation is in a very simple form as follows:

$$(s+\alpha)e^{\tau s} = K_v. \tag{20}$$

In this case, the analytic solution is possible using the formula in Sec. **??**. However, if a PI-controller (delayed velocity and position feedback) is used, the characteristic equation will be in the following form:

$$(s+\alpha)e^{\tau s} = K_p/s + K_v. \tag{21}$$

In this case, the analytical solution for (21) by using the Lambert function is not possible, too. In fact, it is also not possible to get an analytical solution with the Lambert function for the following equation:

$$xe^{x} + ax = b. \tag{22}$$

We remark that for a first-order plant with a PD-controller plus a delay, its characteristic equation can be transformed into the form of (22). In what follows, we will introduce some numerical schemes for the solution of the transcendental equations such as (22) with the application of the Lambert function.

3.2. Numerical Schemes

3.2.1. Direct Iteration

Let us re-write (22) as

$$\frac{1}{b}xe^x + \frac{a}{b}x = 1. \tag{23}$$

Let $\frac{1}{b}xe^x = \alpha_k$. Then, $\frac{a}{b}x = 1 - \alpha_k$. We can use the following iteration to get a fixed-point on α_k. A good starting guess is $\alpha_0 = 0.5$.

```
% xexp(x)+ax=b via Lambert function
a=1; b=1; akm1=0.5; wind=0;
iexit=1; eps_exit=1e-15;
while iexit==1
    ak=1-lambertw(wind,akm1*b) *a/b;
    if abs(abs(ak)-abs(akm1)) < eps_exit
        iexit=0;
    else
        akm1=ak
    end
end
% so, final result is
x1=lambertw(wind,ak*b)
x2=(1-ak)*b/a
```

For this example, we get the solution $x = 0.4011$ for the branch 0 (i.e. `wind=0`), $x = 0.0379+3.4264i$ for the branch 1, and $x = 0.0379-3.4264i$ for the branch -1.

3.2.2. Newton Iteration

Equation (22) can be written in terms of the Lambert function as follows:

$$f(x) = x - W(b - ax) = 0. \tag{24}$$

The Newton iteration formula will be

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})} \tag{25}$$

which is implemented in the following

```
% xexp(x)+ax=b via Lambert function
% Newton Iteration
a=1; b=1; akm1=0.5; wind=1;
iexit=1; eps_exit=1e-15;
while iexit==1
    lfk=lambertw(wind,b-akm1*a);
    fk=akm1-lfk;
    dfk=1+a*lfk/(1+lfk)/(b-akm1*a);
    ak=akm1-fk/dfk;
    if abs(abs(ak)-abs(akm1))<eps_exit
        iexit=0;
    else
        akm1=ak
    end
end
```

3.2.3. Accelerated Newton Iteration

The third-order convergent Newton iteration formula is given by

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{(f'(x_k))^2 - \frac{1}{m}f(x_k)f''(x_k)} \tag{26}$$

where m is the smallest integer such that $f^{(m)}(x^*) \neq 0$ with x^* the true solution. m is usually set to 2. The implementation is straightforward.

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4. Concluding Remarks

In this paper, an analytical stability bound has been obtained for a class of high order delay differential equations with constant coefficients. When the poles are not identical, it is shown that even in some simple situations, the analytical results may not be possible. Some simple numerical procedures are introduced with the use of the Lambert function. Examples are presented for illustration.

The following are some of the issues, possibly open problems, currently under investigation for analytical or semi-analytical (involving Lambert function for numerical iteration as in Sec. 3) solutions.

- For multiple delays, how to benefit from using the Lambert function? This is currently an open problem.
- For single but random delay with known upper and lower bounds or known probability density function (PDF) on the delay magnitude. This is particular interesting for control systems involving networking communication delay which might be random but with a known range or distribution.

Thank you!

Q/A session