PROPOSALS FOR FRACTIONAL PI$^\lambda$D$^\mu$ TUNING [1]

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Abstract: The objective of this work is to find out optimum settings for a fractional $PI^\lambda D^\mu$ controller in order to fulfil five different design specifications for the closed-loop system, taking advantage of the fractional orders, λ and μ. Since these fractional controllers have two parameters more than the conventional PID controller, two more specifications can be met, improving the performance of the system. For the tuning of the controller an iterative optimization method has been used, based on a nonlinear function minimization. Illustrative examples are presented and simulation results show the effectiveness of this kind of controllers.

Keywords: Fractional Calculus, Fractional Order Controller, Robust Controller, Design Specifications, Tuning Optimization Method

1. INTRODUCTION

The PID controller is by far the most dominating form of feedback in use today. Due to its functional simplicity and performance robustness, the proportional-integral-derivative controller has been widely used in the process industries. Design and tuning of PID controllers have been a large research area ever since Ziegler and Nichols presented their methods in 1942 (see (Ziegler and Nichols 1942)). Specifications, stability, design, applications and performance of the PID controller have been widely treated since then (see (Chen *et al.* 2003) and (Asröm and Hägglund 2000) for additional references).

On the other hand, in recent years it is remarkable the increasing number of studies related with the application of fractional controllers in many areas of science and engineering. This fact is due to a better understanding of the fractional calculus (FC) potentialities revealed by many phenomena such as viscoelasticity and damping, chaos, diffusion and wave propagation, percolation and irreversibility.

In what concerns automatic control theory the FC concepts were adapted to frequency-based methods. The frequency response and the tran-

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sient response of the non-integer integral and its application to control systems was introduced by Manabe (see (Manabe 1961)) and more recently in (Barbosa *et al.* 2003). Oustaloup studied the fractional order algorithms for the control of dynamic systems and demonstrated the superior performance of the $CRONE$ (Commande Robuste d'Ordre Non Entier) method over the PID controller. More recently, Podlubny (see (Podlubny 1999)) proposed a generalization of the PID controller, namely the $PI^{\lambda}D^{\mu}$ controller, involving an integrator of order λ and a differentiator of order μ. He also demonstrated the better response of this type of controller, in comparison with the classical PID controller, when used for the control of fractional order systems. A frequency domain approach by using fractional PID controllers is also studied in (Vinagre *et al.* 2000).

Further research activities are running in order to develope new tuning rules for fractional controllers, studying previously the effects of the non integer order of the derivative and integral parts to design a more effective controller to be used in real-life models. Some of these technics are based on an extension of the classical PID control theory. To this respect, in (Caponetto *et al.* 2002) the extension of derivation and integration order from integer to non integer numbers provides a more flexible tuning strategy and therefore an easier achieving of control requirements with respect to classical controllers. In (Leu *et al.* 2002) an optimal fractional order PID controller based on specified gain margin and phase margin with a minimum ISE criterion has been designed by using a differential evolution algorithm.

A more extensive work has been developed in (Sánchez 1999), where a fractional PID control has been applied for active reduction of vertical tail buffeting. A fractional order control strategy has also been successfully applied in the control of a power electronic buck converter (see (Calderón *et al.* 2003*a*) and (Calderón *et al.* 2003*b*)), more concretely a fractional sliding mode control.

Another approach is the use of a new control strategy to control first-order systems with delay (see (Monje *et al.* 2002)) based on a $D^{\beta}I^{\alpha}$ controller with fractional order integral and derivative parts. Besides, it is being developed another method for plants with long dead-time based on the use of a PI^{α} controller with a fractional integral part of order α (see (Monje *et al.* 2003) and (Chen *et al.* 2004)). From the results obtained, it can be concluded that the system controlled with this type of controller is more robust to gain changes.

In this work it is studied the problem of designing a non integer order $PI^{\lambda}D^{\mu}$ controller of the form:

$$C(s) = k_p + \frac{k_i}{s^{\lambda}} + k_d s^{\mu} \quad (1)$$

The interest of this kind of controller is justified by a better flexibility, since it exhibits fractional powers (λ and μ) of the integral and derivative parts, respectively. Thus, five parameters can be tuned in this structure (λ, μ, K_p, K_i and K_d), that is, two more parameters than in the case of a conventional PID controller ($\lambda = 1$ and $\mu = 1$). The fractional orders λ and μ can be used to fulfil additional specifications of design or other interesting requirements for the controlled system.

However, if the controller is designed using the form defined by (1), it is obvious that its time-domain simulation or implementation will require to band-limit its fractional effects. Low and high frequency band-limitations avoid the use of an infinite number of rational modes to approximate the fractional parts, and furthermore, the high-frequency band-limitation of the derivative effect limits its high-frequency gain and thus the control effort provided by the controller.

The paper is organized as follows. Section 2 reviews different design specifications of interest for a closed-loop system, justifying the five specifications required in our case and formulating the compensation problem using a fractional $PI^{\lambda}D^{\mu}$ controller. In section 3, the optimization method used for the tuning of the fractional controller is commented, describing shortly the problem of nonlinear minimization. In sections 4 some illustrative examples are presented, concluding with some remarks in section 5.

2. DESIGN SPECIFICATIONS AND COMPENSATION PROBLEM

Some interesting specifications to be met by the fractional controlled system are proposed:

- **No steady-state error.** Properly implemented a fractional integrator of order $k + \lambda, k \in N, 0 < \lambda < 1$, is, for steady-state error cancellation, as efficient as an integer order integrator of order $k + 1$ (see (Axtel and Bise 1990)).
- **Phase margin (ϕ_m) and gain crossover frequency (ω_{cg}) specifications.** Next conditions must be fulfilled:

$$\begin{aligned} Arg(F(j\omega_{cg})) &= Arg(C(j\omega_{cg})G(j\omega_{cg})) = \\ &= -\pi + \phi_m \end{aligned} \quad (2)$$

$$\begin{aligned} |F(j\omega_{cg})|_{dB} &= |C(j\omega_{cg})G(j\omega_{cg})|_{dB} = \\ &= 0dB \end{aligned} \quad (3)$$

where $F(s)$ is the open-loop transfer function of the system.

- **Gain margin (g_m) and phase crossover frequency (ω_{cp}) specifications.** Next condition must be fulfilled:

$$\frac{1}{|C(j\omega_{cp})G(j\omega_{cp})|} = g_m \quad (4)$$

- **Robustness to variations in the gain of the plant.** To this respect, the next constraint must be fulfilled (see (Chen *et al.* 2003) and (Chen *et al.* 2004)):

$$\left(\frac{d(Arg(C(j\omega)G(j\omega)))}{d\omega}\right)_{\omega=\omega_{cg}} = 0 \quad (5)$$

With this condition the phase is forced to be flat at ω_{cg} and so, to be almost constant within an interval around ω_{cg}. It means that the system is more robust to gain changes and the overshoot of the response is almost constant within the interval.

- **Robustness to high frequency noise.** To ensure a good measurement noise rejection, it must be fulfilled the condition:

$$\left|T(j\omega) = \frac{C(j\omega)G(j\omega)}{1 + C(j\omega)G(j\omega)}\right|_{dB} \leq AdB, \quad (6)$$

$$\forall\ \omega \geq \omega_t rad/s \Rightarrow |T(j\omega_t)|_{dB} = AdB$$

where A is the desired noise attenuation for frequencies $\omega \geq \omega_t\ rad/s$.

- **To ensure a good output disturbance rejection.** The next constraint must be reached:

$$\left|S(j\omega) = \frac{1}{1 + C(j\omega)G(j\omega)}\right|_{dB} \leq BdB, \quad (7)$$

$$\forall\ \omega \leq \omega_s rad/s \Rightarrow |S(j\omega_s)|_{dB} = BdB$$

with B the desired value of the sensitivity function for frequencies $\omega \leq \omega_s\ rad/s$ (desired frequency range).

A set of five of these six specifications can be met by the closed-loop system, since the fractional controller $C(s)$ has five parameters to tune. In our case, the specifications considered are those in equations (2), (3), (5), (6) and (7), ensuring a robust performance of the controlled system to gain changes and noise. The condition of no steady-state error is fulfilled just with the introduction of the fractional integrator properly implemented, as commented before.

The method proposed to solve this set of nonlinear equations is commented in section 3.

3. THE PROBLEM OF NONLINEAR MINIMIZATION

From the specifications above, a set of five nonlinear equations (2, 3, 6, 5 and 7) with five unknown parameters (λ, μ, k_p, k_d and k_i) is obtained. The complexity of this set of nonlinear equations is very significant, specially when it is used a $PI^\lambda D^\mu$ controller and fractional orders of the Laplace variable s are introduced. Thus, the optimization toolbox of Matlab has been used to reach out the better solution with the minimum error. The function used for this purpose is called FMINCON, which finds the constrained minimum of a function of several variables. It solves problems of the form MIN$_{\text{X}}$F(X) subject to: C(X)<=0, Ceq(X)=0, LB<=X<=UB, where F is the function to minimize; C and C_{eq} represent the nonlinear inequalities and equalities, respectively (nonlinear constraints); X is the minimum looked for; LB and UB define a set of lower and upper bounds on the design variables, X.

In our case, the specification in equation (3) is taken as the main function to minimize, and the rest of specifications (2, 5, 6, 7) are taken as constrains for the minimization, all of them subjected to the optimization parameters defined within the function FMINCON.

4. ILLUSTRATIVE EXAMPLES

This section shows the results obtained when using the fractional $PI^\lambda D^\mu$ controller to control a first-order plant, a first-order plant plus an integrator and a first-order plant with a time delay. The reason why these three plants have been selected is that a lot of process can by modelled by them.

4.1 First-Order Plant

The plant to control is $G_1(s) = \frac{k}{\tau s+1} = \frac{0.55}{62s+1}$, typical of a velocity servo. The design specifications required for the controlled system are the following ones:

- Gain crossover frequency, ω_{cg}=1rad/sec (equation (3)).
- Phase margin, $\phi_m = 0.44\pi \approx 80°$ deg (equation (2)).
- Robustness to variations in the gain of the plant must be fulfilled (equation (5)).
- $|T_1(j\omega)|_{dB} \leq -20dB$, $\forall\ \omega \geq \omega_t$=10rad/sec (equation (6)).
- $|S_1(j\omega)|_{dB} \leq -20dB, \forall\ \omega \leq \omega_s$=0.01rad/sec (equation (7)).

Setting all these specifications and using the tuning optimization method commented above, the fractional $PI^\lambda D^\mu$ controller to control this system is:

$$C_1(s) = 15.2864 + \frac{98.1268}{s^{0.1578}} + 1.1625s^{1.0148} \quad (8)$$

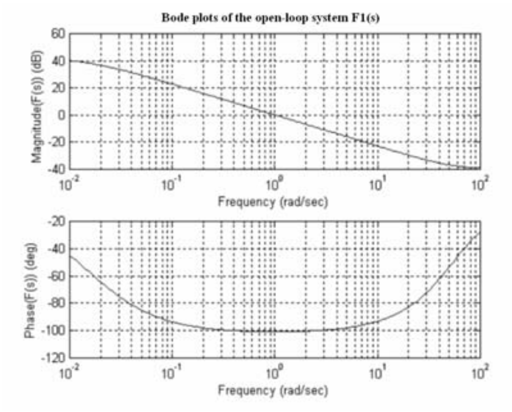

Fig. 1. Bode plots of the open-loop system $F_1(s)$

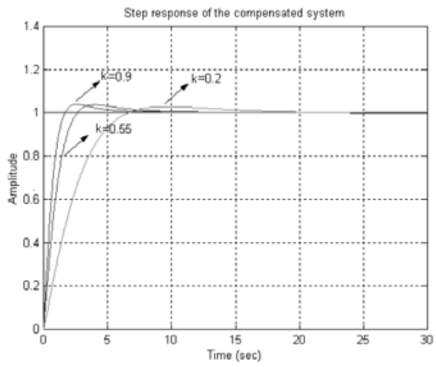

Fig. 2. Step responses of the closed-loop system with controller $C_1(s)$ for $0.2 \le k \le 0.9$

Though the final value theorem states that the fractional system exhibits null steady state error if $\lambda > 0$, the fact of being $\lambda < 1$ makes the output converge to its final value more slowly than in the case of an integer controller. Furthermore, the fractional effect need to be band-limited when it is implemented. Therefore, the fractional integrator must be implemented as $\frac{1}{s^{\lambda}} = \frac{1}{s}s^{1-\lambda}$, ensuring this way the effect of an integer integrator $1/s$ at very-low frequency. In this particular example of application, as well as in the following ones, the fractional integral and derivative parts have been implemented by the Oustaloup continuous approximation of the fractional integrator (see (Oustaloup 1995)), choosing a frequency band from $0.01Hz$ to $100Hz$ and an order of the approximation equals to 5 (number of poles and zeros).

The Bode diagrams of the open-loop system $F_1(s) = C_1(s)G_1(s)$ are shown in figure 1.

As it can be seen, the gain crossover frequency specification, ω_{cg}=$1rad/sec$, and the phase margin specification, $\phi_m = 80°$ deg, are fulfilled. Besides, the phase of the system is forced to be flat at ω_{cg} and so, to be almost constant within an interval around ω_{cg}. It means that the systems is more robust to gain changes and the overshoot of the response is almost constant within this interval, as it can be seen in figure 2.

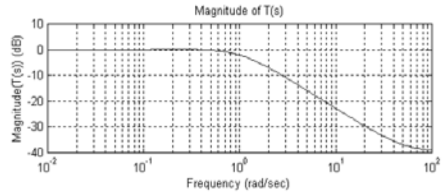

Fig. 3. Magnitude of $T_1(s)$

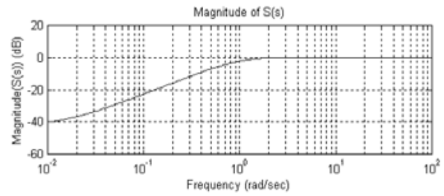

Fig. 4. Magnitude of $S_1(s)$

The magnitudes of the functions $T_1(s)$ and $S_1(s)$ are shown in figures 3 and 4, respectively. As it can be observed, $|T_1(j\omega)|_{dB} \le -20dB$ for $\omega \ge \omega_t = 10rad/\sec$, and $|S_1(j\omega)|_{dB} \le -20dB$ for $\omega \le \omega_s = 0.01rad/\sec$, fulfilling the specifications.

4.2 First-Order Plant Plus an Integrator

Now the plant to control is $G_2(s) = \frac{k}{s(\tau s+1)} = \frac{0.25}{s(s+1)}$, typical of a position servo. The following design specifications for the controlled system have been considered:

- Gain crossover frequency, ω_{cg}=$1rad/sec$.
- Phase margin, $\phi_m = 0.27\pi \approx 48.5°$ deg.
- Robustness to variations in the gain of the plant must be fulfilled.
- $|T_2(j\omega)|_{dB} \le -20dB, \forall\ \omega \ge \omega_t$=10rad/sec.
- $|S_2(j\omega)|_{dB} \le -20dB, \forall\ \omega \le \omega_s$=0.01rad/sec.

In this case, the fractional $PI^{\lambda}D^{\mu}$ controller to control this system is:

$$C_2(s) = 3.8159 + \frac{2.1199}{s^{0.6264}} + 2.2195s^{0.8090} \qquad (9)$$

The implementation of the fractional integral and derivative parts is carried out as commented in the previous example.

The Bode diagrams for the open-loop system $F_2(s) = C_2(s)G_2(s)$ are shown in figure 5.

As it can be seen, the gain crossover frequency specification, ω_{cg}=$1rad/sec$, and the phase margin specification, $\phi_m = 48.5°$ deg, are fulfilled. Besides, the robustness of the controlled system is ensured since the condition of *flat* phase is met, keeping the overshoot constant to variations of the gain of the plant, as it can be seen in figure 6.

The magnitudes of the functions $T_2(s)$ and $S_2(s)$ are shown in figures 7 and 8, respectively, fulfilling the specifications.

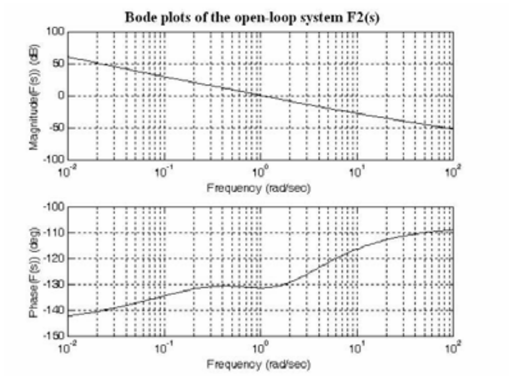

Fig. 5. Bode plots of the open-loop system F_2(s)

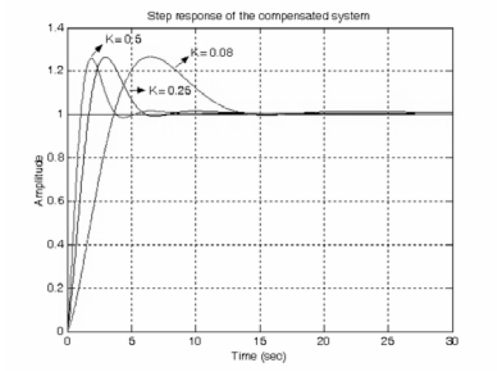

Fig. 6. Step responses of the closed-loop system with controller C_2(s) for $0.08 \leq k \leq 0.5$

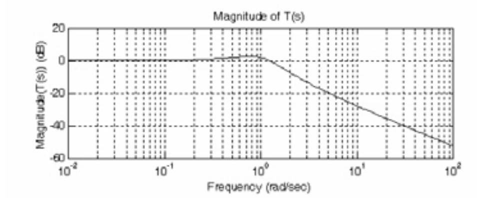

Fig. 7. Magnitude of T_2(s)

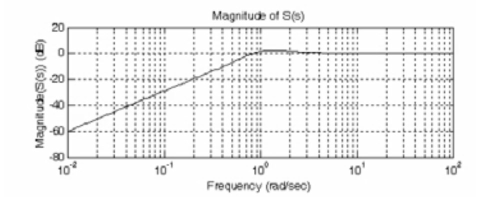

Fig. 8. Magnitude of S_2(s)

4.3 First-Order Plant with a Time Delay

Finally, the plant $G_3(s) = \frac{k}{\tau s+1}e^{-Ls} = \frac{0.55}{62s+1}e^{-1s}$ will be controlled, which corresponds to the pH dynamic model of a real sugar cane raw juice neutralization process (see (Feliu *et al.* 2002)). The design specifications required are:

- Gain crossover frequency, ω_{cg}=0.08$rad/sec.$
- Phase margin, $\phi_m = 0.44\pi \approx 80°$ deg.
- Robustness to variations in the gain of the plant must be fulfilled.
- $|T_3(j\omega)|_{dB} \leq -20dB$, $\forall\, \omega \geq \omega_t$=10rad/sec.

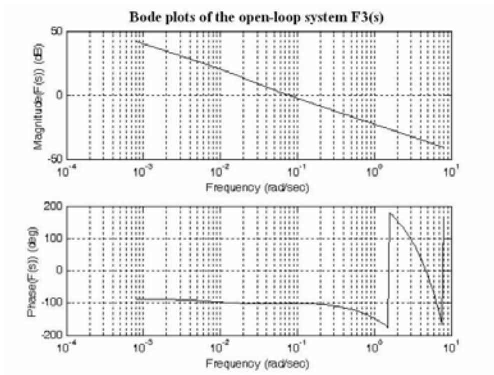

Fig. 9. Bode plots of the open-loop system F_3(s)

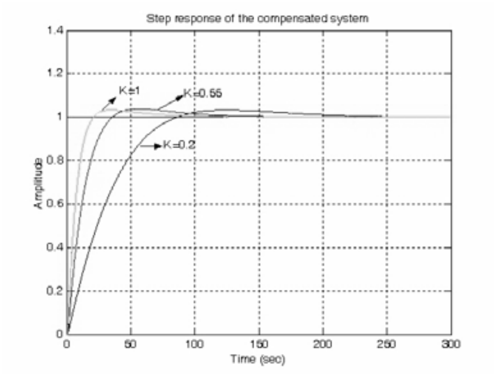

Fig. 10. Step responses of the closed-loop system with controller C_3(s) for $0.2 \leq k \leq 1$

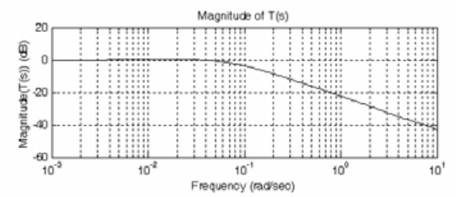

Fig. 11. Magnitude of T_3(s)

- $|S_3(j\omega)|_{dB} \leq -20dB, \forall\, \omega \leq \omega_s$=0.01rad/sec.

In this case, the fractional $PI^{\lambda}D^{\mu}$ controller to control this system is:

$$C_3(s) = 7.9619 + \frac{0.2299}{s^{0.9646}} + 0.1504s^{0.0150} \quad (10)$$

The implementation of the fractional integral and derivative parts is carried out as commented before.

The Bode diagrams for the open-loop system $F_3(s) = C_3(s)G_3(s)$ are shown in figure 9.

Again, all the design specifications are fulfilled, together with the *flat* phase condition, keeping almost constant the overshoot of the closed-loop system (see figure 10).

The magnitudes of the functions $T_3(s)$ and $S_3(s)$ are shown in figures 11 and 12, respectively, fulfilling the specifications.

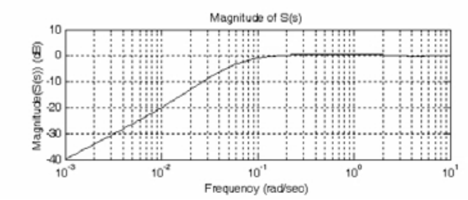

Fig. 12. Magnitude of $S_3(s)$

5. CONCLUDING REMARKS

In this paper it has been proposed a fractional $PI^{\lambda}D^{\mu}$ controller in order to fulfil five different design specifications for the closed-loop system, that is, two more specifications than in the case of a conventional PID controller. An optimization method to tune the controller has been used, based on a nonlinear function minimization subject to some given nonlinear constraints. Simulation results show that the requirements are totally fulfilled for the three different plants proposed. Thus, it has been taken advantage of the fractional orders λ and μ to fulfil additional specifications of design and other interesting requirements for the controlled system.

It would be of interest to apply this kind of controllers to other types of plants, such as nonlinear plants, for instance, and to study other interesting design specifications that could be required for each plant in particular. These and other aspects will be taken into account in further works.

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