A New Discretization Method for Fractional Order Differentiators Via Continued Fraction Expansion

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Outline of the presentation

- **Discretization Methods for Fractional Derivatives**
- A New Method (Mixed Tustin and Simpson)
- A Special Case (Tustin)
- Concluding Remarks

How to discretize the fractional order derivative s^r (r is a real number, s is the Laplace transform operator)?

Basically, there are two types of discretization methods

- **Indirect discretization methods**: Approximate s^r in continuous time domain and then `c2d`.
  - Oustaloup's approximation method;
  - stable minimum phase fitting using `FDIDENT` - a Matlab toolbox for frequency domain identification.
  - and so on.
- **Direct discretization methods**: Starting from the approximate generator $\omega(z^{-1})$ for s and then expand $(\omega(z^{-1}))^r$ with truncations.
  - Euler operator based,
  - Tustin operator based,
  - Al-Alaoui operator based,
  - and so on.

Direct discretization methods

- **FIR form**:
  - Generating function: $\omega(z^{-1}) = (1 - z^{-1})/T$ with T the sampling period.

    Power series expansion (PSE) of $(1 - z^{-1})^{\pm r}$ gives the discretization formula in FIR filter form

- **IIR form**:
  - Generating function: (Tustin operator based) $(\omega(z^{-1})) = \left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)$. Use CFE (Continued Fraction Expansion) or recursive expansion to get IIR approximation.
  - Generating function: (Al-Alaoui operator based) $(\omega(z^{-1})) = \left(\frac{8}{7T}\frac{1-z^{-1}}{1+z^{-1}/7}\right)$. Use CFE to expand it.
  - **A new generating function in this work**.

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Step-1: Get a new second order digital integrator first.

A fact: the ideal integrator $1/s$ lies between that of the Simpson and trapezoidal digital integrators.
"Interpolate" the Simpson and trapezoidal digital integrators to compromise the high frequency accuracy in frequency response, i.e.,

$$H(z) = aH_S(z) + (1-a)H_T(z), \quad a \in [0,1] \tag{1}$$

where a is actually a weighting factor or a tuning knob. $H_S(z)$ and $H_T(z)$ are the z-transfer functions of the Simpson's and the trapezoidal integrators given respectively as follows:

$$H_S(z) = \frac{T(z^2+4z+1)}{3(z^2-1)} \tag{2}$$

and

$$H_T(z) = \frac{T(z+1)}{2(z-1)}. \tag{3}$$

The overall weighted digital integrator with the tuning parameter a is hence given by

$$\begin{aligned} H(z) &= \frac{T(3-a)\{z^2 + [2(3+a)/(3-a)]z + 1\}}{6(z^2-1)} \\ &= \frac{T(3-a)(z+r_1)(z+r_2)}{6(z^2-1)} \end{aligned} \quad (4)$$

where

$$r_1 = \frac{3+a+2\sqrt{3a}}{3-a}, \quad r_2 = \frac{3+a-2\sqrt{3a}}{3-a}.$$

It is interesting to note the fact that $r_1 = \frac{1}{r_2}$ and $r_1 = r_2 = 1$ only when $a = 0$ (trapezoidal). For $a \neq 0$, $H(z)$ must have one non-minimum phase (NMP) zero.

Step-2: Get a new second order digital differentiator.

Be careful: Direct inversion of $H(z)$ will give an unstable filter since $H(z)$ has an NMP zero r_1.
Solution: By reflecting the NMP r_1 to $1/r_1$, i.e. r_2, we have

$$\tilde{H}(z) = K\frac{T(3-a)(z+r_2)^2}{6(z^2-1)}.$$

Question: What is K then?
Solution: let the final value of the impulse responses of $H(z)$ and $\tilde{H}(z)$ be the same, i.e., $\lim_{z\to 1}(z-1)H(z) = \lim_{z\to 1}(z-1)\tilde{H}(z)$, which gives $K = r_1$. Therefore, the new family of the digital differentiators are given by

$$\omega(z) = \frac{1}{\tilde{H}(z)} = \frac{6(z^2-1)}{r_1 T(3-a)(z+r_2)^2} = \frac{6r_2(z^2-1)}{T(3-a)(z+r_2)^2}. \qquad (5)$$

We can regard $\omega(z)$ in (5) as the generating function introduced in the last section. Finally, we can obtain the expression for the DFOD as

$$G(z^{-1}) = \left(\omega(z^{-1})\right)^r = k_0 \left(\frac{1 - z^{-2}}{(1 + bz^{-1})^2}\right)^r \qquad (6)$$

where $r \in [0, 1]$, $k_0 = (\frac{6r_2}{T(3-a)})^r$ and $b = r_2$.

Step-3: Do CFE (continued fractional expansion) for $(\omega(z^{-1}))^r = k_0 \left(\frac{1-z^{-2}}{(1+bz^{-1})^2}\right)^r$. Do CFE expansion by MATLAB Symbolic Toolbox. Let $x = z^{-1}$. Then do

$$\text{CFE}\left(\frac{1-x^2}{(1+bx)^2}\right)^r$$

to the desired order n. MATLAB script: (`p1` and `q1`, respectively, the numerator and denominator polynomials in `x` or z^{-1} with their coefficients being functions of `b` and `r`.)

```
clear all;close all;syms x r b;maple('with(numtheory)');
aas = ( (1-x*x)/(1+b*x)^2  )^r;  n=3; n2=2*n;
maple(['cfe := cfrac(' char(aas) ',x,n2);']);
pq=maple('P_over_Q := nthconver','cfe',n2);
p0=maple('P := nthnumer','cfe',n2);
q0=maple('Q := nthdenom','cfe',n2);
p=(p0(5:length(p0))); q=(q0(5:length(q0)));
p1=collect(sym(p),x); q1=collect(sym(q),x);
```

EXAMPLES

$r = 0.5$. The values of the truncation order n and the weighting factor a are denoted as subscripts of $G_{(n,a)}(z)$. $T = 0.001$sec.

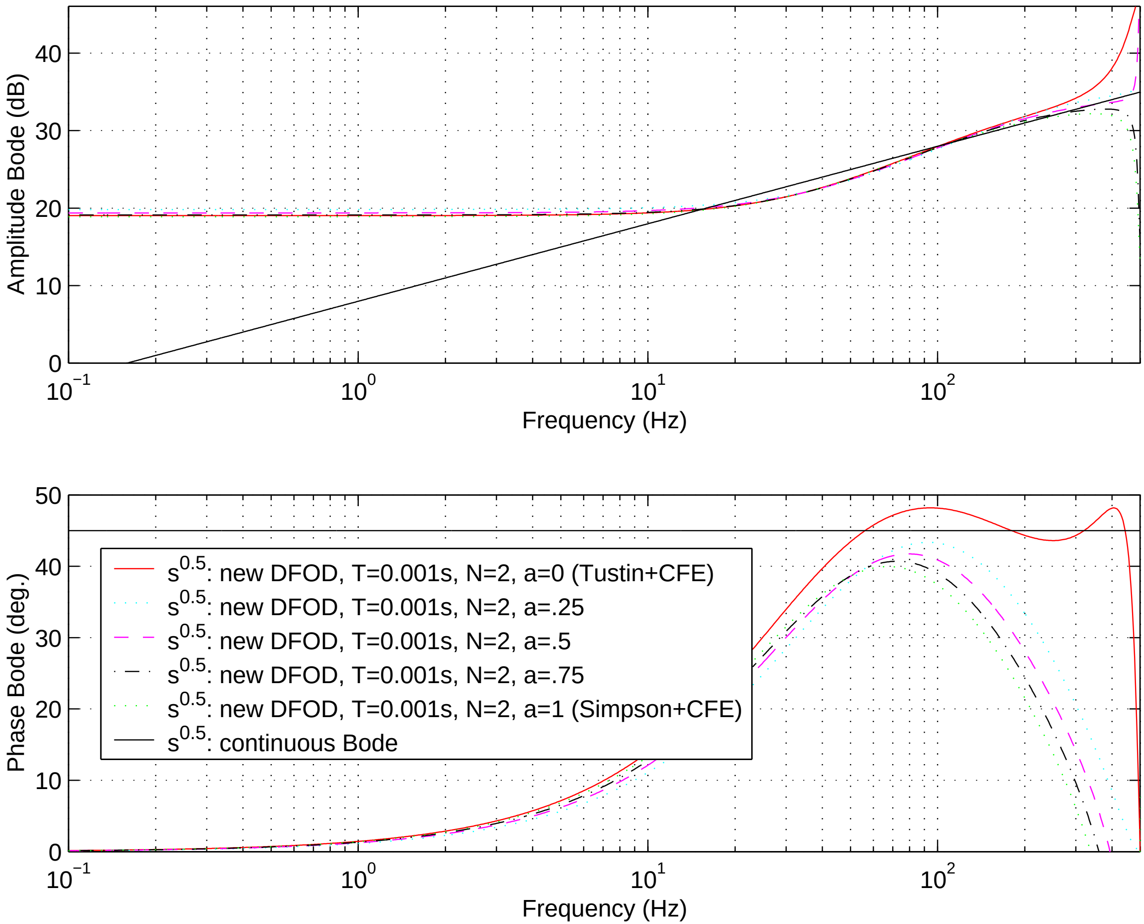
$$
\begin{aligned}
G_{(2,0.00)}(z^{-1}) &= \frac{178.9 - 89.44z^-1 - 44.72z^-2}{4 + 2z^-1 - z^-2} \\
G_{(2,0.25)}(z^{-1}) &= \frac{138.8 + 98.07z^{-1} - 158.2z^{-2}}{4 + 5.034z^{-1} - z^{-2}} \\
G_{(2,0.50)}(z^{-1}) &= \frac{127 + 41.26z^{-1} - 112.6z^{-2}}{4 + 2.98z^{-1} - z^{-2}} \\
G_{(2,0.75)}(z^{-1}) &= \frac{119.3 + 25.56z^{-1} - 97.96z^{-2}}{4 + 2.19z^{-1} - z^{-2}} \\
G_{(2,1.00)}(z^{-1}) &= \frac{113.4 + 17.74z^{-1} - 89.81z^{-2}}{4 + 1.698z^{-1} - z^{-2}}
\end{aligned}
\tag{7}
$$

$r = 0.5$. The values of the truncation order n and the weighting factor a are denoted as subscripts of $G_{(n,a)}(z)$. $T = 0.001$sec.

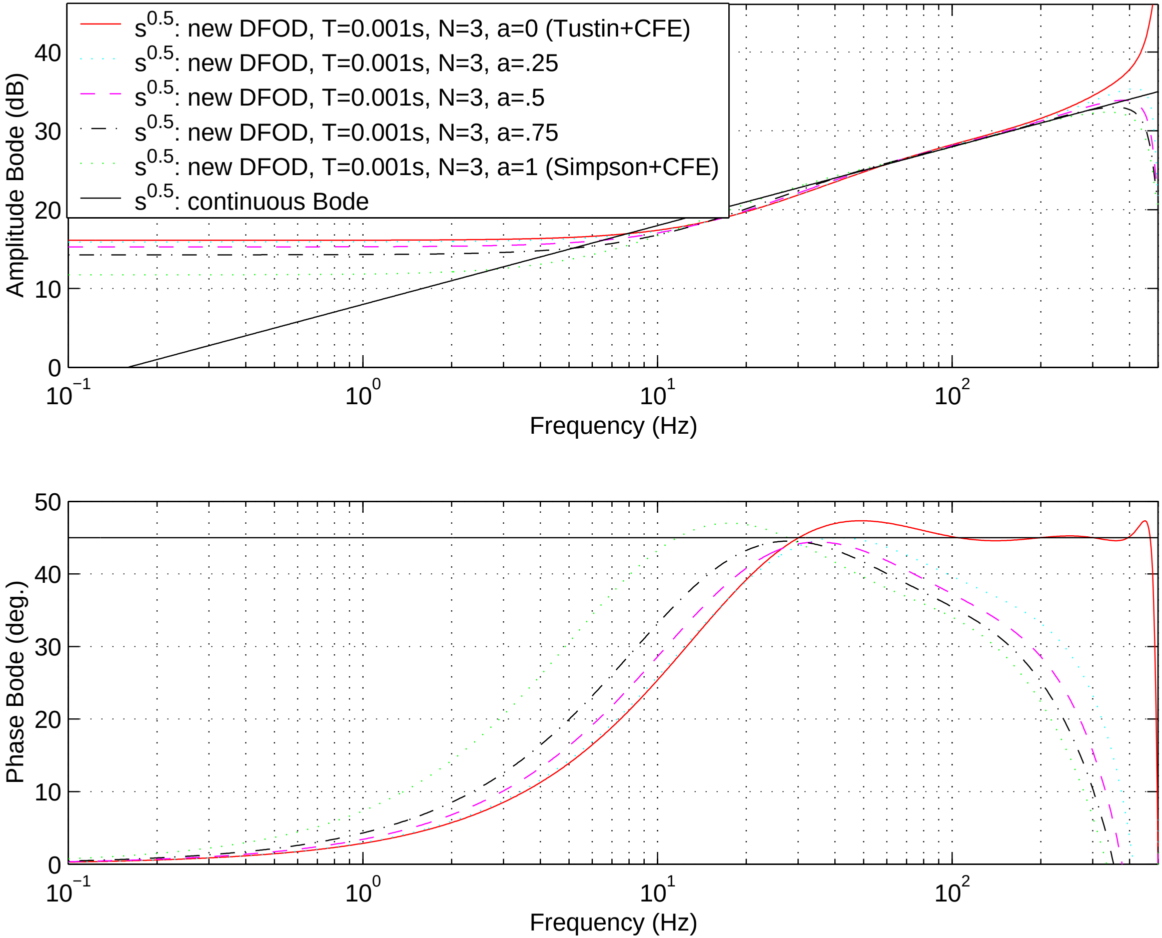
$$\begin{aligned}
G_{(3,0.00)}(z^{-1}) &= \frac{357.8 - 178.9z^{-1} - 178.9z^{-2} + 44.72z^{-3}}{8 + 4z^{-1} - 4z^{-2} - z^{-3}} \\
G_{(3,0.25)}(z^{-1}) &= \frac{392.9 - 78.04z^{-1} - 349.8z^{-2} + 88.97z^{-3}}{11.32 + 4z^{-1} - 5.66z^{-2} - z^{-3}} \\
G_{(3,0.50)}(z^{-1}) &= \frac{1501 - 503.6z^{-1} - 1289z^{-2} + 446.5z^{-3}}{47.26 + 4z^{-1} - 23.63z^{-2} - z^{-3}} \\
G_{(3,0.75)}(z^{-1}) &= \frac{968.1 - 442z^{-1} - 820.8z^{-2} + 363z^{-3}}{32.47 - 4z^{-1} - 16.24z^{-2} + z^{-3}} \\
G_{(3,1.00)}(z^{-1}) &= \frac{353.1 - 208z^{-1} - 297.4z^{-2} + 164.7z^{-3}}{12.46 - 4z^{-1} - 6.228z^{-2} + z^{-3}}
\end{aligned} \tag{8}$$

$r = 0.5$. The values of the truncation order n and the weighting factor a are denoted as subscripts of $G_{(n,a)}(z)$. $T = 0.001$sec.

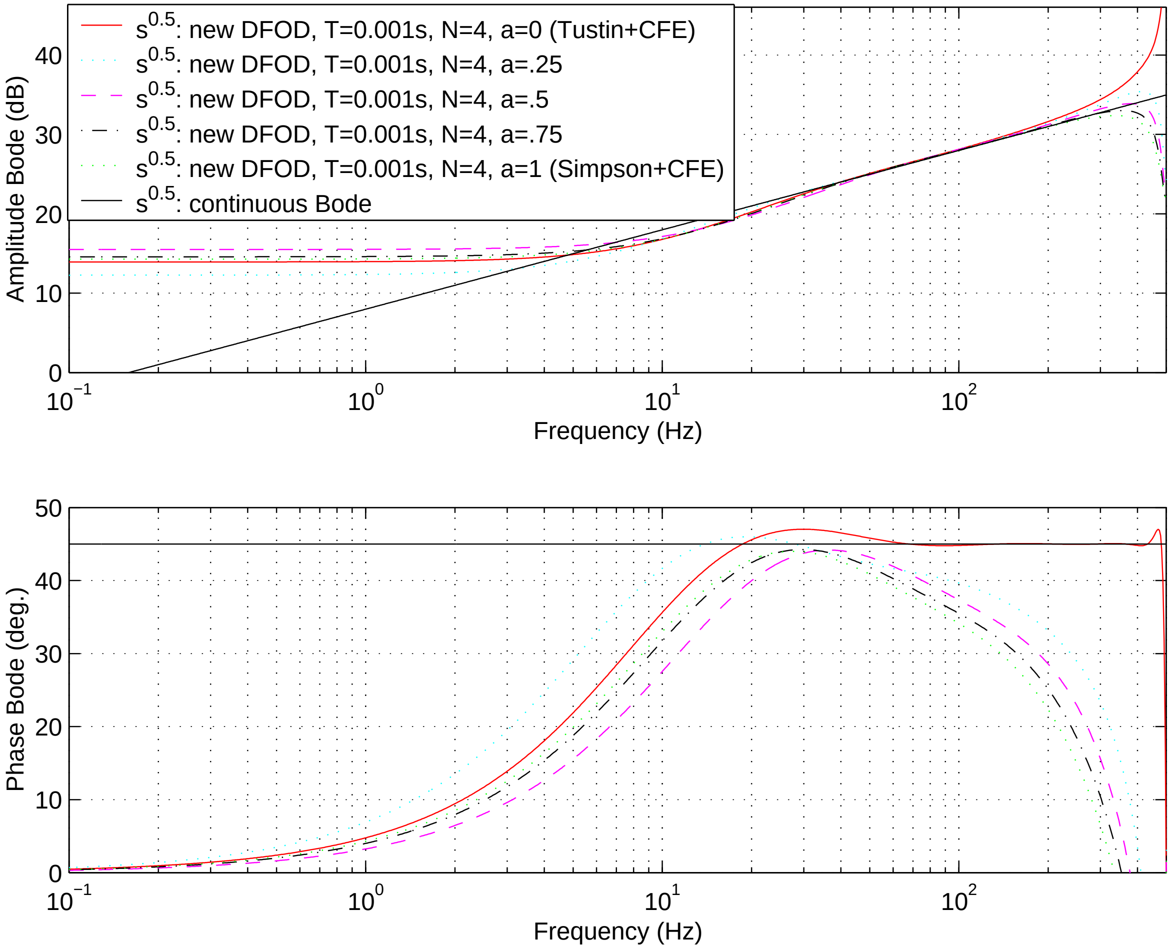
$$
\begin{aligned}
G_{(4,0.00)}(z^{-1}) &= \frac{715.5 - 357.8z^{-1} - 536.7z^{-2} + 178.9z^{-3} + 44.72z^{-4}}{16 + 8z^{-1} - 12z^{-2} - 4z^{-3} + z^{-4}} \\
G_{(4,0.25)}(z^{-1}) &= \frac{555.3 - 392.9z^{-1} - 477.2z^{-2} + 349.8z^{-3} - 19.56z^{-4}}{16 - 2.489z^{-1} - 12z^{-2} + 1.245z^{-3} + z^{-4}} \\
G_{(4,0.50)}(z^{-1}) &= \frac{508.1 - 1501z^{-1} - 4.478z^{-2} + 1289z^{-3} - 382.9z^{-4}}{16 - 40.54z^{-1} - 12z^{-2} + 20.27z^{-3} + z^{-4}} \\
G_{(4,0.75)}(z^{-1}) &= \frac{477 + 968.1z^{-1} - 919z^{-2} - 820.8z^{-3} + 422.7z^{-4}}{16 + 37.8z^{-1} - 12z^{-2} - 18.9z^{-3} + z^{-4}} \\
G_{(4,1.00)}(z^{-1}) &= \frac{453.6 + 353.1z^{-1} - 661.7z^{-2} - 297.4z^{-3} + 221.5z^{-4}}{16 + 16.74z^{-1} - 12z^{-2} - 8.371z^{-3} + z^{-4}}
\end{aligned} \tag{9}
$$



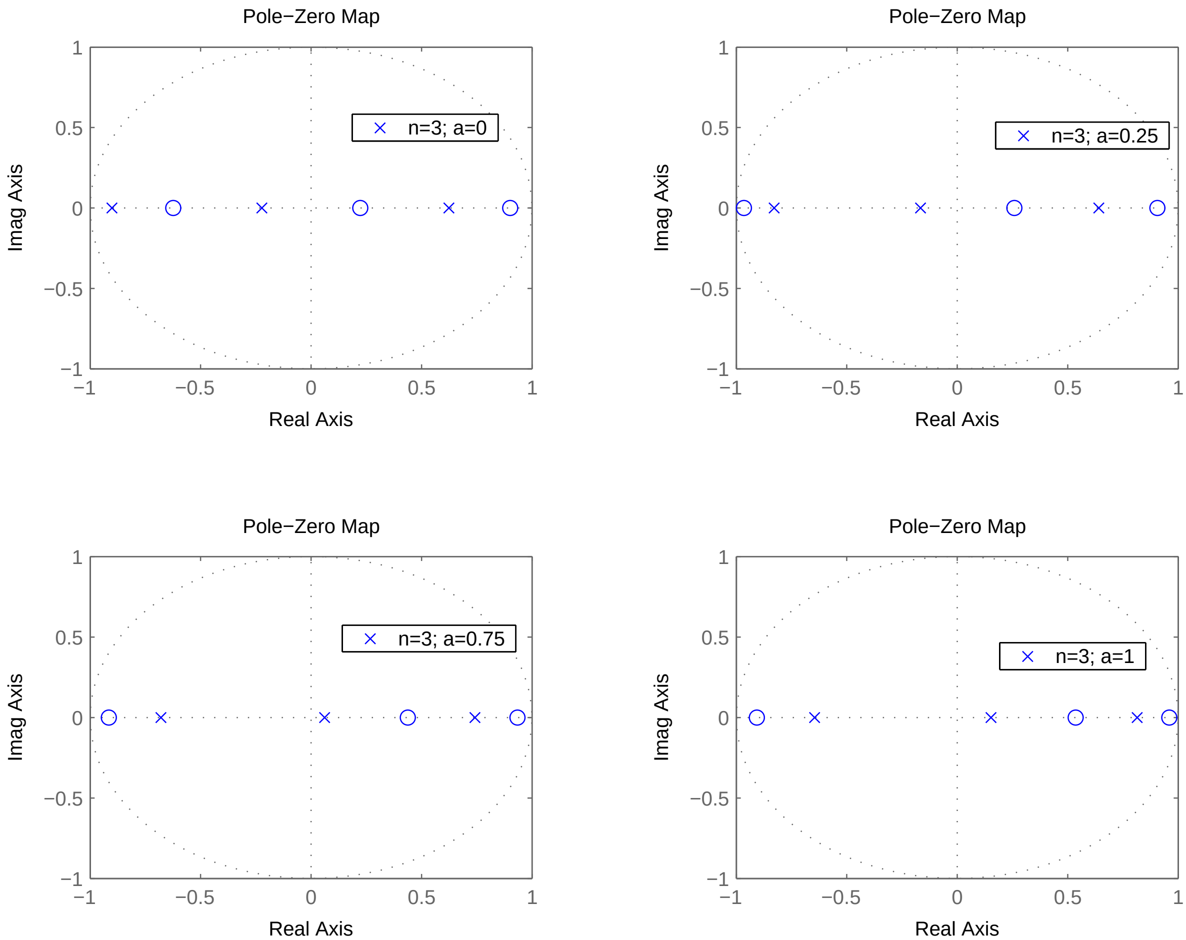
Bode plot comparison for $r = 0.5$, $n = 2$ and $a = 0, .25, .5, .75, 1$



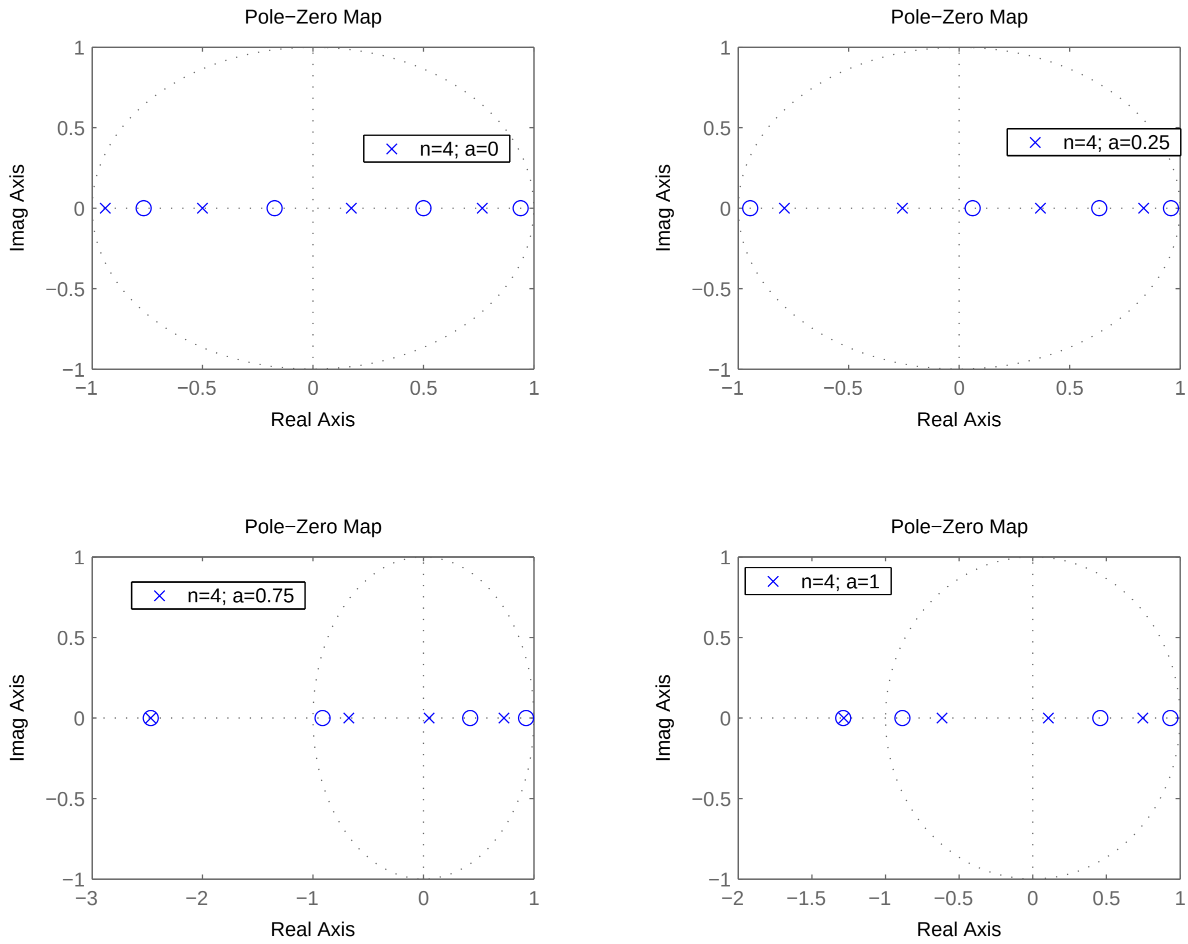
Bode plot comparison for $r = 0.5$, $n = 3$ and $a = 0, .25, .5, .75, 1$



Bode plot comparison for $r = 0.5$, $n = 4$ and $a = 0, .25, .5, .75, 1$



Pole-zero maps for $r = 0.5$, $n = 3$ and $a = 0, .25, .75, 1$



Pole-zero maps for $r = 0.5$, $n = 4$ and $a = 0, .25, .75, 1$

Observations:

- No complex conjugate poles or zeros. We can further observe that for odd order of CFE ($n = 3$), the pole-zero maps are nicely behaved, that is, all the poles and zeros lie inside the unit circle and the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$.

- However, when n is even, and when a is near to 1, there may have one cancelling pole-zero pair which may not be desirable. We suggest to use an odd n when applying the new discretization scheme of this paper.

- When $a = 0$, the pole-zero map is always inside the unit circle in an interlacing way along the segment of the real axis corresponding to $z \in (-1, 1)$.

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Consider the special case for $a = 0$ (Tustin operator).

$$
\begin{aligned}
D^{\pm r}(z) &= \frac{Y(z)}{F(z)} = \left(\frac{2}{T}\right)^{\pm r} \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm r}\right\}_{p,q} \\
&= \left(\frac{2}{T}\right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})} \qquad (10)
\end{aligned}
$$

By using the MAPLE call

```
Drp:=cfrac(((1-x)/(1+x))^r,x,p)
```

where $x = z^{-1}$, the obtained symbolic approximation has the following form:

$$
D^r(z) = 1 + \cfrac{z^{-1}}{-\frac{1}{2}\frac{1}{r} + \cfrac{z^{-1}}{-2 + \cfrac{z^{-1}}{\frac{3}{2}\frac{r}{r^2-1} + \cfrac{z^{-1}}{2 + \cfrac{z^{-1}}{-\frac{5}{2}\frac{r^2-1}{r(-4+r^2)} + \cfrac{z^{-1}}{-2+\cdots}}}}}}. \qquad (11)
$$

The general expressions for numerator $\mathbf{P}_p(z^{-1})$ and denominator $\mathbf{Q}_q(z^{-1})$ of $D^r(z)$ in (10) for $p = q = 1, 3, 5, 7, 9$.

p=q	$\mathbf{P}_p(z^{-1})$ $(k=1)$, and $\mathbf{Q}_q(z^{-1})(k=0)$
1	$(-1)^k z^{-1}r + 1$
3	$(-1)^k (r^3 - 4r)z^{-3} + (6r^2 - 9)z^{-2} + (-1)^k 15z^{-1}r + 15$
5	$(-1)^k (r^5 - 20r^3 + 64r)z^{-5} + (-195r^2 + 15r^4 + 225)z^{-4} + (-1)^k (105r^3 - 735r)z^{-3} + (420r^2 - 1050)z^{-2} + (-1)^k 945z^{-1}r + 945$
7	$(-1)^k (784r^3 + r^7 - 56r^5 - 2304r)z^{-7} + (10612r^2 - 1190r^4 - 11025 + 28r^6)z^{-6} + (-1)^k (53487r + 378r^5 - 11340r^3)z^{-5} + (99225 - 59850r^2 + 3150r^4)z^{-4} + (-1)^k (17325r^3 - 173250r)z^{-3} + (-218295 + 62370r^2)z^{-2} + (-1)^k 135135z^{-1}r + 135135$
9	$(-1)^k (-52480r^3 + 147456r + r^9 - 120r^7 + 4368r^5)z^{-9} + (45r^8 + 120330r^4 - 909765r^2 - 4410r^6 + 893025)z^{-8} + (-1)^k (-5742495r - 76230r^5 + 1451835r^3 + 990r^7)z^{-7} + (-13097700 + 9514890r^2 - 796950r^4 + 13860r^6)z^{-6} + (-1)^k (33648615r - 5405400r^3 + 135135r^5)z^{-5} + (-23648625r^2 + 51081030 + 945945r^4)z^{-4} + (-1)^k (-61486425r + 4729725r^3)z^{-3} + (16216200r^2 - 72972900)z^{-2} + (-1)^k 34459425z^{-1}r + 34459425$

With $r = 0.5$ and $T = 0.001$ sec. the approximate models for $p = q = 1, 3, 7, 9$ are:

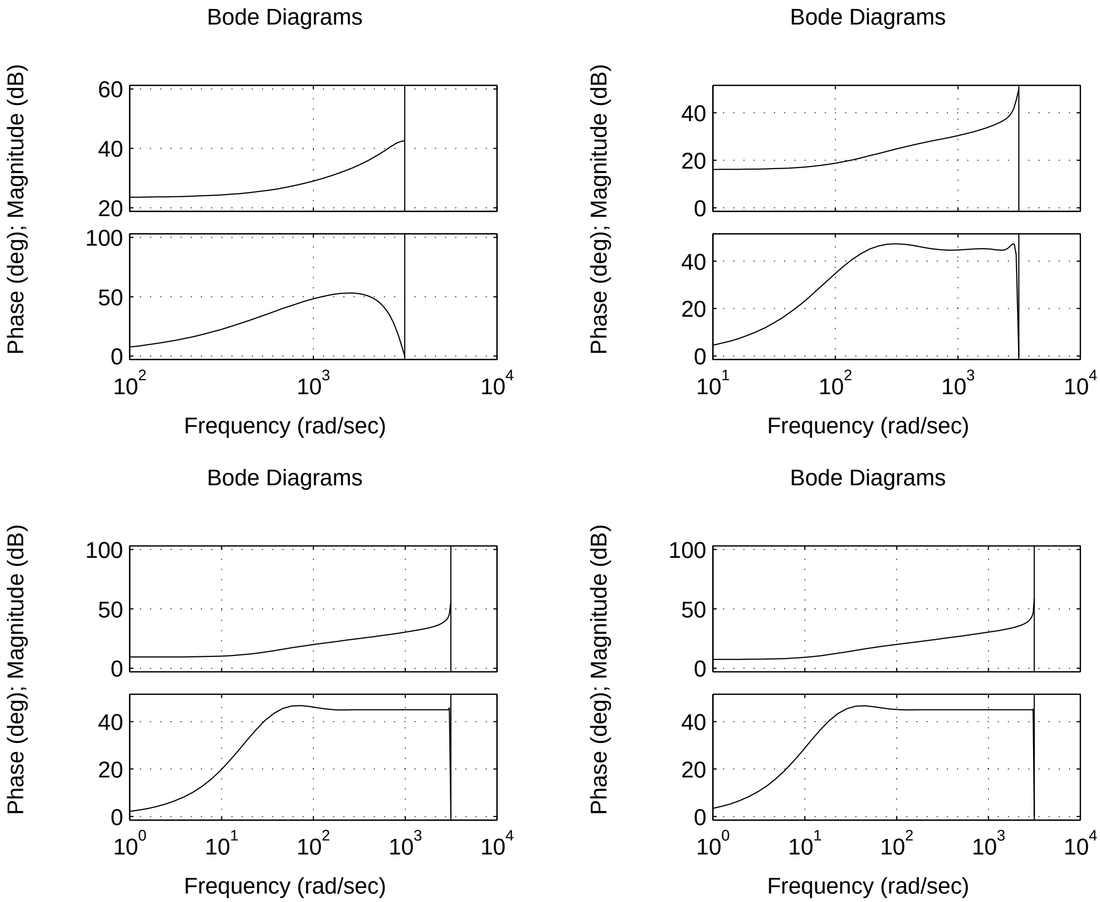
$$G_1(z) = 44.72\frac{z-0.5}{z+0.5}, \quad G_3(z) = 44.72\frac{z^3-0.5z^2-0.5z+0.125}{z^3+0.5z^2-0.5z-0.125}$$

$$G_7(z) = 44.72\frac{z^7-0.5z^6-1.5z^5+0.625z^4+0.625z^3-0.1875z^2-0.0625z+0.007813}{z^7+0.5z^6-1.5z^5-0.625z^4+0.625z^3+0.1875z^2-0.0625z-0.007813}$$

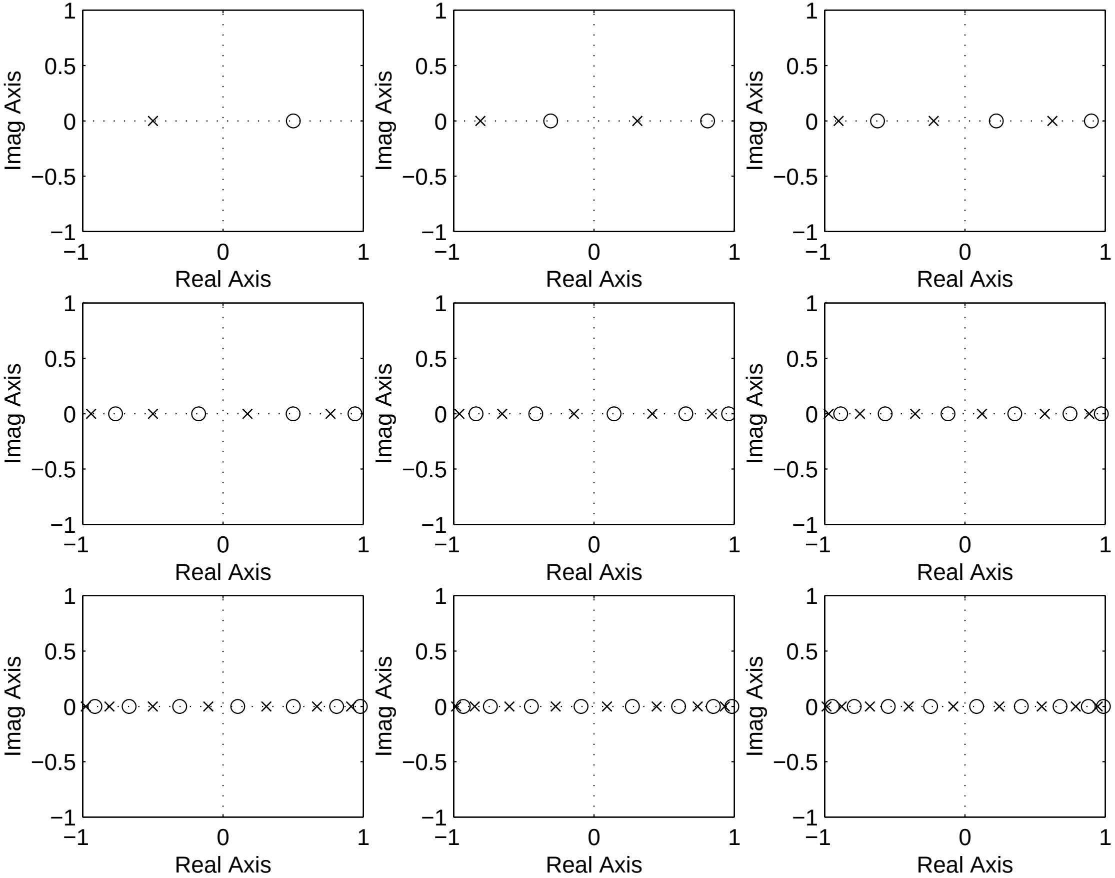
$$G_9(z) = 44.72\frac{z^9-0.5z^8-2z^7+0.875z^6+1.313z^5-0.4688z^4-0.3125z^3+0.07813z^2+0.01953z-0.001953}{z^9+0.5z^8-2z^7-0.875z^6+1.313z^5+0.4688z^4-0.3125z^3-0.07813z^2+0.01953z+0.001953}$$

In MATLAB Symbolic Toolbox, we can get the same result by the following script:

```
syms x r;maple('with(numtheory)');
f = ((1-x)/(1+x))^r;
maple(['cf := cfrac(' char(f) ',x,10);'])
maple('nd5 := nthconver','cf',10)
maple('num5 := nthnumer','cf',10)
maple('den5 := nthdenom','cf',10)
```



Bode plots (approximation orders 1,3,7,9) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.



Zero-pole distribution (approximation orders 1,2,$\cdots$,9) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.

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Concluding Remarks

- A new direct discretization method is presented for fractional derivatives s^r.
- A tuning knob within the generating function is introduced. This generating function is the weighted sum of Tustin and Simpson operators.
- Matlab code is ready to use.
- Further investigation: optimal balanced time and frequency domain approximation of s^r.

Thank you!

Q/A session

FOC web pages:
http://mechatronics.ece.usu.edu/foc

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