Optimal Switching Control Via Direct Search Optimization

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Outline of the presentation

- **Optimal Switching Control - Problem Formulation**
- Direct Search Methods - LJ Procedure and IDP
- Two Illustrative Examples
- Concluding Remarks
1. Optimal Switching Control - Problem Formulation

A **switched system**: a tuple $S = \{\mathcal{D}, \mathcal{F}\}$ where

- $\mathcal{D} = (I, E)$ is a directed graph indicating the discrete structure of the system with its node set $I = \{1, 2, \cdots, M\}$ for subsystem indices and

- $\mathcal{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m, i \in I\}$ is a set of vector fields with $f_i$ describing the vector field for the $i$-th subsystem $\dot{x} = f_i(x, u)$.

So, if an event $e = (i_1, i_2)$ occurs, the subsystem $i_1$ switches to the subsystem $i_2$; $\mathcal{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m, i \in I\}$ is a set of vector fields with $f_i$ describing the vector field for the $i$-th subsystem $\dot{x} = f_i(x, u)$. 
General switched linear quadratic optimal control problems (GSLQ): a special switched system $S$ with all linear subsystems $\dot{x} = A_i x + B_i u$, $i \in I$.

- **Given** a fixed time interval $[t_0, t_f]$,
- **find** a continuous input signal $u(t), t \in [t_0, t_f]$ and a switching sequence $\sigma = ((t_0, i_0), (t_1, e_1), (t_2, e_2), \cdots, (t_K, e_K))$ with $K$ the total number of switches, $i_0 \in I, t_{i-1} < t_i$ and $e_i = (t_{i-1}, t_i) \in E$ for $i = 0, 1, \cdots, K$,
- **such that**

$$J = \frac{1}{2} x^T(t_f)Q_f x(t_f) + M_f x(t_f) + W_f + \int_{t_0}^{t_f} \left( \frac{1}{2} x^T Q x + x^T V u + \frac{1}{2} u^T R u + M x + N u + W \right) dt$$

where $t_0, t_f$ and $x(t_0) = x_0$ are given; $Q_f, M_f, W_f, Q, V, R, M, N, W$ are matrices of appropriate dimensions; $Q_f$ and $Q$ are positive semidefinite and $R$ is positive definite.
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Luus-Jaakola (LJ) Optimization Procedure:


Two key ideas in LJ optimization procedure:

- 1. Randomly chosen test points are chosen over a search region at each iteration.

- 2. As iterations proceed, the search is made more intensive by reducing the region size over which the test points are chosen.

Several papers have been written since 1973 to address the problem of how to choose the region over which the randomly chosen test points should be chosen, how to choose the number of test points to be used at each iteration, and how to handle constraints.
Detailed LJ Procedure:

1. Given some initial point $\mathbf{x}^*$, choose $R$ random points in the $n$-dimensional space through the equation $\mathbf{x} = \mathbf{x}^* + \mathbf{D}r$ where $\mathbf{D}$ is a diagonal matrix, where randomly chosen diagonal elements lie in the interval $[-1, +1]$, and $\mathbf{r}$ is the region size vector.

2. Check the feasibility of each such randomly chosen point with respect to the inequality constraints in Eq. (??). For each feasible point evaluate the performance index $I$ in Eq. (??), and keep the best $\mathbf{x}$-value.

3. An iteration is defined by Steps 1 and 2. At the end of each iteration, $\mathbf{x}^*$ is replaced by the best feasible $\mathbf{x}$-value obtained in step 2, and the region size vector $\mathbf{r}$ is reduced by $\gamma$ through $\mathbf{r}^{j+1} = \gamma \mathbf{r}^j$ where $\gamma$ is a region contraction factor such as 0.95, and $j$ is the iteration index.
Some Key References for Iterative Dynamic Programming (IDP):


Optimal Control Problem

Continuous dynamic system: \( \frac{dx}{dt} = f(x, u) \) where \( x(0) \) is given. The state vector \( x \) is an \((n \times 1)\) vector and \( u \) is an \((m \times 1)\) control vector bounded by

\[
\alpha_j \leq u_j \leq \beta_j, \quad j = 1, 2, \cdots, m.
\]

Optimal control problem: Minimize (or maximize) the performance index

\[
I = \Phi(x(t_f))
\]

where the final time \( t_f \) is specified.
Iterative Dynamic Programming Algorithm

• 1. Divide the time interval $[0, t_f]$ into $P$ time stages $(0, t_1), (t_1, t_2), \cdots, (t_{k-1}, t_k), \cdots, (t_{P-1}, t_P)$, each of equal length $L$.

• 2. Choose the number of grid points $N$ at each time stage, the number of test values for $u$ denoted by $R$, initial region size $r^{(0)}$, initial control policy $u^{(0)}$, region contraction factor $\gamma$ used after every iteration, region restoration factor $\eta$ used after every pass, number of iterations used in every pass and the number of passes.

• 3. Set the pass number index $q = 1$ and the iteration number index $j = 1$.

• 4. By choosing $N$ control values evenly distributed inside the allowable region $r_j$, integrate the state equation from $t = 0$ to $t = t_f$ with the given initial condition, to generate $N$ trajectories. The $N$ values of $x$ at the beginning of each time stage constitute the $N$ grid points at each time stage, except the first one for which there is a single grid point consisting of the given initial condition.
5. Starting at the beginning of the last time stage $P$, corresponding to the time $t_f - L$, for each grid point, integrate the state equation from $t_f - L$ to $t_f$ once with each of the $R$ allowable control values calculated from

$$u(P) = u^{*j}(P) + Dr^j(P)$$

where $u^{*j}(P)$ is the best value obtained in the previous iteration and $D$ is an $(m \times m)$ diagonal matrix of different random numbers between -1 and 1. Compare the $R$ values of the performance index and choose the control that gives the best value to the performance index. This best control is stored for use in step 6.

6. Proceed to stage $P - 1$, corresponding to time $t_f - 2L$. For each grid point integrate the state equation once with each of the $R$ allowable control values from $t_f - 2L$ to $t_f - L$. Continue integration from $t_f - L$ to $t_f$ with the control from the Step 5 that corresponds to the grid point that is closest to the resulting $x$ at $t_f - L$. Compare the $R$ values of the performance index and store the control value that gives the best value.
• 7. Repeat the procedure for stages $P - 2$, $P - 3$, etc., up to stage 1 which consists of the single grid point. This concludes one iteration.

• 8. Reduce the region $r$ by a factor $\gamma$:

$$r^{j+1}(k) = \gamma r^j(k), \quad k = 1, 2, \ldots, P$$

where $0 < \gamma < 1$ and $j$ is the iteration index.

• 9. Increment the iteration index $j$ by 1 and go to step 4. Continue the procedure for the number of iterations specified in step 2 to finish a pass.

• 10. Increment the pass number index $q$ by 1. Restore each region size to a fraction $\eta$ of its value at the beginning of the pass. Go to step 4, continue the procedure for the number of passes specified in step 2 and interpret the results.
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Example 1

A switched system (GSLQ) consisting of three subsystems

\begin{equation}
\dot{x} = A_i x + B_i u, \quad i = 1, 2, 3,
\end{equation}

\begin{align*}
A_1 &= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, & B_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0.5 & 5.3 \\ -5.3 & 0.5 \end{pmatrix}, & B_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}, & B_3 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{align*}

In this case, \( t_0 = 0 \), \( t_f = 3 \) and the system switches at \( t = t_1 \) from subsystem 1 to 2 and at \( t = t_2 \) from subsystem 2 to 3 \((0 \leq t_1 \leq t_2 \leq 3)\). Find the optimal switching time instants \( t_1 \) and \( t_2 \), and the control \( u \) for \( t \in [t_0, t_f] \) such that \( J = [(x_1(3) + 4.1437)^2 + (x_2(3) - 9.3569)^2] + 0.5 \int_0^3 u^2(t) dt \) is minimized. The initial state \( x(0) = [4, 4]^T \). No constraint is assumed on \( u(t) \).
Effect of the number of random numbers for convergence
to $t_1 = 0.59395$, $t_2 = 2.78328$, $u = 0$
Effect of the number of random numbers for convergence to $t_1 = 1.000, t_2 = 2.000, u = 0$
Observations:

- Solution-1: $J = 3.787 \times 10^{-11}$ with $t_1 = 1.00000$, $t_2 = 2.00000$
  

- Solution-2: $J = 1.142 \times 10^{-11}$ with $t_1 = 0.59396$, $t_2 = 2.78329$
  
  – This result is new and a surprise.

- LJ+IDP direct search method is powerful and can reveal some interesting results.
Example 2

Consider a switched system consisting of

subsystem 1: \[
\dot{x} = \begin{bmatrix}
0.6 & 1.2 \\
-0.8 & 3.4
\end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u;
\] (3)

and

subsystem 2: \[
\dot{x} = \begin{bmatrix}
4 & 3 \\
-1 & 0
\end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u.
\] (4)

It is known that \(t_0 = 0\) and \(t_f = 2\). The system switches once at time instant \(t = t_1\) and \(t_1 \in [0, 2]\) from “subsystem 1” to “subsystem 2”. The control objective here is to find the optimal \(t_1\) and the control signal \(u(t)\) (\(t \in [0, 2]\)) such that

\[
J = \frac{1}{2}(x_1(2) - 1)^2 + \frac{1}{2}(x_2(2) - 2)^2
\]  

\[+
\frac{1}{2} \int_0^2 [(x_2(t) - 2)^2 + u^2(t)] \, dt
\] (5)

is minimized. The initial state is given as \(x(0) = [0, 2]^T\).
Use of dichotomous search to obtain $t_1 = 0.190$ with the use of 20 stages
Control policy for Example 2, with the use of 20 time stages, yielding $J = 9.80750$
Control policy for Example 2, with the use of 40 time stages, yielding $J = 9.78095$
Control policy for Example 2, with the use of 80 time stages, yielding $J = 9.77578$
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Concluding Remarks

- For switched systems, the optimal switching times and the optimal control policy can be established by incorporating the Luus-Jaakola optimization procedure into the iterative dynamic programming formulation.

- This works well if there is not a very large change in the control policy when the switching from one system to another takes place.

- However, if there is a very large change in the optimal control policy when switching from one system to another, the optimization can be done separately, as illustrated by the second example.

- The approaches presented here provide alternative procedures to existing methods and are applicable to nonlinear systems, and constraints can be readily handled.

- **Further investigation:** For the case with unknown number of switches.
Thank you!

Q/A session

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