Proceedings of DETC'2003
2003 ASME Design Engineering Technical Conferences
September 2-6, 2003, Chicago, Illinois, USA

DETC2003/VIB-48391

A NEW DISCRETIZATION METHOD FOR FRACTIONAL ORDER DIFFERENTIATORS VIA CONTINUED FRACTION EXPANSION

YangQuan Chen *
CSOIS, Department of Electrical and Computer Engineering,
Utah State University
4120 Old Main Hill,
Logan, Utah 84322-4120, USA
Email: yqchen@ece.usu.edu

Blas M. Vinagre
Department of Electronic and Electromechanical Engineering
Industrial Engineering School,
University of Extramadura, Avda. De Elvas s/n, 06071-Badajoz, Spain
Email: bvinagre@unex.es

Igor Podlubny
Department of Informatics and Process Control, BERG Faculty,
Technical University of Kosice,
B. Nemcovej 3, 042 00 Kosice,
Slovak Republic
Email: Igor.Podlubny@tuke.sk

ABSTRACT

In this contribution, to discretize the fractional order differentiators in continuous time domain, a new IIR (infinite impulse response) type digital fractional order differentiator (DFOD) is proposed by using a new family of first order digital differentiators expressed in the second order IIR filter form. The integer first order digital differentiators are obtained by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule. The distinguishing point of the proposed DFOD lies in an additional tuning knob to compromise the high frequency approximation accuracy.

Key words: Fractional differentiator, fractional-order dynamic systems, fractional-order differentiator, discretization, Tustin operator, Al-Alaoui operator, recursive.

INTRODUCTION

Fractional calculus is a 300-years-old topic. The theory of fractional-order derivative was developed mainly in the 19-th century. Recent books (Oldham and Spanier, 1974; Samko *et al.*, 1987; Miller and Ross, 1993; Podlubny, 1994) provide a good source of references on fractional calculus. However, applying fractional-order calculus to dynamic systems control is just a recent focus of interest (Lurie, 1994; Podlubny, 1999; Oustaloup *et al.*, 1995, 1996; Raynaud and ZergaInoh, 2000). For pioneering works, we cite (Manabe, 1960, 1961; Oustaloup, 1981; Axtell and Bise, 1990).

In theory, the control systems can include both the fractional order dynamic system or plant to be controlled and the fractional-order controller (FOC). However, in control practice, it is more common to consider the fractional-order controller. This is due to the fact that the plant model may have already been obtained as an integer order model in the classical sense. In most cases, our objective is to apply fractional order control to enhance the system control performance. For example, as in the CRONE [1] control (Oustaloup, 1995; Oustaloup *et al.*, 1995, 1996), *fractal robustness* is pursued. The desired frequency template leads to fractional transmittance (Oustaloup *et al.*, 1999; Oustaloup and Mathieu, 1999) on which the CRONE controller synthesis is based. In the CRONE controller, the major ingredient is the fractional-order derivative s^r, where r is a real number and s is the Laplacian operator. Another example is the $PI^{\lambda}D^{\mu}$ controller (Podlubny, 1999; Petráš, 1999), an extension of PID controller. In general form, the transfer function of $PI^{\lambda}D^{\mu}$ is given by $K_p + T_i s^{-\lambda} + T_d s^{\mu}$, where λ and μ are positive real numbers; K_p is the proportional gain, T_i the integration constant and T_d the differentiation constant. Clearly, taking $\lambda = 1$ and $\mu = 1$, we

*Corresponding author. Center for Self-Organizing and Intelligent Systems (CSOIS), UMC 4160, College of Engineering, Utah State University, Logan, Utah 84322-4160, USA. Tel. 1(435)797-0148; Fax: 1(435)797-3054. URL: http://www.csois.usu.edu/

[1] CRONE is a French abbreviation for "*Contrôle Robuste d'Ordre Non Entier*" (which means non-integer order robust control).

obtain a classical PID controller. If $\lambda = 0$ $(T_i = 0)$ we obtain a PD^μ controller, etc. All these types of controllers are particular cases of the $PI^\lambda D^\mu$ controller. It can be expected that the $PI^\lambda D^\mu$ controller may enhance the systems control performance due to more tuning knobs introduced. Actually, in theory, $PI^\lambda D^\mu$ itself is an infinite dimensional linear filter due to the fractional order in the differentiator or integrator. It should be pointed out that a band-limit implementation of FOC is important in practice, i.e., the finite dimensional approximation of the FOC should be done in a proper range of frequencies of practical interest (Oustaloup *et al.*, 2000; Oustaloup and Mathieu, 1999). Moreover, the fractional order can be a complex number as discussed in (Oustaloup *et al.*, 2000). In this paper, we focus on the case when the fractional order is a real number.

The key step in digital implementation of a FOC is the numerical evaluation or discretization of the fractional-order differentiator s^r. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods (Oustaloup *et al.*, 2000), two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit *s*-transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization. Existing *direct discretization* methods include the application of the direct power series expansion (PSE) of the Euler operator (Machado, 1997; Vinagre *et al.*, 2001, 2000a,b), continuous fractional expansion (CFE) of the Tustin operator (Vinagre *et al.*, 2001, 2000a,b; Chen and Moore, 2002), and numerical integration based method (Machado, 1997; Chen and Moore, 2002). However, as pointed out in (Al-Alaoui, 1993, 1995, 1997), the Tustin operator based discretization scheme exhibits large errors in high frequency range. A new mixed scheme of Euler and Tustin operators is proposed in (Chen and Moore, 2002) which yields the Al-Alaoui operator (Al-Alaoui, 1993).

The above discretization methods for s^r lead naturally to the DFOD's usually in IIR form. Recently, there are some methods to directly obtain the DFOD's in FIR (finite impulse response) form (Tseng *et al.*, 2000; Tseng, 2001). However, using an FIR filter to approximate s^r may be less efficient due to the very high order of the FIR filter. In this paper, a new IIR (infinite impulse response) type digital fractional order differentiator (FOD) is proposed by using a new family of first order digital differentiators expressed in the second order IIR filter form. The integer first order digital differentiators are obtained by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule (Al-Alaoui, 1995). The distinguishing point of the proposed FOD lies in an additional tuning knob to compromise the high frequency approximation accuracy.

FRACTIONAL-ORDER DERIVATIVE AND ITS DISCRETIZATION

Fractional calculus is a generalization of integration and differentiation to non-integer (fractional) order fundamental operator ${}_aD_t^r$, where a and t are the limits and r, $(r \in \mathbb{R})$ the order of the operation. The two definitions used for the general fractional integrodifferential are the Grünwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition (Oldham and Spanier, 1974), (Podlubny, 1994). The GL definition is that

$$ {}_aD_t^r f(t) = \lim_{h\to 0} h^{-r} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{r}{j} f(t-jh) \tag{1} $$

where $[\cdot]$ means the integer part while the RL definition

$$ {}_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} d\tau \tag{2} $$

for $(n-1 < r < n)$ and where $\Gamma(\cdot)$ is the Euler's *gamma* function.

For convenience, Laplace domain notion is usually used to describe the fractional integro-differential operation(Podlubny, 1994). The Laplace transform of the RL fractional derivative/integral (2) under zero initial conditions for order r, $(0 < r < 1)$ is given by (Oldham and Spanier, 1974):

$$ \pounds\{{}_aD_t^{\pm r} f(t); s\} = s^{\pm r} F(s) \tag{3} $$

The key point in digital implementation of a FOC is the numerical evaluation or discretization of the fractional-order differentiator. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods, two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit *s*-transfer function. In this paper, we focus on the *direct discretization* method.

The simplest and most straightforward method is the direct discretization using finite memory length expansion from GL definition (1). This approach is based on the fact that, for a wide class of functions, the two definitions - GL (1) and RL (2) - are equivalent (Podlubny, 1994). In general, the discretization of fractional-order differentiator/integrator $s^{\pm r}$, $(r \in \mathbb{R})$ can be expressed by the so-called generating function $s = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients (Lubich, 1986). For example, when a backward difference rule is used, i.e., $\omega(z^{-1}) = (1-z^{-1})/T$, performing the PSE of $(1-z^{-1})^{\pm r}$ gives the discretization formula for GL formula (1). By using the short memory principle (Podlubny, 1994), the discrete equivalent of the fractional-order integro-differential operator, $(\omega(z^{-1}))^{\pm r}$, is given by

$$(\omega(z^{-1}))^{\pm r} = T^{\mp r} z^{-[L/T]} \sum_{j=0}^{[L/T]} (-1)^j \binom{\pm r}{j} z^{[L/T]-j} \qquad (4)$$

where T is the sampling period, L is the memory length and $(-1)^j \binom{\pm r}{j}$ are binomial coefficients $c_j^{(r)}$, $(j = 0, 1, \ldots)$ where

$$c_0^{(r)} = 1, \qquad c_j^{(r)} = \left(1 - \frac{1 + (\pm r)}{j}\right) c_{j-1}^{(r)}. \qquad (5)$$

It is very important to note that PSE scheme leads to approximations in the form of polynomials, that is, the discretized fractional order derivative is in the form of FIR filters. Taking into account that our aim is to obtain discrete equivalents to the fractional integrodifferential operators in the Laplace domain, $s^{\pm r}$, the following considerations have to be made:

1. s^r, $(0 < r < 1)$, viewed as an operator, has a branch cut along the negative real axis for arguments of s on $(-\pi, \pi)$ but is free of poles and zeros.
2. A dense interlacing of simple poles and zeros along a line in the s plane is, in some way, equivalent to a branch cut.
3. It is well known that, for interpolation or evaluation purposes, rational functions are sometimes superior to polynomials, roughly speaking, because of their ability to model functions with zeros and poles. In other words, for evaluation purposes, rational approximations frequently converge much more rapidly than PSE and have a wider domain of convergence in the complex plane.
4. Trapezoidal rule maps adequately the stability regions of the s plane on the z plane, and maps the points $s = 0, s = -\infty$ to the points $z = 1$ and $z = -1$ respectively.

THE CONCEPT OF GENERATING FUNCTION

In general, the discretization of the fractional-order differentiator s^r (r is a real number) can be expressed by the so-called generating function $s = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients (Lubich, 1986). For example, as shown in the last section, when a backward difference rule is used, i.e., $\omega(z^{-1}) = (1 - z^{-1})/T$ with T the sampling period, performing the power series expansion (PSE) of $(1 - z^{-1})^{\pm r}$ gives the discretization formula which is actually in FIR filter form (Machado, 1997; Vinagre *et al.*, 2001). In (Vinagre *et al.*, 2000b; Chen and Moore, 2002), the trapezoidal (Tustin) rule is used as a generating function

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{\pm r}. \qquad (6)$$

The DFOD is then obtained by using the CFE (Vinagre *et al.*, 2000b) or a new recursive expansion formula (Chen and Moore, 2002). It is interesting to note that in (Chen and Moore, 2002), the so-called Al-Alaoui operator is used which is a mixed scheme of Euler and Tustin operators (Al-Alaoui, 1993). Correspondingly, the generating function for discretization is

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{8}{7T}\frac{1 - z^{-1}}{1 + z^{-1}/7}\right)^{\pm r}. \qquad (7)$$

Clearly, (7) is an infinite order of rational discrete-time transfer function. To approximate it with a finite order rational one, continued fraction expansion (CFE) is an efficient way. In general, any function $G(z)$ can be represented by continued fractions in the form of

$$G(z) \simeq a_0(z) + \cfrac{b_1(z)}{a_1(z) + \cfrac{b_2(z)}{a_2(z) + \cfrac{b_3(z)}{a_3(z) + \ldots}}} \qquad (8)$$

where the coefficients a_i and b_i are either rational functions of the variable z or constants. By truncation, an approximate rational function, $\widehat{G}(z)$, can be obtained.

The basic idea of this paper is to define a new generating function by applying a new family of first order digital differentiators expressed in the second order IIR filter form (Al-Alaoui, 1995) via the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule.

INTEGER ORDER IIR TYPE DIGITAL INTEGRATOR

It was pointed out in (Al-Alaoui, 1993, 1997) that the magnitude of the frequency response of the ideal integrator $1/s$ lies between that of the Simpson and trapezoidal digital integrators. It is reasonable to "interpolate" the Simpson and trapezoidal digital integrators to compromise the high frequency accuracy in frequency response. This leads to the following hybrid digital integrator

$$H(z) = aH_S(z) + (1 - a)H_T(z), \quad a \in [0, 1] \qquad (9)$$

where a is actually a weighting factor or tuning knob. $H_S(z)$ and $H_T(z)$ are the z-transfer functions of the Simpson's and the trapezoidal integrators given respectively as follows:

$$H_S(z) = \frac{T(z^2+4z+1)}{3(z^2-1)} \tag{10}$$

and

$$H_T(z) = \frac{T(z+1)}{2(z-1)}. \tag{11}$$

The overall weighted digital integrator with the tuning parameter a is hence given by

$$\begin{aligned} H(z) &= \frac{T(3-a)\{z^2+[2(3+a)/(3-a)]z+1\}}{6(z^2-1)} \\ &= \frac{T(3-a)(z+r_1)(z+r_2)}{6(z^2-1)} \end{aligned} \tag{12}$$

where

$$r_1 = \frac{3+a+2\sqrt{3a}}{3-a}, \quad r_2 = \frac{3+a-2\sqrt{3a}}{3-a}.$$

It is interesting to note the fact that $r_1 = \frac{1}{r_2}$ and $r_1 = r_2 = 1$ only when $a = 0$ (trapezoidal). For $a \neq 0$, $H(z)$ must have one non-minimum phase (NMP) zero.

NEW DISCRETIZATION SCHEME: IIR TYPE FRACTIONAL ORDER DIGITAL DIFFERENTIATOR

Firstly, we can obtain a family of new integer order digital differentiators from the digital integrators introduced in the last section. Direct inversion of $H(z)$ will give an unstable filter since $H(z)$ has an NMP zero r_1. By reflecting the NMP r_1 to $1/r_1$, i.e. r_2, we have

$$\tilde{H}(z) = K\frac{T(3-a)(z+r_2)^2}{6(z^2-1)}.$$

To determine K, let the final value of the impulse responses of $H(z)$ and $\tilde{H}(z)$ be the same, i.e., $\lim_{z\to 1}(z-1)H(z) = \lim_{z\to 1}(z-1)\tilde{H}(z)$, which gives $K = r_1$. Therefore, the new family of the digital differentiators are given by

$$\omega(z) = \frac{1}{\tilde{H}(z)} = \frac{6(z^2-1)}{r_1 T(3-a)(z+r_2)^2} = \frac{6r_2(z^2-1)}{T(3-a)(z+r_2)^2}. \tag{13}$$

We can regard $\omega(z)$ in (13) as the generating function introduced in the last section. Finally, we can obtain the expression for the DFOD as

$$G(z^{-1}) = \left(\omega(z^{-1})\right)^r = k_0\left(\frac{1-z^{-2}}{(1+bz^{-1})^2}\right)^r \tag{14}$$

where $r \in [0,1]$, $k_0 = (\frac{6r_2}{T(3-a)})^r$ and $b = r_2$.

It is well known that, compared to the power series expansion method, the Continued Fraction Expansion (CFE) is a method for evaluation of functions with faster convergence and larger domain of convergence in the complex plane. Using CFE, an approximation for a irrational function $G(z^{-1})$ can be expressed in the form of (8). Similar to (7), here, the irrational transfer function $G(z^{-1})$ in (14) can be expressed by an infinite order of rational discrete-time transfer function by CFE method as shown in (8).

The CFE expansion can be automated by using a symbolic computation tool such as the MATLAB Symbolic Toolbox. To illustrate, let us denote $x = z^{-1}$. Referring to (14), the task is to perform the following expansion:

$$\text{CFE}\left(\frac{1-x^2}{(1+bx)^2}\right)^r$$

to the desired order n. The following MATLAB script will generate the above CFE with `p1` and `q1` containing, respectively, the numerator and denominator polynomials in `x` or z^{-1} with their coefficients being functions of `b` and `r`.

```
clear all;close all;syms x r b;maple('with(numtheory)');
aas = ( (1-x*x)/(1+b*x)^2  )^r;  n=3; n2=2*n;
maple(['cfe := cfrac(' char(aas) ',x,n2);']);
pq=maple('P_over_Q := nthconver','cfe',n2);
p0=maple('P := nthnumer','cfe',n2);
q0=maple('Q := nthdenom','cfe',n2);
p=(p0(5:length(p0))); q=(q0(5:length(q0)));
p1=collect(sym(p),x); q1=collect(sym(q),x);
```

ILLUSTRATIVE EXAMPLES

Here we present some results for $r = 0.5$. The values of the truncation order n and the weighting factor a are denoted as subscripts of $G_{(n,a)}(z)$. Let $T = 0.001$sec. We have the following:

$$\begin{aligned} G_{(2,0.00)}(z^{-1}) &= \frac{178.9-89.44z^-1-44.72z^-2}{4+2z^-1-z^-2} \\ G_{(2,0.25)}(z^{-1}) &= \frac{138.8+98.07z^{-1}-158.2z^{-2}}{4+5.034z^{-1}-z^{-2}} \\ G_{(2,0.50)}(z^{-1}) &= \frac{127+41.26z^{-1}-112.6z^{-2}}{4+2.98z^{-1}-z^{-2}} \end{aligned}$$

$$G_{(2,0.75)}(z^{-1}) = \frac{119.3+25.56z^{-1}-97.96z^{-2}}{4+2.19z^{-1}-z^{-2}}$$
$$G_{(2,1.00)}(z^{-1}) = \frac{113.4+17.74z^{-1}-89.81z^{-2}}{4+1.698z^{-1}-z^{-2}} \tag{15}$$

$$G_{(3,0.00)}(z^{-1}) = \frac{357.8-178.9z^{-1}-178.9z^{-2}+44.72z^{-3}}{8+4z^{-1}-4z^{-2}-z^{-3}}$$
$$G_{(3,0.25)}(z^{-1}) = \frac{392.9-78.04z^{-1}-349.8z^{-2}+88.97z^{-3}}{11.32+4z^{-1}-5.66z^{-2}-z^{-3}}$$
$$G_{(3,0.50)}(z^{-1}) = \frac{1501-503.6z^{-1}-1289z^{-2}+446.5z^{-3}}{47.26+4z^{-1}-23.63z^{-2}-z^{-3}}$$
$$G_{(3,0.75)}(z^{-1}) = \frac{968.1-442z^{-1}-820.8z^{-2}+363z^{-3}}{32.47-4z^{-1}-16.24z^{-2}+z^{-3}}$$
$$G_{(3,1.00)}(z^{-1}) = \frac{353.1-208z^{-1}-297.4z^{-2}+164.7z^{-3}}{12.46-4z^{-1}-6.228z^{-2}+z^{-3}} \tag{16}$$

$$G_{(4,0.00)}(z^{-1}) = \frac{715.5-357.8z^{-1}-536.7z^{-2}+178.9z^{-3}+44.72z^{-4}}{16+8z^{-1}-12z^{-2}-4z^{-3}+z^{-4}}$$
$$G_{(4,0.25)}(z^{-1}) = \frac{555.3-392.9z^{-1}-477.2z^{-2}+349.8z^{-3}-19.56z^{-4}}{16-2.489z^{-1}-12z^{-2}+1.245z^{-3}+z^{-4}}$$
$$G_{(4,0.50)}(z^{-1}) = \frac{508.1-1501z^{-1}-4.478z^{-2}+1289z^{-3}-382.9z^{-4}}{16-40.54z^{-1}-12z^{-2}+20.27z^{-3}+z^{-4}}$$
$$G_{(4,0.75)}(z^{-1}) = \frac{477+968.1z^{-1}-919z^{-2}-820.8z^{-3}+422.7z^{-4}}{16+37.8z^{-1}-12z^{-2}-18.9z^{-3}+z^{-4}}$$
$$G_{(4,1.00)}(z^{-1}) = \frac{453.6+353.1z^{-1}-661.7z^{-2}-297.4z^{-3}+221.5z^{-4}}{16+16.74z^{-1}-12z^{-2}-8.371z^{-3}+z^{-4}} \tag{17}$$

The Bode plot comparisons for the above three groups of approximate fractional order digital differentiators are summarized in Figure 1, Figure 2 and Figure 3 respectively. We can clearly observe the improvement in high frequency magnitude response. If trapezoidal scheme is used, the high frequency magnitude response is far from the ideal one. The role of the tuning knob a is obviously useful in some applications. MATLAB code for this new DFOD is available upon request.

For $n=3$ and $n=4$, the pole-zero maps are shown respectively in Figure 4 and Figure 5 with some different values of a. First of all, we observe that there is no complex conjugate poles or zeros. We can further observe that for odd order of CFE ($n=3$), the pole-zero maps are nicely behaved, that is, all the poles and zeros lie inside the unit circle and the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1,1)$. However, when n is even, and when a is near to 1, there may have one cancelling pole-zero pair as seen in Figure 5 which may not be desirable. We suggest to use an odd n when applying the new discretization scheme of this paper. As can be seen in the next section, when $a=0$, the pole-zero map is always inside the unit circle in an interlacing way along the segment of the real axis corresponding to $z \in (-1,1)$.

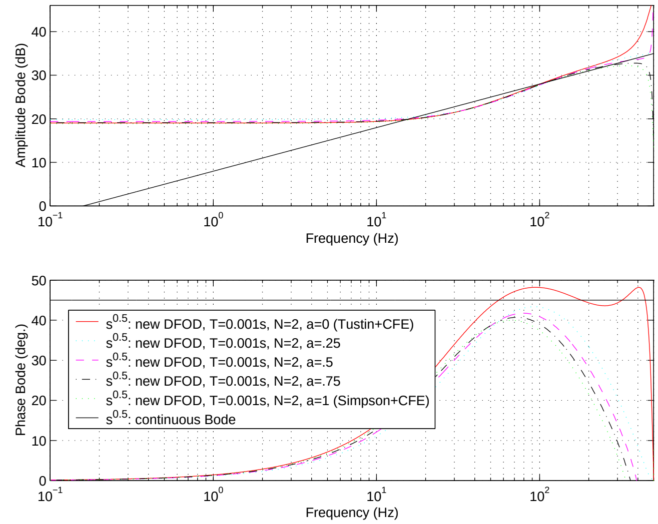

Figure 1. Bode plot comparison for $r=0.5$, $n=2$ and $a=0,.25,.5,.75,1$

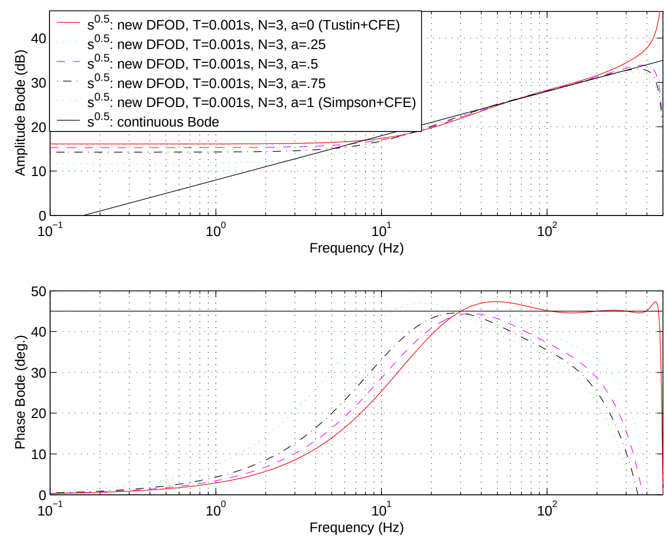

Figure 2. Bode plot comparison for $r=0.5$, $n=3$ and $a=0,.25,.5,.75,1$

THE SPECIAL CASE: TUSTIN OPERATOR ALONE

Consider the special case for $a=0$ in the general scheme(14). This is the CFE of Tustin operator. Let the resulting discrete transfer function, approximating fractional-order opera-

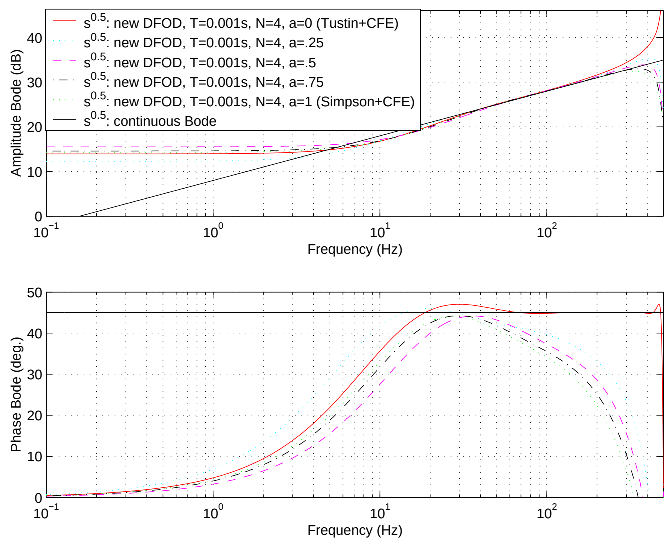

Figure 3. Bode plot comparison for $r = 0.5$, $n = 4$ and $a = 0, .25, .5, .75, 1$

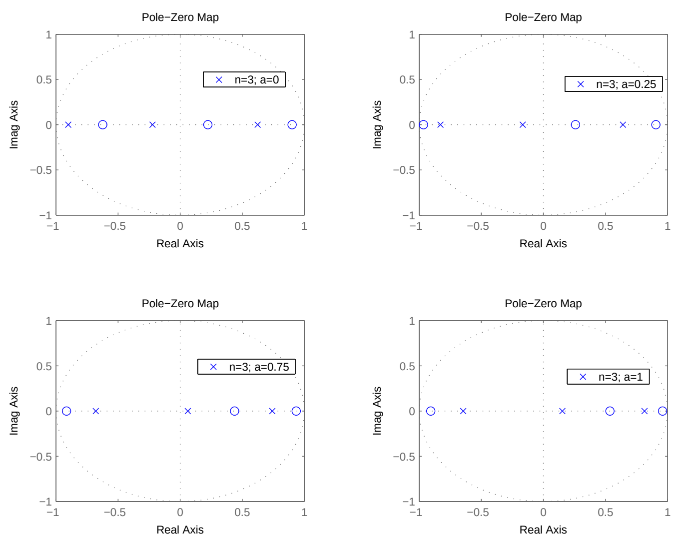

Figure 4. Pole-zero maps for $r = 0.5$, $n = 3$ and $a = 0, .25, .75, 1$

tors, be expressed by

$$
\begin{aligned}
D^{\pm r}(z) &= \frac{Y(z)}{F(z)} = \left(\frac{2}{T}\right)^{\pm r} \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm r}\right\}_{p,q} \\
&= \left(\frac{2}{T}\right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})}
\end{aligned} \tag{18}
$$

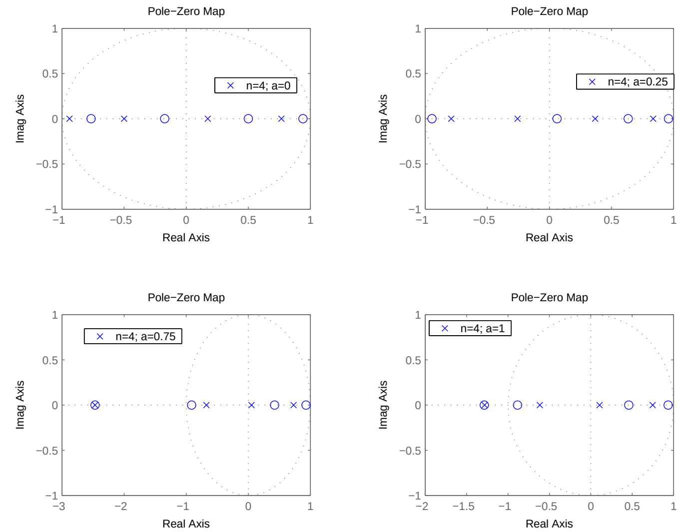

Figure 5. Pole-zero maps for $r = 0.5$, $n = 4$ and $a = 0, .25, .75, 1$

where T is the sample period, $\mathrm{CFE}\{u\}$ denotes the function resulting from applying the continued fraction expansion to the function u, $Y(z)$ is the Z transform of the output sequence $y(nT)$, $F(z)$ is the Z transform of the input sequence $f(nT)$, p and q are the orders of the approximation, and P and Q are polynomials of degrees p and q, correspondingly, in the variable z^{-1}.

By using the MAPLE call

```
Drp:=cfrac(((1-x)/(1+x))^r,x,p)
```

where $x = z^{-1}$, the obtained symbolic approximation has the following form:

$$
D^r(z) = 1 + \cfrac{z^{-1}}{-\cfrac{1}{2}\cfrac{1}{r} + \cfrac{z^{-1}}{-2 + \cfrac{z^{-1}}{\cfrac{3}{2}\cfrac{r}{r^2-1} + \cfrac{z^{-1}}{2 + \cfrac{z^{-1}}{-\cfrac{5}{2}\cfrac{r^2-1}{r(-4+r^2)} + \cfrac{z^{-1}}{-2+\cdots}}}}}}. \tag{19}
$$

In MATLAB Symbolic Toolbox, we can get the same result by the following script:

```
syms x r;maple('with(numtheory)');
f = ((1-x)/(1+x))^r;
maple(['cf := cfrac(' char(f) ',x,10);'])
maple('nd5 := nthconver','cf',10)
maple('num5 := nthnumer','cf',10)
maple('den5 := nthdenom','cf',10)
```

In Table , the general expressions for numerator and denominator of $D^r(z)$ in (18) are listed for $p = q = 1, 3, 5, 7, 9$.

With $r = 0.5$ and $T = 0.001$ sec. the approximate models

Table 1. $D^r(z)$

p=q	$\mathbf{P}_p(z^{-1})$ **(**$k=1$**), and** $\mathbf{Q}_q(z^{-1})(k=0)$
1	$(-1)^k z^{-1} r + 1$
3	$(-1)^k (r^3 - 4r) z^{-3} + (6r^2 - 9) z^{-2} + (-1)^k 15 z^{-1} r + 15$
5	$(-1)^k (r^5 - 20r^3 + 64r) z^{-5} + (-195r^2 + 15r^4 + 225) z^{-4} + (-1)^k (105r^3 - 735r) z^{-3} + (420r^2 - 1050) z^{-2} + (-1)^k 945 z^{-1} r + 945$
7	$(-1)^k (784r^3 + r^7 - 56r^5 - 2304r) z^{-7} + (10612r^2 - 1190r^4 - 11025 + 28r^6) z^{-6} + (-1)^k (53487r + 378r^5 - 11340r^3) z^{-5} + (99225 - 59850r^2 + 3150r^4) z^{-4} + (-1)^k (17325r^3 - 173250r) z^{-3} + (-218295 + 62370r^2) z^{-2} + (-1)^k 135135 z^{-1} r + 135135$
9	$(-1)^k (-52480r^3 + 147456r + r^9 - 120r^7 + 4368r^5) z^{-9} + (45r^8 + 120330r^4 - 909765r^2 - 4410r^6 + 893025) z^{-8} + (-1)^k (-5742495r - 76230r^5 + 1451835r^3 + 990r^7) z^{-7} + (-13097700 + 9514890r^2 - 796950r^4 + 13860r^6) z^{-6} + (-1)^k (33648615r - 5405400r^3 + 135135r^5) z^{-5} + (-23648625r^2 + 51081030 + 945945r^4) z^{-4} + (-1)^k (-61486425r + 4729725r^3) z^{-3} + (16216200r^2 - 72972900) z^{-2} + (-1)^k 34459425 z^{-1} r + 34459425$

for $p = q = 1,3,7,9$ are:

$$G_1(z) = 44.72\frac{z-0.5}{z+0.5}, \quad G_3(z) = 44.72\frac{z^3-0.5z^2-0.5z+0.125}{z^3+0.5z^2-0.5z-0.125}$$

$$G_7(z) = 44.72\frac{z^7-0.5z^6-1.5z^5+0.625z^4+0.625z^3-0.1875z^2-0.0625z+0.007813}{z^7+0.5z^6-1.5z^5-0.625z^4+0.625z^3+0.1875z^2-0.0625z-0.007813}$$

$$G_9(z) = 44.72\frac{z^9-0.5z^8-2z^7+0.875z^6+1.313z^5-0.4688z^4-0.3125z^3+0.07813z^2+0.01953z-0.001953}{z^9+0.5z^8-2z^7-0.875z^6+1.313z^5+0.4688z^4-0.3125z^3-0.07813z^2+0.01953z+0.001953}$$

In Figure 6, the Bode plots and the distributions of zeros and poles of the approximations are presented. In Figure 6a the effectiveness of the approximations fitting the ideal responses in a wide range of frequencies, in both magnitude and phase, can be observed. In Figure 7b it can be observed that the approximations fulfill the two desired properties: (i) all the poles and zeros lie inside the unit circle, and (ii) the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1,1)$.

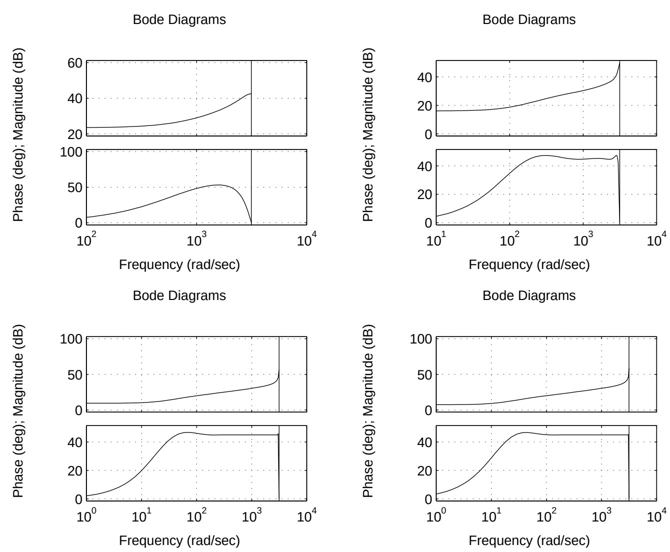

Figure 6. Bode plots (approximation orders 1,3,7,9) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.

CONCLUSIONS

We have presented a new IIR (infinite impulse response) type digital fractional order differentiator (DFOD) with a tuning knob to compromise the high frequency approximation accuracy. It is a direct discretization of fractional order differentiators in continuous time domain. The basic idea is the use of a new family of first order digital differentiators expressed in the second order IIR filter form (Al-Alaoui, 1995) via the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule.

The MATLAB codes developed in this paper are available from the authors upon email request.

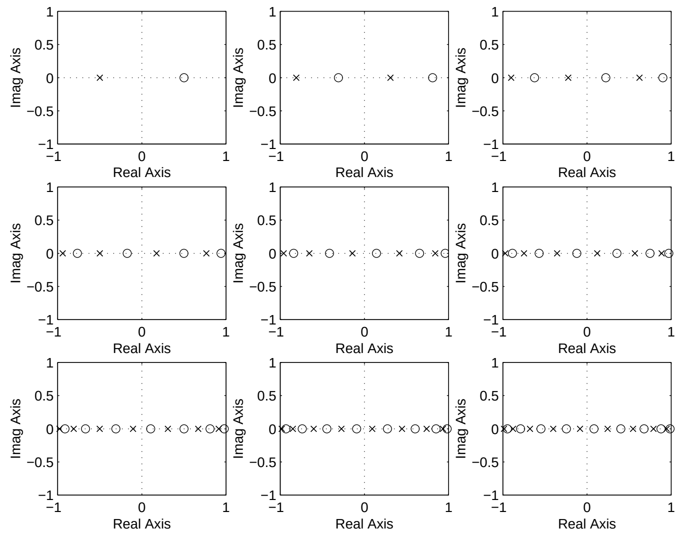

Figure 7. Zero-pole distribution (approximation orders 1,2,$\cdots$,9) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.

ACKNOWLEDGMENT

This project has been funded in part by the National Academy of Sciences under the Collaboration in Basic Science and Engineering Program/Twinning Program supported by Contract No. INT-0002341 from the National Science Foundation. The contents of this publication do not necessarily reflect the views or policies of the National Academy of Sciences or the National Science Foundation, nor does mention of trade names, commercial products or organizations imply endorsement by the National Academy of Sciences or the National Science Foundation.

Blas M. Vinagre is partially supported by the Research Grant 2PR02A024 (Junta de Extremadura and FEDER).

REFERENCES

Al-Alaoui, M. A. (1993). Novel digital integrator and differentiator. *Electronics Letters*, **29**(4), 376–378.

Al-Alaoui, M. A. (1995). A class of second-order integrators and low-pass differentiators. *IEEE Trans. on Circuit and Systems I: Fundamental Theory and Applications*, **42**(4), 220–223.

Al-Alaoui, M. A. (1997). Filling the gap between the bilinear and the backward difference transforms: an interactive design approach. *Int. J. of Electrical Engineering Education*, **34**(4), 331–337.

Axtell, M. and Bise, E. M. (1990). Fractional calculus applications in control systems. In *Proceedings of the IEEE 1990 Nat. Aerospace and Electronics Conf.*, pages 563–566, New York, USA.

Chen, Y. Q. and Moore, K. L. (2002). Discretization Schemes for Fractional-Order Differentiators and Integrators. *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, **49**(3), 363–367.

Lubich, C. H. (1986). Discretized fractional calculus. *SIAM J. Math. Anal.*, **17**(3), 704–719.

Lurie, B. J. (1994). Three-parameter tunable tilt-integral-derivative (TID) controller. *US Patent US5371670*.

Machado, J. A. T. (1997). Analysis and design of fractional-order digital control systems. *J. of Systems Analysis-Modeling-Simulation*, **27**, 107–122.

Manabe, S. (1960). The non-integer integral and its application to control systems. *JIEE (Japanese Institute of Electrical Engineers) Journal*, **80**(860), 589–597.

Manabe, S. (1961). The non-integer integral and its application to control systems. *ETJ of Japan*, **6**(3-4), 83–87.

Miller, K. S. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York.

Oldham, K. B. and Spanier, J. (1974). *The Fractional Calculus*. Academic Press, New York.

Oustaloup, A. (1981). Fractional order sinusoidal oscillators: Optimization and their use in highly linear FM modulators. *IEEE Transactions on Circuits and Systems*, **28**(10), 1007–1009.

Oustaloup, A. (1995). *La Dérivation non Entière*. HERMES, Paris.

Oustaloup, A. and Mathieu, B. (1999). *La commande CRONE : du scalaire au multivariable*. HERMES, Paris.

Oustaloup, A., Mathieu, B., and Lanusse, P. (1995). The CRONE control of resonant plants: application to a flexible transmission. *European Journal of Control*, **1**(2).

Oustaloup, A., Moreau, X., and Nouillant, M. (1996). The CRONE suspension. *Control Engineering Practice*, **4**(8), 1101–1108.

Oustaloup, A., Sabatier, J., and Lanusse, P. (1999). From fractal robustness to CRONE control. *Fractionnal Calculus and Applied Analysis*, **2**(1), 1–30.

Oustaloup, A., Levron, F., Nanot, F., and Mathieu, B. (2000). Frequency band complex non integer differentiator : characterization and synthesis. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, **47**(1), 25–40.

Petráš, I. (1999). The fractional-order controllers: Methods for their synthesis and application. *Journal of Electrical Engineering*, **50**(9-10), 284–288.

Podlubny, I. (1994). Fractional-order systems and fractional-order controllers. Technical Report UEF-03-94, Slovak Academy of Sciences. Institute of Experimental Physics, Department of Control Engineering. Faculty of Mining, University of Tecnology. Kosice.

Podlubny, I. (1999). Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers. *IEEE Trans. Automatic Control*, **44**(1), 208–214.

Raynaud, H. and ZergaInoh, A. (2000). State-space representation for fractional order controllers. *Automatica*, **36**, 1017–1021.

Samko, S. G., Kilbas, A. A., and Marichev, O. I. (1987). *Fractional integrals and derivatives and some of their applications*. Nauka i technika, Minsk.

Tseng, C. (2001). Design of fractional order digital FIR differentiator. *IEEE Signal Processing Letters*, **8**(3), 77–79.

Tseng, C., Pei, S., and Hsia, S. (2000). Computation of Fractional Derivatives using Fourier Transform and Digital FIR Differentiator. *Signal Processing*, **80**, 151–159.

Vinagre, B. M., Podlubny, I., Hernandez, A., and Feliu, V. (2000a). On realization of fractional-order controllers. In *Proc. of the Conference Internationale Francophone d'Automatique*, Lille, France.

Vinagre, B. M., Podlubny, I., Hernandez, A., and Feliu, V. (2000b). Some approximations of fractional order operators used in control theory and applications. *Fractional Calculus and Applied Analysis*, **3**(3), 231–248.

Vinagre, B. M., Petras, I., Merchan, P., and Dorcak, L. (2001). Two digital realisation of fractional controllers: application to temperature control of a solid. In *Proc. of the European Control Conference (ECC2001)*, pages 1764–1767, Porto, Portugal.

