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Two direct Tustin discretization methods for fractional-order differentiator/integrator

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Abstract

This paper deals with fractional calculus and its approximate discretization. Two direct discretization methods useful in control and digital filtering are presented for discretizing the fractional-order differentiator or integrator. Detailed mathematical formulae and tables are given. An illustrative example is presented to show the practically usefulness of the two proposed discretization schemes. Comparative remarks between the two methods are also given.

Keywords: Fractional differentiator; Fractional-order controllers; Tustin operator; Power series expansion; Continued fraction expansion; Discretization

1. Introduction

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with correspondence between Leibniz and L'Hospital in 1695. There are many applications of the fractional-order calculus such as physical system modeling [1], control theory (e.g.

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[2–8]), to name a few. For the latest development of fractional calculus in automatic control and robotics, we cite [9]. For practical application of the fractional-order models (e.g. for realization of fractional-order controllers (FOC)), one needs to discretize the FOC. It is well known that the fractional-order systems involve unlimited memory (infinite dimensional) while the integer-order systems are with limited memory (finite dimensional). It is important to approximately describe the fractional-order system using a finite difference equation. To do so, rational approximations [10] are often used mainly in continuous-time domain. In practice, direct discretization is more preferred. The work of this paper provides a way to achieve direct discretization of fractional-order operator using Tustin operator.

In this paper, two practically useful direct discretization methods are presented and compared by some illustrative examples. The first one is the recursive Tustin discretization scheme based on Muir's recursion (`Tustin+Muir`). The other scheme is the continued fraction expansion (CFE) of the Tustin operator (`Tustin+CFE`). Two direct discretization schemes are then applied, as an illustrative example, to a double integrator plant with an uncertain gain. The robustness of FOC is demonstrated and the two direct discretization schemes are compared. It is found that `Tustin+CFE` scheme is better in terms of accuracy while `Tustin+Muir` is attractive for its nice closed-form recursion. Both schemes presented are applicable in FOC implementation. Note that the discretization schemes presented in [11] were based on different operators and therefore, the reported results were hard to compare. In this paper, based on the *same* Tustin operator, the two discretization schemes are now comparable. Moreover, in this paper, an illustrative example is included to demonstrate how a fractional-order controller can be applied to a double integrator plant with an uncertain gain.

This paper is organized as follows: in Section 2, fractional-order derivative and its discretization are briefly reviewed; Section 3 presents a new direct discretization scheme based on Tustin operator and Muir recursion; Section 4 details another direct discretization scheme based on the Tustin operator and the continued fraction expansion method; Section 5 presents a fractional-order control of a double integrator plant with possible uncertainty in the plant gain. Section 6 concludes this paper with some remarks.

2. Fractional-order derivative and its discretization

Fractional calculus is a generalization of integration and differentiation to a fractional, or non-integer, order fundamental operator ${}_aD_t^r$, where a and t are the limits and r, ($r \in \mathbb{R}$) the order of the operation. Two commonly used definitions for the general fractional integrodifferential are the Grünwald–Letnikov (GL) definition and the Riemann–Liouville (RL) definition [8,12]. The GL definition is that

$$ {}_aD_t^r f(t) = \lim_{h \to 0} h^{-r} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{r}{j} f(t-jh), \tag{1} $$

where $[\cdot]$ means the integer part, $\binom{r}{j}$ is the fractional binomial coefficient. while the RL definition is

$$_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} \mathrm{d}\tau \tag{2}$$

for $(n-1<r<n)$ and where $\Gamma(\cdot)$ is the Euler's *gamma* function.

For convenience, Laplace domain notion is usually used to describe the fractional integro-differential operation [8]. The Laplace transform of the RL fractional derivative/integral (2) under zero initial conditions for order r, $(0<r<1)$ is given by [12]:

$$\pounds\{ _aD_t^{\pm r} f(t); s\} = s^{\pm r} F(s), \tag{3}$$

where $F(s)$ is the normal Laplace transform of $f(t)$ and $a=0$.

The key point in digital implementation of an FOC is the numerical evaluation or discretization of the fractional-order differentiator s^r. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods, two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s-transfer function. In this paper, we focus on the *direct discretization* method.

The simplest and most straightforward method is the direct discretization using finite memory length expansion from GL definition (1). This approach is based on the fact that, for a wide class of functions, the two definitions—GL (1) and RL (2)—are equivalent [8]. In general, the discretization of fractional-order differentiator/integrator $s^{\pm r}$, $(r \in \mathbb{R})$ can be expressed by the so-called generating function $s = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients [13]. For example, when a backward difference rule is used, i.e., $\omega(z^{-1}) = (1-z^{-1})/T$, performing the power series expansion (PSE) of $(1-z^{-1})^{\pm r}$ gives the discretization formula for the GL definition (1). By using the short memory principle [8], the discrete equivalent of the fractional-order integro-differential operator, $(\omega(z^{-1}))^{\pm r}$, is given by

$$D^{\pm(r)}(z) = (\omega(z^{-1}))^{\pm r} = T^{\mp r} z^{-[L/T]} \sum_{j=0}^{[L/T]} (-1)^j \binom{\pm r}{j} z^{[L/T]-j}, \tag{4}$$

where T is the sampling period, L is the memory length and $(-1)^j \binom{\pm r}{j}$ are binomial coefficients $c_j^{(r)}$, $(j = 0, 1, \ldots)$ where

$$c_0^{(r)} = 1, \qquad c_j^{(r)} = \left(1 - \frac{1+(\pm r)}{j}\right) c_{j-1}^{(r)}. \tag{5}$$

It is very important to note that the PSE scheme leads to approximations in the form of polynomials, that is, the discretized fractional-order derivative is in the form of FIR (finite impulse response) filters. Taking into account that our aim is to obtain discrete equivalents to the fractional integrodifferential operators in the Laplace domain, $s^{\pm r}$, the following considerations have to be made:

(1) s^r, $(0<r<1)$, viewed as an operator, has a branch cut along the negative real axis for arguments of s on $(-\pi, \pi)$ but is free of poles and zeros.

(2) A dense interlacing of simple poles and zeros along a line in the s plane is, in some way, equivalent to a branch cut.
(3) It is well known that, for interpolation or evaluation purposes, rational functions are sometimes superior to polynomials, roughly speaking, because of their ability to model functions with zeros and poles. In other words, for evaluation purposes, compared to PSE, the rational approximation usually converges much more rapidly and has a wider domain of convergence in the complex plane.
(4) The Tustin transformation or the trapezoidal rule maps adequately the stability regions of the s plane on the z plane, and maps the points $s = 0$, $s = -\infty$ to the points $z = 1$ and -1, respectively.

Therefore, in this paper, for the direct discretization of s^r, $(0<r<1)$, we shall concentrate on the trapezoidal rule or Tustin operator as the generating function as follows:

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm r}. \tag{6}$$

In expanding the above into a rational function, we shall use two techniques. The first one is based on Muir-recursion applied to numerator and denominator of the Tustin operator and the second one is by the continued fraction expansion. It should be pointed out that, for control applications, the obtained approximate discrete-time rational transfer function should be stable and minimum phase. Furthermore, for a better fit to the continuous frequency response, it would be of high interest to obtain discrete approximations with poles an zeros interlaced along the line $z\in(-1,1)$ of the z plane. As it will be shown later, the two direct discretization approximations proposed in this paper enjoy the above desirable properties. In the next sections, we first introduce the new direct discretization scheme by recursive Tustin transformation followed by the second direct discretization scheme by continued fraction expansion of the Tustin operator.

3. Direct discretization by recursive Tustin transformation

One of the key points of Tustin discretization of fractional-order differentiator is how to get a recursive formula similar to (5) in the preceeding subsection. Here, we introduce the so-called Muir-recursion scheme, which was originally used in geophysical data processing with applications to petroleum prospecting [14]. The Muir-recursion was motivated in computing the vertical plane wave reflection response via the impedance of a stack of n-layered earth. This scheme can be used in recursive discretization of fractional-order differentiator of Tustin generating function. In the following, without loss of generality, assume that $r\in[-1,1]$. In order to simplify the presentation, we only give the recursive formula for positive r

$$(\omega(z^{-1}))^{r} = \left(\frac{2}{T}\right)^{r}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{r} = \left(\frac{2}{T}\right)^{r}\lim_{n\to\infty}\frac{A_n(z^{-1},r)}{A_n(z^{-1},-r)}, \tag{7}$$

where

$$A_0(z^{-1},r)=1,\quad A_n(z^{-1},r)=A_{n-1}(z^{-1},r)-c_n z^n A_{n-1}(z,r) \tag{8}$$

and

$$c_n=\begin{cases} r/n, & n \text{ is odd},\\ 0, & n \text{ is even}.\end{cases} \tag{9}$$

For any given order of approximation n, we can use MATLAB symbolic toolbox to generate an expression for $A_n(z^{-1},r)$. Therefore,

$$s^r \approx \left(\frac{2}{T}\right)^r \frac{A_n(z^{-1},r)}{A_n(z^{-1},-r)}.$$

For a ready reference, we listed $A_n(z^{-1},r)$ in Table 1 up to $n=9$, which should be sufficient in many applications.

Remark 3.1. To examine the correctness of the Muir-recursion used for the recursive discretization of the fractional-order derivative operator, one can compare the symbolic Taylor expansion of (6). It has been verified that the proposed recursive formula is as correct as Taylor series expansion till the order of approximation.

As an example, using the recursive method described in this section, the discretization of $s^{0.5}$ sampled at 0.001 s is studied numerically, and the approximate models are

$$G_1(z)=\frac{44.72z-22.36}{z+0.5},\quad G_3(z)=\frac{44.72z^3-22.36z^2+3.727z-7.454}{z^3+0.5z^2+0.08333z+0.1667},$$

Table 1
Table of formulae $A_n(z^{-1},r)$ for $n=1,\ldots,9$

n	$A_n(z^{-1},r)$
0	1
1	$-rz^{-1}+1$
3	$-\frac{1}{3}rz^{-3}+\frac{1}{3}r^2z^{-2}-rz^{-1}+1$
5	$-\frac{1}{5}rz^{-5}+\frac{1}{5}r^2z^{-4}-\left(\frac{1}{3}r+\frac{1}{15}r^3\right)z^3+\frac{2}{5}r^2z^{-2}-rz^{-1}+1$
7	$-\frac{1}{7}rz^{-7}+\frac{1}{7}r^2z^{-6}-\left(\frac{1}{5}r+\frac{2}{35}r^3\right)z^{-5}+\left(\frac{26}{105}r^2+\frac{1}{105}r^4\right)z^{-4}$ $-\left(\frac{1}{3}r+\frac{2}{21}r^3\right)z^{-3}+\frac{3}{7}r^2z^{-2}-rz^{-1}+1$
9	$-\frac{1}{9}rz^{-9}+\frac{1}{9}r^2z^{-8}-\left(\frac{1}{7}r+\frac{1}{21}r^3\right)z^{-7}+\left(\frac{34}{189}r^2+\frac{2}{189}r^4\right)z^{-6}$ $-\left(\frac{1}{5}r+\frac{16}{189}r^3+\frac{1}{945}r^5\right)z^{-5}+\left(\frac{17}{63}r^2+\frac{1}{63}r^4\right)z^{-4}$ $-\left(\frac{1}{3}r+\frac{1}{9}r^3\right)z^{-3}+\frac{4}{9}r^2z^{-2}-rz^{-1}+1$

$$G_7(z) = \frac{44.72z^7 - 22.36z^6 + 4.792z^5 - 7.986z^4 + 2.795z^3 - 4.792z^2 + 1.597z - 3.194}{z^7 + 0.5z^6 + 0.1071z^5 + 0.1786z^4 + 0.0625z^3 + 0.1071z^2 + 0.0357z + 0.07143},$$

$$G_9(z) = \frac{\begin{array}{c}44.72z^9 - 22.36z^8 + 4.969z^7 - 8.075z^6 + 3.061z^5 - 4.947z^4 + \\ 2.041z^3 - 3.461z^2 + 1.242z - 2.485\end{array}}{\begin{array}{c}z^9 + 0.5z^8 + 0.1111z^7 + 0.1806z^6 + 0.06845z^5 + 0.1106z^4 + \\ 0.04563z^3 + 0.07738z^2 + 0.02778z + 0.05556\end{array}}.$$

We present four plots as shown in Fig. 1 and to show the effectiveness of the approximate discretization with Z-transfer function given above, the approximations are compared with the exact solution (straight lines).

It should be pointed out that the direct discretization method introduced above always gives a Z-transfer function with stable minimum phase characteristics.

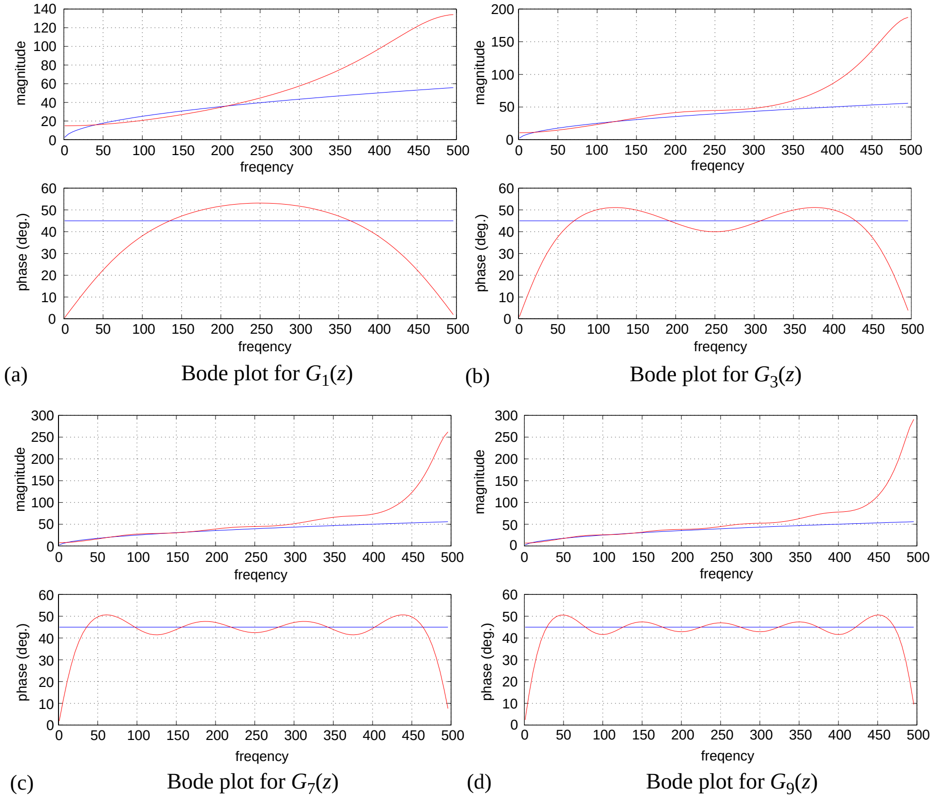

Fig. 1. Approximate discretization of $s^{0.5}$ at $T = 0.001$ s.

4. Direct discretization by continued fraction expansion of Tustin transformation

It is well known that, compared to the power series expansion method, the continued fraction expansion (CFE) is a method for evaluation of functions with faster convergence and larger domain of convergence in the complex plane [15,16]. Using CFE, an approximation for any irrational function $G(z)$ can be expressed in the form

$$\begin{aligned} G(z) &\simeq a_0(z) + \cfrac{b_1(z)}{a_1(z) + \cfrac{b_2(z)}{a_2(z) + \cfrac{b_3(z)}{a_3(z) + \cdots}}} \\ &= a_0(z) + \frac{b_1(z)}{a_1(z)+}\,\frac{b_2(z)}{a_2(z)+}\,\frac{b_3(z)}{a_3(z)+}\cdots, \end{aligned} \tag{10}$$

where a_i's and b_i's are either rational functions of the variable z or constants. The application of the method yields a rational function, $\hat{G}(z)$, which is an approximation of the irrational function $G(z)$.

The resulting discrete transfer function, approximating fractional-order operators, can be expressed as

$$\begin{aligned} D_{\pm r}(z) &= \frac{Y(z)}{F(z)} = \left(\frac{2}{T}\right)^{\pm r} \mathrm{CFE}\left\{ \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm r} \right\}_{p,q} \\ &= \left(\frac{2}{T}\right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})}, \end{aligned} \tag{11}$$

where T is the sampling period, $\mathrm{CFE}\{u\}$ denotes the function from applying the continued fraction expansion to the function u, $Y(z)$ is the Z transform of the output sequence $y(nT)$, $F(z)$ is the Z transform of the input sequence $f(nT)$, p and q are the orders of the approximation, and P and Q are polynomials of degrees p and q, correspondingly, in the variable z^{-1}.

By using MAPLE or MATLAB Symbolic Math Toolbox, the obtained symbolic approximation has the following form:

$$D_r(z) = 1 + \cfrac{z^{-1}}{-\frac{1}{2}\frac{1}{r} + \cfrac{z^{-1}}{-2 + \cfrac{z^{-1}}{\frac{3}{2}\frac{r}{r^2-1} + \cfrac{z^{-1}}{2 + \cfrac{z^{-1}}{-\frac{5}{2}\frac{r^2-1}{r(-4+r^2)} + \cfrac{z^{-1}}{-2+\cdots}}}}}}. \tag{12}$$

In Table 2, the general expressions for numerator and denominator of $D_r(z)$ in (11) are listed for $p = q = 1, 3, 5, 7, 9$.

Table 2
Numerators and denominators of $D_r(z)$ (12) for different r

$p=q$	$P_p(z^{-1})$ $(k=1)$, and $Q_q(z^{-1})(k=0)$
1	$(-1)^k z^{-1} r + 1$
3	$(-1)^k (r^3 - 4r) z^{-3} + (6r^2 - 9) z^{-2} + (-1)^k 15 z^{-1} r + 15$
5	$(-1)^k (r^5 - 20r^3 + 64r) z^{-5} + (-195r^2 + 15r^4 + 225) z^{-4} + (-1)^k (105r^3 - 735r) z^{-3} + (420r^2 - 1050) z^{-2} + (-1)^k 945 z^{-1} r + 945$
7	$(-1)^k (784r^3 + r^7 - 56r^5 - 2304r) z^{-7} + (10612r^2 - 1190r^4 - 11025 + 28r^6) z^{-6} + (-1)^k (53487r + 378r^5 - 11340r^3) z^{-5} + (99225 - 59850r^2 + 3150r^4) z^{-4} + (-1)^k (17325r^3 - 173250r) z^{-3} + (-218295 + 62370r^2) z^{-2} + (-1)^k 135135 z^{-1} r + 135135$
9	$(-1)^k (-52480r^3 + 147456r + r^9 - 120r^7 + 4368r^5) z^{-9} + (45r^8 + 120330r^4 - 909765r^2 - 4410r^6 + 893025) z^{-8} + (-1)^k (-5742495r - 76230r^5 + 1451835r^3 + 990r^7) z^{-7} + (-13097700 + 9514890r^2 - 796950r^4 + 13860r^6) z^{-6} + (-1)^k (33648615r - 5405400r^3 + 135135r^5) z^{-5} + (-23648625r^2 + 51081030 + 945945r^4) z^{-4} + (-1)^k (-61486425r + 4729725r^3) z^{-3} + (16216200r^2 - 72972900) z^{-2} + (-1)^k 34459425 z^{-1} r + 34459425$

With $r = 0.5$ and $T = 0.001$ s, the approximate models for $p = q = 1, 3, 7, 9$ are

$$G_1(z) = 44.72\frac{z - 0.5}{z + 0.5}, \quad G_3(z) = 44.72\frac{z^3 - 0.5z^2 - 0.5z + 0.125}{z^3 + 0.5z^2 - 0.5z - 0.125},$$

$$G_7(z) = 44.72\frac{z^7 - 0.5z^6 - 1.5z^5 + 0.625z^4 + 0.625z^3 - 0.1875z^2 - 0.0625z + 0.007813}{z^7 + 0.5z^6 - 1.5z^5 - 0.625z^4 + 0.625z^3 + 0.1875z^2 - 0.0625z - 0.007813},$$

$$G_9(z) = 44.72\frac{z^9 - 0.5z^8 - 2z^7 + 0.875z^6 + 1.313z^5 - 0.4688z^4 - 0.3125z^3 + 0.07813z^2 + 0.01953z - 0.001953}{z^9 + 0.5z^8 - 2z^7 - 0.875z^6 + 1.313z^5 + 0.4688z^4 - 0.3125z^3 - 0.07813z^2 + 0.01953z + 0.001953}.$$

In Figs. 2 and 3, the Bode plots and the distributions of zeros and poles of the approximations are presented. In Fig. 2, the effectiveness of the approximations fitting the ideal responses in a wide range of frequencies, in both magnitude and phase, can be observed. In Fig. 3, it can be observed that the approximations fulfill the two desired properties: (i) all the poles and zeros lie inside the unit circle, and

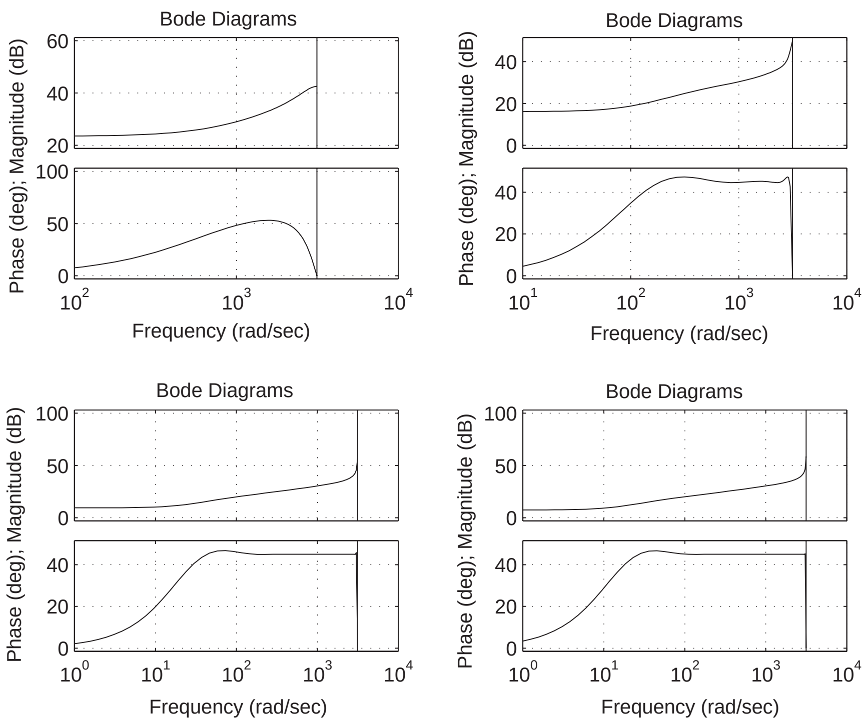

Fig. 2. CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ s. Bode plots (approximation orders 1, 3, 7, 9).

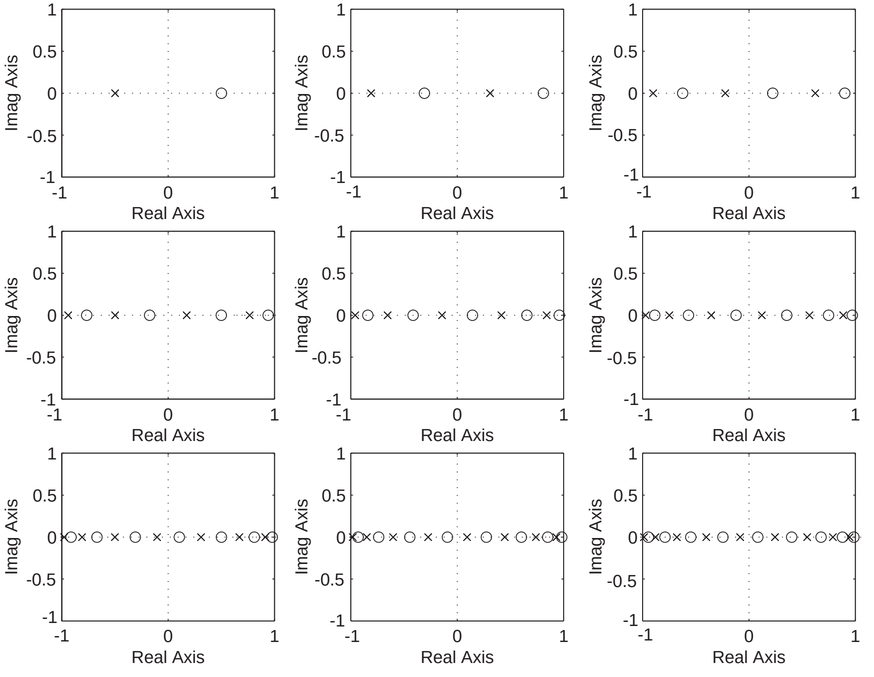

Fig. 3. CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ s. Zero-pole distribution (approximation orders $1, 2, \ldots, 9$).

(ii) the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$.

5. An illustrative application example

Consider a system with the following transfer function in the form of a double integrator

$$H(s) = \frac{A}{s^2},$$

where the gain A is uncertain.

As can be seen in [17], this plant is one of the most fundamental systems in control applications, representing single-degree-of-freedom translational and rotational motion, with applications in many practical problems (see [17] and references included): low-friction, free rigid-body motion, single-axis spacecraft rotation and rotary crane motion. The double integrator plant model has also been used in flexible robotics (see [18]).

Suppose a fractional-order controller of the form

$$D(s) = s^r, \quad 0 < r < 1 \tag{13}$$

is to be used. The open-loop transfer function of the controlled system will be of the form

$$F_o(s) = D(s)G(s) = \frac{A}{s^{2-r}}.$$

The above transfer function is in the form of the Bode's ideal transfer function [8] with the following properties:

(a) *Open loop*:
   1. the amplitude curve has a constant slope of $-(2-r)$;
   2. the crossover frequency depends only on A;
   3. the phase curve is a horizontal line at $-(2-r)(\pi/2)$;
   4. the Nyquist curve is a straight line through the origin with argument $-(2-r)(\pi/2)$

(b) *Closed loop with unity negative feedback*:
   1. the transfer function has the form
   $$F_c(s) = \frac{A}{s^{2-r} + A}; \tag{14}$$
   2. the gain margin is infinite;
   3. the phase margin is constant, $\Phi_m = \pi\left(1 - \frac{2-r}{2}\right)$;
   4. the step response has the expression (see [8,19]):
   $$y(t) = At^{2-r}E_{2-r,2-r+1}(-At^{2-r}),$$
   where $E_{2-r,2-r+1}(-At^{2-r})$ is the Mittag–Leffler function in two parameters. Assuming $A \in \mathbb{R}^+$, such a step response exhibits an overshoot independent of parameter A and dependent only on the parameter r, the fractional-order. This is a desired property in some applications such as car suspension control system, etc.

If $A = 100$, $r = 0.5$, the following properties can be achieved:

- phase margin, $\Phi_m = 45^\circ$,
- rise time, $t_r = 0.018$ s,
- overshoot, $M_p = 35\%$,
- peak time, $t_p = 0.029$ s.

In Figs. 4–6, the following results are displayed: Bode plots for $D_1(z)$ and $D_2(z)$ (Fig. 4), the two discrete approximations of the controller $D(s)$ (13) where $D_1(z)$ is via `Tustin+Muir` scheme with $n = 7$ and $D_2(z)$ the `Tustin+CFE` scheme with $p = q = 7$; Bode plots for the controlled system with several values of A (Fig. 5), and the step responses of the controlled system for several values of A (Fig. 6). As can be observed, the `Tustin+CFE` scheme performs a better frequency-domain approximation with a more flat phase response in a wider frequency range. This leads to the

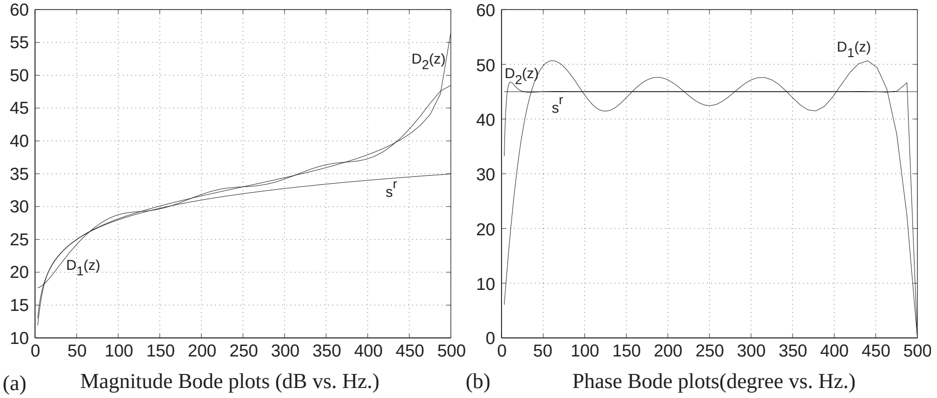

Fig. 4. Comparison of the two direct discretization schemes ($D_1(z)$: `Tustin+Muir`; $D_2(z)$: `Tustin+CFE`): (a) Magnitude Bode plots (dB vs. Hz); (b) phase Bode plots (degree vs. Hz).

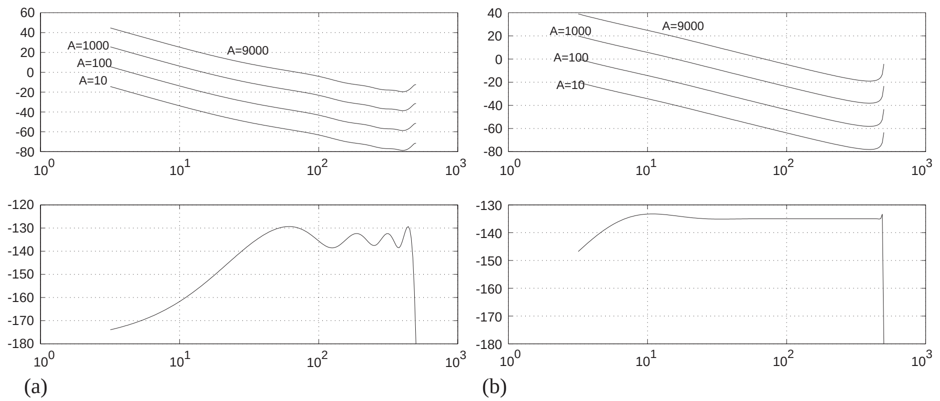

Fig. 5. Comparison of the two direct discretization schemes at different gains ($D_1(z)$:`Tustin+Muir`; $D_2(z)$:`Tustin+CFE`): (a) Bode plots for `Tustin+Muir` (top: amplitude (dB) vs. frequency (Hz); bottom: phase (deg) vs. frequency (Hz)); (b) Bode plots for `Tustin+CFE` (top: amplitude (dB) vs. frequency (Hz); bottom: phase (deg) vs. frequency (Hz)).

time-domain behavior of the controlled system closer to the theoretical one for a wider range of A values. The most interesting feature of the fractional-order controlled system is the equal-overshoot behavior when A varies. The overshoot is close to 35%, the theoretically predicted value for $A \in (1000, 9000)$ when $D_2(z)$ is used as the controller. When $D_1(z)$ is used, we can see that the 35% overshoot can be maintained only when $A \geqslant 9000$. Clearly, due to a better phase approximation of $D_2(z)$ to $D(s)$, compared to $D_1(z)$, $D_2(z)$ gives better time-domain performance which is also in accordance with the obtained phase margins of the controlled system. We can see that with the introduction of a fractional-order controller, in terms of overshoot and oscillation/damping, the control performance is more robust with

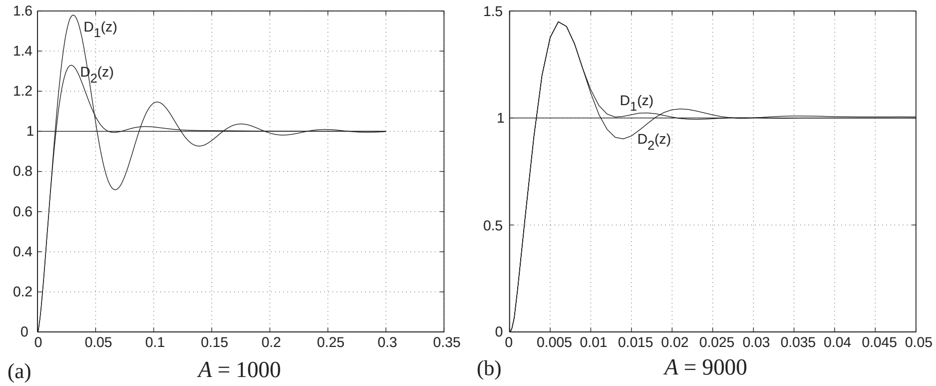

Fig. 6. Step response comparison (output vs. time in second): (a) $A = 1000$; (b) $A = 9000$.

respect to the uncertain plant gain. As we have demonstrated above, the plant gain can be allowed to vary in a very large range. Under the same order of approximation, compared to the `Tustin+Muir` scheme, `Tustin+CFE` scheme gives a better fit to the original continuous FOC. However, `Tustin+Muir` is more attractive in the sense that it has a nice closed-form recursive expansion formulae which may be useful when the order of approximation of FOC would be determined in real time.

6. Concluding remarks

We have presented two direct discretization schemes for implementation of fractional-order controller. The first scheme uses the Muir recursion formula for recursive Tustin operator expansion while the other scheme is by the continued fraction expansion. Practically useful formula and tables are given. Illustrative examples are presented to show the practically usefulness of the two proposed discretization schemes. Comparative remarks between the two methods are also given.

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