IDENTIFICATION OF A FRACTIONAL LINEAR DIFFUSION-WAVE EQUATION FROM NOISY BOUNDARY MEASUREMENTS [1]

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Abstract: In this paper, a hybrid symbolic and numerical method is used to identify the unknown wave constant, fractional order and boundary profile of a fractional order diffusion-wave equation based on boundary measurements. The unknown boundary profile is parameterized by a polynomial. The measurement noise is also considered. Two extreme cases for both noise-free and noisy measurements are also presented to show the robustness of the proposed identification method. The effectiveness and advantages are demonstrated via simulation examples.

Keywords: Parameter estimation, fractional diffusion-wave equation, numerical optimization, symbolic computation.

1. INTRODUCTION

In this paper, a hybrid symbolic and numerical method is used to identify the unknown wave constant, fractional order and boundary profile of a fractional order diffusion-wave equation based on boundary measurements possibly corrupted with measurement noise. Fractional wave equations are obtained from the classical diffusion (resp. wave) equations by replacing the first (resp. second) order time derivative term by a fractional-order derivative with order $1 < \alpha < 2$. Since many of the universal phenomenons can be modelled accurately using the fractional diffusion-wave equations (see (Nigmatullin, 1986)), there has been a growing interest in investigating the solutions and properties of these equations. Research has been focused on the analytical solution to the fractional diffusion and wave equations (see (Wyss, 1986), (Schneider, 1989), (Mainardi and Paradisi, 1997), and (Agrawal, 2002)). Compared with the publications on the control of the wave equation (see (Morgül, 2002), (Chen, 1979), (Morgül, 1998), and (Datko et al., 1986)), research results on the control of fractional wave equations are very few (see (Liang et al., 2003)). In control engineering, system identification and parameter estimation play an important role, because the more knowledge we have about the plant to be controlled, the better the controller we can design. To the best of the authors' knowledge, this is the first research result on the parameter estimation of systems governed

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by the fractional diffusion-wave equations. We will demonstrate how we can identify the unknown diffusion-wave constant, unknown fractional order and the unknown initial boundary profile simultaneously from the boundary measurements.

The paper is organized as follows. In Sec. 2, the problem formulation is given. In Sec. 3, the system identification method used in this paper is presented. In Sec. 4, we validate the algorithm via simulation examples. Sec. 5 studies two extreme cases. Finally, Sec. 6 concludes this paper.

2. PROBLEM FORMULATION

Consider a cable made from special materials, with one end fixed and the other end free, governed by the following fractional diffusion-wave equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = b^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 1 < \alpha < 2,\ x \in [0,1],\ t \geq 0, \quad (1)$$

where $u(x,t)$ is the displacement of the cable at $x \in [0,1]$ and $t \geq 0$, α is the parameter describing the order of the fractional derivative, b is a constant decided by the tension and the mass per unit length of the cable. Clearly, this is a special type of fractional order PDE system in-between the diffusion equation ($\alpha = 1$) and the wave equation ($\alpha = 2$).

Equation (1) is subject to the following initial and boundary conditions

$$u(0,t) = 0, \quad (2)$$

$$u_x(1,t) = f(t), \quad (3)$$

$$u(x,0) = u_0(x), \quad (4)$$

$$u_t(x,0) = v_0(x), \quad (5)$$

where $f(t)$ is the boundary control force at the free end, $u_0(x)$ and $v_0(x)$ are the initial displacement condition (initial boundary profile) and the initial velocity profile, respectively.

The Caputo definition of fractional derivative of order α of function $f(t)$ (see (Caputo, 1967; Podlubny, 1999)) is used throughout this paper as follows:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < \alpha \leq n). \quad (6)$$

In this paper we assume the values of α and b are not exactly known and need to be estimated. Furthermore, initial profiles $u_0(x)$ and $v_0(x)$ may be unknown and need to be identified, too. We also assume that the displacement of the free end $u(1,t)$ can be measured for the identification task.

3. THE PROPOSED IDENTIFICATION METHOD

For simplicity of our presentation and for practical reasons, we consider a simplified scenario. That is, with the initial shape $u_0(x)$, possibly unknown, initial velocity $v_0(x) = 0$ (cable is initially at rest), and no boundary force $f(t) = 0$, we can measure the displacement of the free end $u(1,t)$ as the system output measurement data for identification.

If equations (1)-(5) can be solved, we can estimate α and b through an optimization program to make the solution fit the measurement data as closely as possible.

There are three problems with the above idea. First, since we want to simulate the parameter estimation problem, how to generate the measurement data given an α, b, and $u_0(x)$? Second, if we want to do a real experiment rather than generating the "measured data" via simulation, it is hard to make the actual initial shape $u_0(x)$ the same as the desired one. Third, how to solve equations (1)-(5)?

The first problem can be solved by numerically solving equations (1)-(5), since the analytical solution is still an unsolved problem. We developed a method combining the symbolic computation and numerical computation to solve the fractional diffusion-wave equation, effective even if $f(t)$, a boundary feedback controller, is included. The solution plus Gaussian noise, which is unavoidable in the actual experiments, can be used as the measured data. It is illustrated below how to solve (1)-(5), assuming $\alpha = 1.75$, $b = 0.5$, $f(x) = 0$, and the initial conditions

$$u_0(x) = -\frac{1}{2}\sin(\frac{1}{2}\pi x), \quad (7)$$

$$v_0(x) = 0. \quad (8)$$

Based on the definition of (see Equation 6), the Laplace transform of the Caputo fractional derivative is (Podlubny, 1999)

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^k(0^+) s^{\alpha-1-k}. \quad (9)$$

Taking the Laplace transform of (1)- (3) with respect to t, we obtain the following ODE (Ordinary Differential Equation) and boundary conditions

$$\frac{\mathrm{d}^2 U(x,s)}{\mathrm{d}x^2} - 4s^{\frac{7}{4}} U(x,s) = 2s^{\frac{3}{4}} \sin(\frac{1}{2}\pi x) \quad (10)$$

$$U(0,s) = 0 \quad (11)$$

$$\frac{\mathrm{d}U(x,s)}{\mathrm{d}x}|_{x=1} = 0 \quad (12)$$

where $U(x,s)$ is the Laplace transform of $u(x,t)$.

Solving (10), we have

$$U(x,s) = e^{-2\,s^{\frac{7}{8}}x}C_1 + e^{2\,s^{\frac{7}{8}}x}C_2 \\ -8\,\frac{s^{3/4}\sin\left(1/2\,\pi\,x\right)}{16\,s^{7/4}+\pi^2}) \qquad (13)$$

where C_1 and C_2 are arbitrary constants. By taking derivative of (13) with respect to x, we have

$$\frac{\mathrm{d}U(x,s)}{\mathrm{d}x} = -2\,s^{\frac{7}{8}}e^{-2s^{\frac{7}{8}}x}C_1 + 2\,s^{\frac{7}{8}}e^{2s^{\frac{7}{8}}x}C_2 \\ -\frac{4s^{\frac{3}{4}}\cos\left(\frac{1}{2}\,\pi\,x\right)\pi}{16\,s^{\frac{7}{4}}+\pi^2}. \qquad (14)$$

Substituting (13) and (14) into (11) and (12), respectively, yields

$$C_1 + C_2 = 0, \qquad (15)$$

$$-e^{-2s^{\frac{7}{8}}}C_1 + e^{2s^{\frac{7}{8}}}C_2 = 0. \qquad (16)$$

Solving (15) and (16) simultaneously, we obtain

$$C_1 = C_2 = 0. \qquad (17)$$

So, finally,

$$U(x,s) = -8\,\frac{s^{3/4}\sin\left(1/2\,\pi\,x\right)}{16\,s^{7/4}+\pi^2}. \qquad (18)$$

So far, all the calculation steps illustrated above can be automated via computer symbolic algebra, such as Matlab Symbolic Math Toolbox (MathWorks, 2003). Now, $u(x,t)$ can be obtained by taking the numerical inverse Laplace transform of (18). Among the existing numeric inverse Laplace transform methods, the FFT (Fast Fourier Transform) method is both accurate and fast (see (Duffy, 1993)). We choose the program in (see (Brančík, 1999)) to take the inverse Laplace transform of $U(x,s)$.

We can solve the second problem by treating the initial shape $u_0(x)$ as the extra parameters to be estimated such that the parameter estimation algorithm does not depend on the exact knowledge of $u_0(x)$. Specifically, we assume the initial shape to be parameterized by the following polynomial, which equals to zero at $x=0$,

$$\tilde{u}_0(x) = \sum_{n=1}^{N} a_n x^n \qquad (19)$$

where a_i is the parameter to be estimated. By increasing N, we expect that the estimated initial shape $\tilde{u}_o(x)$ will converge to $u_0(x)$, the real initial shape.

The third problem can be solved using the same procedure to solve the first problem, except using (19) rather than (7).

Now the parameter estimation problem can be formulate as the following nonlinear programming problem

$$\min_{a_0,\cdots,a_N,\tilde{\alpha},\tilde{b}} J(a_0,\cdots,a_N,\tilde{\alpha},\tilde{b}) = \\ \min_{a_0,\cdots,a_N,\tilde{\alpha},\tilde{b}} \sum_{n=0}^{N_s-1} \left(u(1,n\Delta t) - \tilde{u}(1,n\Delta t)\right)^2, \qquad (20)$$

where $a_0,\ldots,a_N,\tilde{\alpha},\tilde{b}$ are the parameters to be estimated; $u(1,n\Delta t)$ are the measured boundary response data at time $n\Delta t$ with the sampling time Δt; $\tilde{u}(1,n\Delta t)$ are the solution to (1)-(5) based on parameters $a_0,\ldots,a_N,\tilde{\alpha},\tilde{b}$ and N_s is the total number of samples.

At the step, the identification problem has been converted to a numerical optimization problem which can be solved by various existing optimization codes. The optimization program we chosen for this study is `SolvOpt` (see (Kuntsevich and Kappel, 1997)), a free program for local nonlinear optimization problems.

4. SIMULATION RESULTS FOR ALGORITHM VALIDATION

To generate the simulated measurement data, the following parameters and initial conditions are used

$$\alpha = 1.75,\; b = 0.5,\; \Delta t = 0.078s,\; N_s = 256$$

$$u_0(x) = -\frac{1}{2}\sin(\frac{1}{2}\pi x)$$

$$v_0(x) = 0.$$

The corresponding solution, i.e., the boundary response $u(1,t)$, from (1)-(5) is plotted in Fig. 1. To simulate the real signal, Gaussian noise is

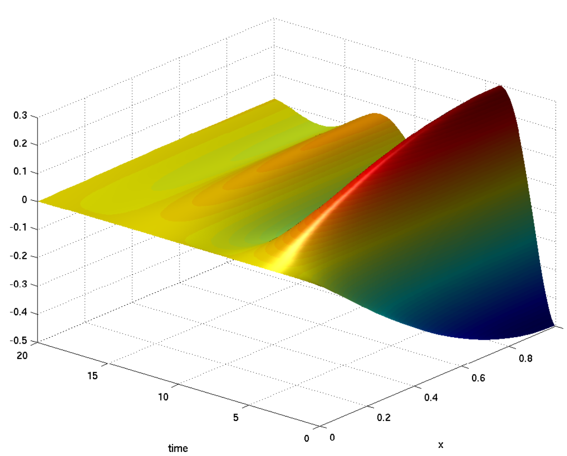

Fig. 1. Displacement of the whole cable $u(x,t)$

added to the displacement of the free end and shown in Fig. 2.

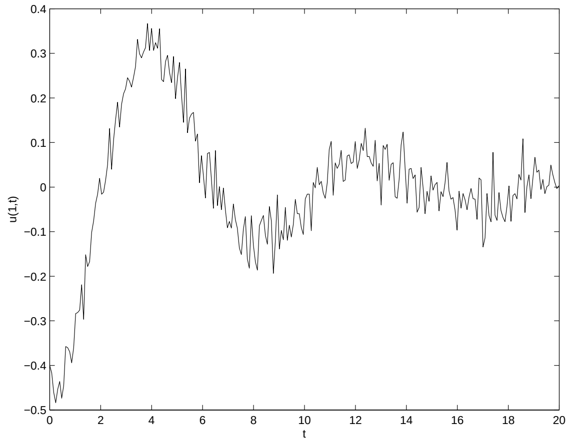

Fig. 2. Boundary measurement data with Gaussian noise added

Table 1. Estimated parameters

	$\tilde{\alpha}$	$\tilde{b}$	α	b
$N = 2$	1.755	0.499	1.75	0.50
$N = 3$	1.751	0.498		

The parameters to be estimated are initialized as follows

$$a_0 = a_1 = \ldots = a_N = 0,\ \tilde{\alpha} = 1.5,\ \tilde{b} = 1.$$

After the optimization process, the estimated initial shapes are plotted in Fig. 3. The estimated parameters are shown in Table 1. Note that, in the sequel, we only report the results for noisy measurement cases if not otherwise stated.

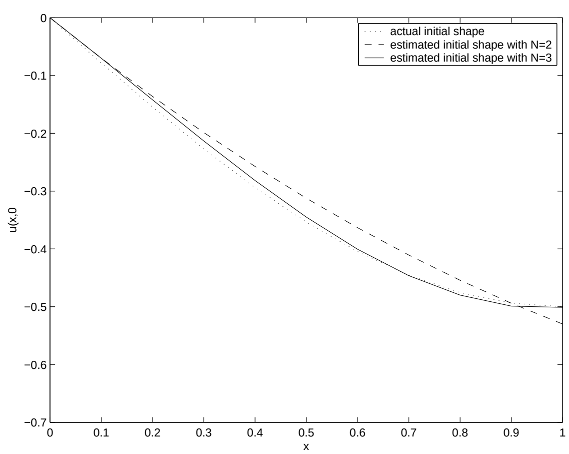

Fig. 3. Estimated initial shape $\tilde{u}_0(x)$ and the actual initial shape $u_0(x)$

We can see that the unknown parameters have been successfully estimated. Although using the third order polynomial generates more accurate initial shape than using the second order polynomial, the estimated α and b are only slightly better, which means this algorithm is not very sensitive to the initial conditions.

Another source of error in this algorithm is from the time mismatch between the measured data and the numerical solution. In (20), $u(1, n\Delta t)$ is desired to be measured at $t = n\Delta t$. However, due to various reasons in practice, especially the inaccuracy of the starting time, the actual time at which $u(1,t)$ is sampled can be slightly different from the desired time instant $n\Delta t$. We simulated the effect of this time mismatch problem by using the following objective function:

Table 2. Estimated parameters for different time mismatch steps

	time mismatch	$\tilde{\alpha}$	$\tilde{b}$	α	b
$N = 2, k = 1$	$0.078s$	1.757	0.503	1.75	0.50
$N = 2, k = 2$	$0.157s$	1.761	0.507		

$$\min\ J(a_0, \ldots, a_N, \tilde{\alpha}, \tilde{b}) = \sum_{n=0}^{N_s-k-1} (u(1,(n+k)\Delta t) - \tilde{u}(1, n\Delta t))^2, \quad (21)$$

i.e., the simulated time mismatch is $k\Delta t$.

Choosing $N = 2$ and $k = 1, 2$, the estimated initial shapes are plotted in Fig. 4. The estimated parameters are summarized in Table 2.

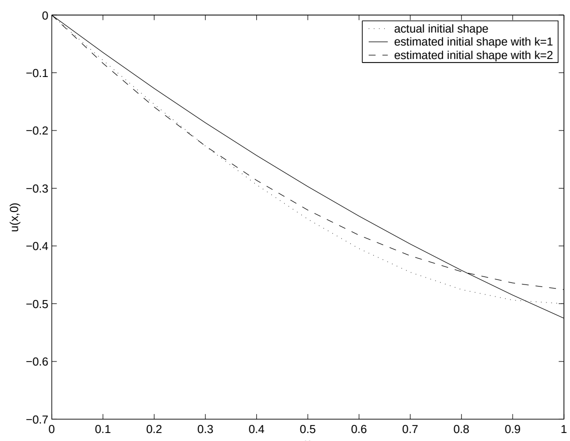

Fig. 4. Estimated initial shape for different time mismatch steps

We can see that even if the time mismatch is relatively big, the accuracy of the estimated parameters is still satisfactory.

5. SIMULATION STUDIES ON TWO EXTREME CASES

In this section, we study two extreme cases, i.e., when α is very close to 1 and when $\alpha = 2$. In many existing schemes, in extreme cases, the estimation accuracy is usually degraded, or even worse, the algorithm may fail. It is meaningful to check the robustness of our proposed algorithms in these extreme cases.

First let us study the parameter estimation when $\alpha = 1.01$. In this case, the fractional wave equation is closer to a diffusion equation than to a wave

Table 3. Estimated parameters when $\alpha = 1.01$

	$\tilde{\alpha}$	$\tilde{b}$	α	b
$N = 2, k = 2$	1.016	0.497	1.01	0.50
$N = 3, k = 2$	1.010	0.506		

equation. All parameters and initial conditions are the same as in Sec. 4 except the α, the fractional order.

The solution to (1)-(5), to be taken as measurement data for system identification, is plotted in Fig. 5. The boundary measurement data with

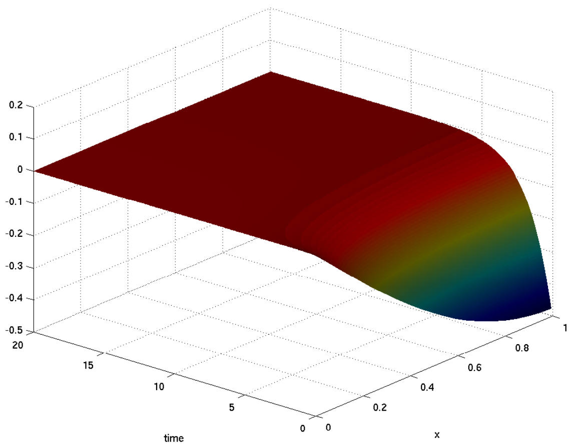

Fig. 5. Displacement of the whole cable $u(x, t)$ when $\alpha = 1.01$

Gaussian noise added is plotted in Fig. 6. The

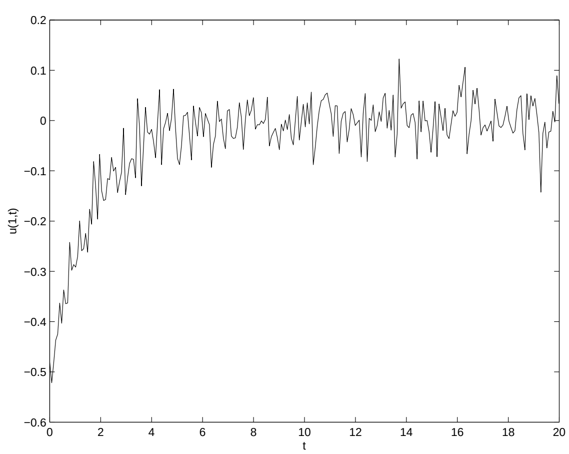

Fig. 6. Boundary measurement data for identification with Gaussian noise added when $\alpha = 1.01$

estimated initial shapes are plotted in Fig. 7. The estimated parameters are shown in Table 3.

Simulation results show that the algorithm works well even if α is very close to 1. The same as before, although it makes the estimation of the initial shape more accurate, increasing N improves the accuracy of the estimated α and b very little.

Next we study the parameter estimation when $\alpha = 2$, i.e., the fractional wave equation becomes the wave equation.

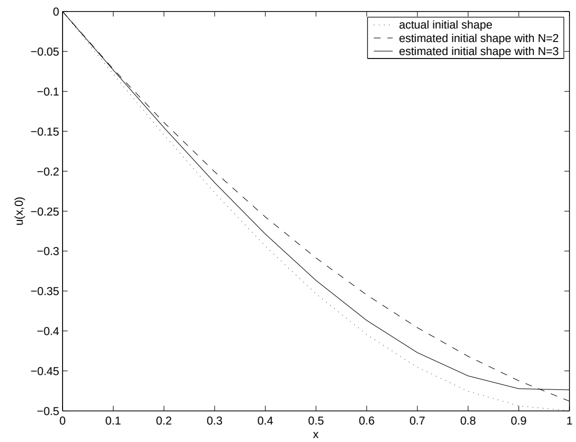

Fig. 7. Estimated initial shape when $\alpha = 1.01$

The solution to (1)-(5) to be used as the measurement data for estimation is plotted in Fig. 8. The

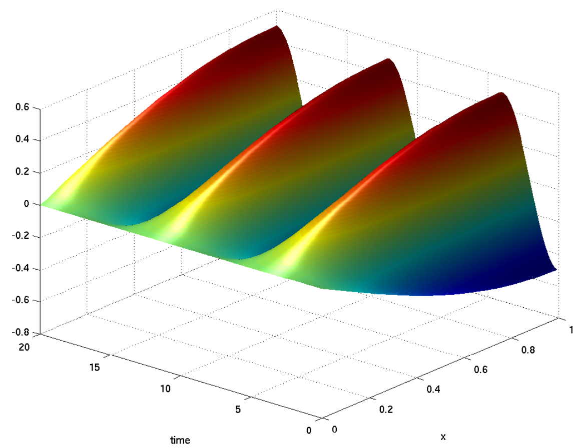

Fig. 8. Displacement of whole cable when $\alpha = 2$

boundary measurement data with Gaussian noise added is plotted in Fig. 9.

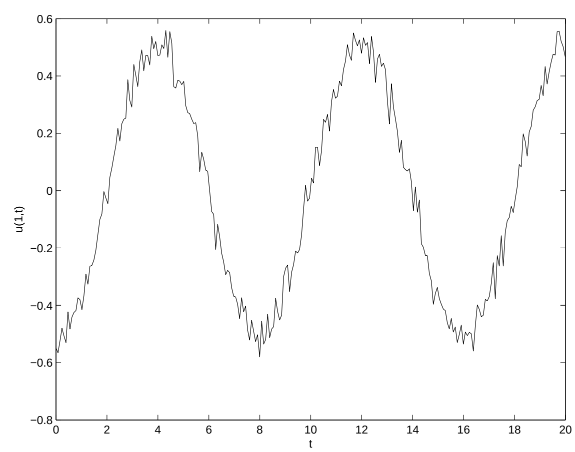

Fig. 9. Boundary measurement data with Gaussian noise added when $\alpha = 1.01$

The estimated initial shapes are plotted in Fig. 10. The estimated parameters are shown in Tab. 4.

In the case of $\alpha = 2$, the algorithm works equally well as in the case $\alpha = 1.01$. It can be concluded

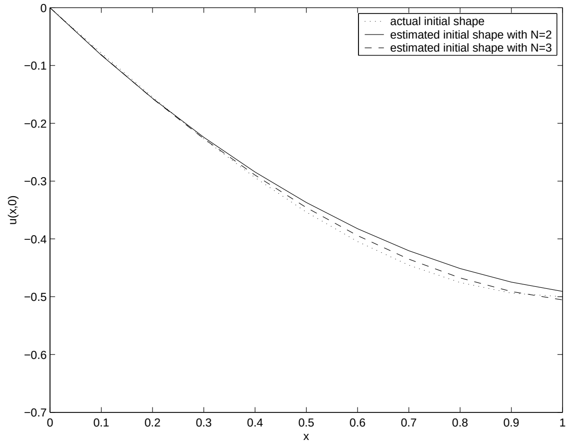

Fig. 10. Estimated initial shape when $\alpha = 2$

Table 4. Estimated parameters when $\alpha = 2$

	$\tilde{\alpha}$	$\tilde{b}$	α	b
$N = 2, k = 2$	2.013	0.508	2.00	0.50
$N = 3, k = 2$	2.002	0.506		

that even if in the extreme cases, the performance does not deteriorate by using our proposed identification algorithm.

6. CONCLUDING REMARKS

Simulation results show the effectiveness of the parameter estimation algorithms proposed in this paper. The algorithm does not rely on the exact knowledge of the actual initial condition and is insensitive to the difference between the actual initial shape in the experiment and the desired initial shape. The accuracy is satisfactory even when facing the relative large time mismatch and extreme cases for fractional order. The presented algorithm is expected to work for the parameter estimation of the temporal-spatial fractional diffusion equations given by $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = b^2 \frac{\partial^\beta u(x,t)}{\partial x^\beta}$ with positive real numbers α and β, which will be our future research efforts.

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