THE FRACTIONAL INTEGRATOR AS REFERENCE FUNCTION [1]

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Abstract: The purpose of this paper is, on one hand, to analyze the fractional integrator as reference system for control by comparison to the position servo in order to find their similitudes and differences, and, on the other hand, to review the approaches previously made for design specifications in every domain when using the fractional integrator as reference system, and to propose and discuss new approaches.

Keywords: Fractional Integrator, Reference Function, Servo Mechanism, Design Specifications

1. INTRODUCTION

Traditionally the position servo has been used as reference open loop system for automatic control purposes (see (Ogata 2002, Özbay 1999)). That is, as reference system for controller design in order to obtain a closed loop controlled system with prescribed behaviour using specifications in the frequency domain (phase margin, crossover frequency, etc.), the complex plane (dominant poles location) or the time domain step response (rise time, overshoot, peak time, settling time, etc.), and the corresponding equivalences between those domains and with the characteristic parameters (damping ratio and natural frequency). On the other hand, from the very beginnings of the use of fractional calculus in control (see (Manabe 1961, Oustaloup 1995)) the fractional integrator has been considered as an alternative reference system for control purposes in order to obtain closed loop controlled systems robust to gain changes. In any case, for using a system as a reference one, it is necessary to know its frequency and time behaviour and to define clearly its characteristic parameters and its relations with useful specifications. The purpose of this paper is, on the one hand, to compare the two reference systems cited above in order to find their similitudes and differences, and, on the other hand, to review the approaches previously made for design specifications in every domain when using the fractional integrator as reference system, and to propose and discuss new approaches.

The paper is organized as follows. Section 2 is devoted to a brief review of the position servo as reference system and the origin and meaning

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of some relations between specifications in every domain of analysis. Section 3 follows an analogous procedure for the analysis of the fractional integrator in order to establish some guidelines when using it as reference system for control. Section 4 gives some conclusions.

2. THE POSITION SERVO

A typical position servo is shown in figure 1, where J and B represent the load elements corresponding to inertia (J) and viscous friction (B) effects. The closed loop transfer function can be expressed as:

$$F(s) = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \tag{1}$$

where $\delta = B/(2\sqrt{JK})$ represents the damping ratio, and $\omega_n = \sqrt{K/J}$ the undamped natural frequency.

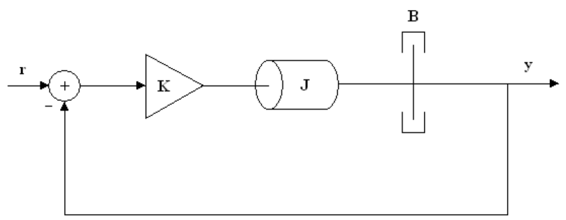

Fig. 1. Typical servo system.

The poles of (1) are in:

$$s_{1,2} = -\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2} \tag{2}$$

being ω_n its magnitude and $\tan^{-1}\frac{\sqrt{1-\delta^2}}{\pm\delta}$ its argument.

The unit step response is of the form:

$$y(t) = 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \cdot \tag{3}$$
$$\cdot \sin\left(\left(\omega_n\sqrt{1-\delta^2}\right)t + \tan^{-1}\frac{\sqrt{1-\delta^2}}{\delta}\right)$$

which, for the usual case of underdamped system ($0 < \delta < 1$), is a damped oscillation with attenuation or decaying factor $\sigma = \delta\omega_n$, and damped or proper frequency $\omega_p = \omega_n\sqrt{1-\delta^2}$, being ω_n the undamped natural frequency and δ the damping ratio or attenuation for unity natural frequency. This unity step response is represented in figure 2 for $\delta \in [0,1)$ and $\omega_n = 1$ (note that $\omega_n \neq 1$ is reflected as a time scaling). In this figure it can be observed that as δ decreases, the rise time, t_r, (time for the response to achieve for the first time its final value), and the peak time, t_p, (the time at which the absolute maximum occurs) descrease, and the settling time, t_s, (the time required for the response to settle within a small deviation from its final value), and the overshoot, M_p, increase. It can be also observed that the slopes of the curves are very close near the origin, and have increasing values as δ decreases.

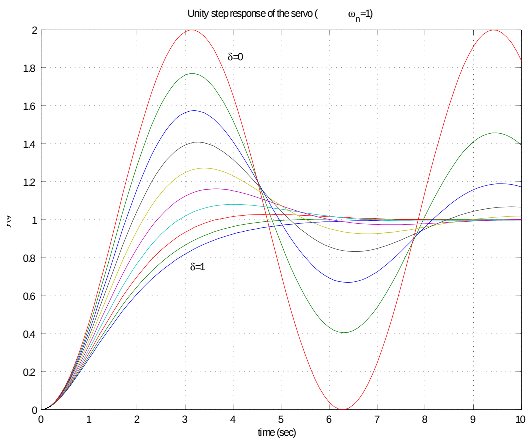

Fig. 2. Step responses of the closed loop servo for $\delta \in [0,1)$.

For the definition of the characteristic parameters and specifications of the system (1) in the frequency domain, it must be taken into account that the starting point of such a transfer function is the damped RLC oscillator, in which the quality factor, Q, is defined for lossy coils ($R \neq 0$) as the ratio of reactance to resistance at frequency $\omega_n = 1/\sqrt{LC}$ (see, e.g., (Van Valkenburg 1982)). The frequency domain characteristics for the normalized case, $|F(j0)| = 1$, can be obtained as follows:

- Quality factor:

$$Q = |F(j\omega_n)| = \frac{1}{2\delta} \tag{4}$$

- Resonant frequency (frequency for maximum of the magnitude):

$$\omega_r = \omega_n\sqrt{1-(1/2Q^2)} = \omega_n\sqrt{1-2\delta^2} \tag{5}$$

- Resonance factor (maximum of the magnitude):

$$M_r = |F(j\omega_r)| = \frac{Q}{\sqrt{1-(1/4Q^2)}} = \tag{6}$$
$$= \frac{1}{2\delta\sqrt{1-\delta^2}}$$

The Bode plots of the closed loop servo are presented in figure 3 for $\delta \in [0,1)$ and $\omega_n = 1$. It can be observed that asymptotically ($\omega \to \infty$) the slope of the magnitude curve is $-40dB/dec$ and the phase is $-\pi$. Furthermore, there is an increasing wideband with the decreasing of δ, and

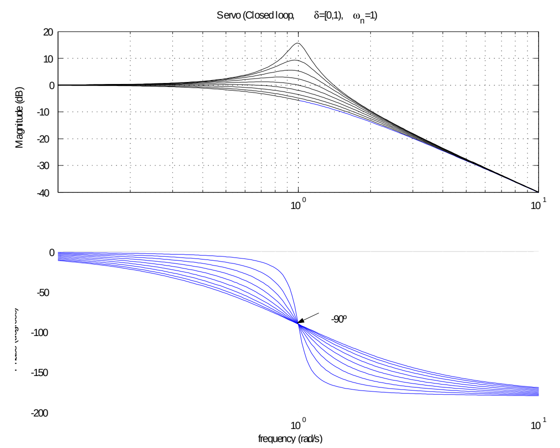

Fig. 3. Bode plots of the closed loop servo for $\delta \in [0, 1)$.

a crossing point of the phase curves at $\omega = 1$ with phase value $-\frac{\pi}{2}$.

Useful relations between time and frequency characteristics can be stablished. For example, the phase margin can be obtained as:

$$PM = \tan^{-1} \frac{2\delta}{\sqrt{\sqrt{1+4\delta^4} - 2\delta^2}} \qquad (7)$$

3. THE FRACTIONAL INTEGRATOR

3.1 Historical Overview

Maybe the first mention of the interest of considering a fractional integrodifferential operator in a feedback loop, though without mention of the term “fractional”, was made by Bode in (Bode 2001), and next in a more comprehensive way in (Bode 1945). A key problem in the design of a feedback amplifier was to come up with a feedback loop so that the performance of the closed loop were invariant to changes in the amplifier gain. Bode presented an elegant solution to this robust design problem, which he called the *ideal cutoff characteristic*, nowadays known as *ideal loop transfer function,* whose Nyquist plot is a straight line through the origin giving a phase margin invariant to gain changes. Clearly, this ideal system is, from our point of view, a fractional integrator with transfer function:

$$G_\alpha(s) = \left(\frac{\omega_g}{s}\right)^\alpha \qquad (8)$$

being ω_g the gain crossover frequency, and the constant phase margin,

$$PM = \pi - \alpha\frac{\pi}{2} \qquad (9)$$

This system is also ideal in the sense that for it the Bode’s gain-phase relations (see (**?**, Özbay 1999)) are exact relations. For (8), the equation:

$$\arg\left(G_\alpha(j\omega_0)\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} M(u)W(u)du \qquad (10)$$

with:

$$u = \ln\left(\frac{\omega}{\omega_0}\right); \quad M(u) = \frac{d}{du}\left|G_\alpha(j\omega_0 e^u)\right|; \qquad (11)$$

$$W(u) = \ln\left(\coth\left(\frac{|u|}{2}\right)\right) \qquad (12)$$

gives:

$$\arg\left(G_\alpha(j\omega_0)\right) = -\alpha\frac{\pi}{2} \qquad (13)$$

that is, the phase at any frequency is constant, and so, is exactly proportional to the derivative of the magnitude (slope) on a logarithmic frequency scale.

Though the notions introduced by Bode in (Bode 2001) (minimum phase, phase margin, Bode plots, etc.) became key elements of automatic control, Bode’s ideal loop transfer function was not much discussed in the literature for a long time, although it can be considered a natural beginning of robust control. The first proposal for the use of the fractional integrator (in fact the Bode’s ideal system) was made in (Tustin *et al.* 1958). In this work, the requirements for the design of large position-control systems lead to constraints on the form of the frequency characteristics, and the ideal one results to be a fractional order integrodifferentiator fulfilling the Bode’s gain-phase relationship 10. It also mentioned in that paper the *ideal servo mechanism* (no friction) having a transfer function

$$C(s) = \frac{k}{s^2} \qquad (14)$$

which is a limit case of (8) with $\alpha = 2$.

The first explicit mention of the fractional integrator in control can be found in (Manabe 1961). In that paper the fractional integrator is studied for the first time as a reference system for control, its frequency and step responses are analyzed, and expressions for characteristic parameters and usual time and frequency specifications are given. Definitive apportations for the use of the fractional integrator as reference system for control were made during the seventies in two ways. The first one, recovering the Bode’s idea of robustness and extending it to other parameter’s variations, led to the Qualitative Control Theory of Horowitz (see, e.g., (Horowitz 1993),(Åström 1999)). The second one, particularly interesting for us, was the introduction of the Robust Control of Non Integer Order (CRONE control (Oustaloup 1995)), in which, taking the fractional integrator as reference system for loop shaping (particularly in the second generation CRONE control), systematic procedures for systems analysis, controllers design, and controllers implementation have been developed during two decades, becoming an alternative paradigm in robust control.

In what follows the fractional integrator in closed loop will be analyzed in the different domains in order to compare the results with the obtained for the servo.

3.2 Complex plane analysis

The fractional integrator (8) in closed loop, has a transfer function of the form:

$$F_\alpha(s) = \frac{\left(\frac{\omega_g}{s}\right)^\alpha}{1+\left(\frac{\omega_g}{s}\right)^\alpha} = \frac{1}{\left(\frac{s}{\omega_g}\right)^\alpha + 1} \tag{15}$$

and the closed loop poles, for $1 < \alpha < 2$ (equivalent to the underdamped servo, $0 < \delta < 1$), are:

$$s_{1,2} = \omega_g e^{\pm j\frac{\pi}{\alpha}} = \omega_g \left(\cos\frac{\pi}{\alpha} \pm j\sin\frac{\pi}{\alpha}\right) \tag{16}$$

being the crossover frequency, ω_g, its magnitude and $\tan^{-1}\frac{\pi}{\alpha}$ its argument (It must be noted that the considered solutions of the characteristic equation in 15 are only the corresponding to the main sheet of the Riemann surface defined by $-\pi < \arg(s) < \pi$). By comparing expressions (2) and (16), expressions for characteristic parameters can be obtained as follows:

- Natural frequency as magnitude of the poles:

$$\omega_{\alpha n} = |s_{1,2}| = \omega_g \tag{17}$$

- Attenuation as real part of the poles:

$$\sigma = \text{real}(s_{1,2}) = -\omega_g \cos\frac{\pi}{\alpha} \tag{18a}$$

- Damping ratio as attenuation for $\omega_n = 1$:

$$\delta_\alpha = \text{real}(s_{1,2})_{\omega_g=1} = -\cos\frac{\pi}{\alpha}; \tag{19}$$

- Damped frequency as imaginary part of the poles:

$$\omega_d = \text{Im}\,(s_{1,2}) = \omega_g \sin\frac{\pi}{\alpha} = \omega_{\alpha n}\sqrt{1-\delta_\alpha^2}. \tag{20}$$

As can be observed, the characteristic parameters have expressions analogous to the case of the servo, with the damping ratio depending only on α. But in this case, the natural frequency (which is the oscillation frequency for no damping case, $\alpha = 2$) is equal to the crossover frequency, which is not the case of the servo. The former expressions, (17) to (20), are the expressions used in (Oustaloup 1995).

3.3 Frequency domain analysis

The expression for the frequency response of (15) is:

$$F_\alpha(j\omega) = \frac{1}{\left(1+\left(\frac{\omega}{\omega_g}\right)^\alpha \cos\frac{\alpha\pi}{2}\right) + j\left(\frac{\omega}{\omega_g}\right)^\alpha \sin\frac{\alpha\pi}{2}} \tag{21}$$

From (21) and with the usual definitions, the following expressions can be obtained for the characteristic performances:

- Quality factor (magnitude for natural frequency):

$$Q_a = |F_\alpha(j\omega_g)| = \frac{1/\sqrt{2}}{\sqrt{1+\cos\frac{\alpha\pi}{2}}} \tag{22}$$

- Resonant frequency (frequency for maximum magnitude):

$$\omega_{\alpha r} = \omega_g\left(-\cos\frac{\alpha\pi}{2}\right) \tag{23}$$

- Resonance factor (maximum magnitude):

$$M_{\alpha r} = |F_\alpha(j\omega_{\alpha r})| = \frac{1}{\sin\frac{\alpha\pi}{2}} \tag{24}$$

The last two expressions, (23) and (24), are used in (Oustaloup 1995) and (Manabe 1961), but in the last paper an alternative definition of damping ratio is introduced based on complex plane-frequency relations of the position servo (equations (5) and (6)). By doing so, the expression for the damping ratio is:

$$\delta_{m\alpha} = \left(\frac{1}{\sqrt{2}}\sqrt{1+\cos\frac{\alpha\pi}{2}}\right) \tag{25}$$

being this equivalent damping ratio, $\delta_{m\alpha}$, the damping ratio of a servo whose resonance peak is equal to the resonant peak of the fractional integrator. It must be noted that, in correspondence with the domain of definition, for $\alpha = 1$ the damping is 0.707 (frequency domain, (25)), or 1 (complex plane, (19)). The frequency response of the fractional integrator in closed loop is shown in figure 4. As can be observed, there are important differences between these frequency responses and the corresponding to the servo. In this case, there is no crossing point in phase, but there is a crossing point in magnitude in $-6dB$, that is, there is a constant $-6dB$ wideband. The asymptotic values are $\arg(F_\alpha(j\infty) = -\alpha\pi$ for phase, and the asymptotic slopes of the magnitude curves are $|F_\alpha(j\infty)| = -\alpha 20dB/dec$ (it must noted that for the servo case, these asymptotic values were independent of δ). Of course, these differences will be reflected in the time domain unity step response.

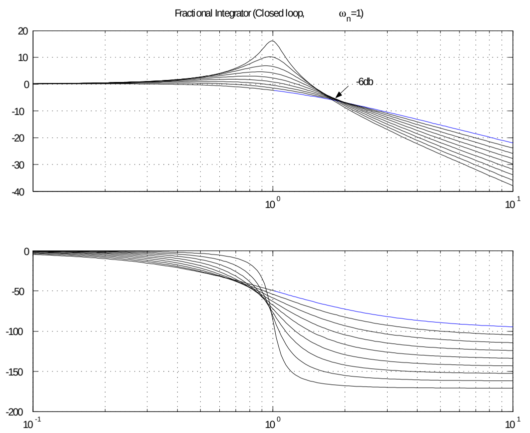

Fig. 4. Bode plots of the fractional integrator in closed loop for $\alpha \in (1, 2]$.

3.4 Time domain analysis

The unity step response of the closed loop system (15) is of the form (see (Gorenflo and Mainardi 1996, Barbosa *et al.* 2003)):

$$y_\alpha(t) = \pounds^{-1}\left[\frac{1}{s}\frac{1}{\left(\frac{s}{\omega_g}\right)^\alpha + 1}\right] = 1 - E_\alpha(-(\omega_g t)^\alpha) \tag{26}$$

denoting by $\pounds^{-1}[\cdot]$ the inverse Laplace transformation, and being $E_\alpha(\cdot)$ the one-parameter Mittag-Leffler function (Podlubny 1999, Gorenflo and Mainardi 1996). Following (Gorenflo and Mainardi 1996), for the case of $1 < \alpha < 2$, the function $E_\alpha(-t^\alpha)$ can be expressed on the form:

$$E_\alpha(-t^\alpha) = f_\alpha(t) + g_\alpha(t) \tag{27}$$

being the function $f_\alpha(t)$,

$$f_\alpha(t) = -\frac{1}{\pi}\int_0^\infty \frac{s^{\alpha-1}\sin(\alpha\pi)}{s^{2\alpha} + 2s^\alpha\cos(\alpha\pi) + 1}e^{-st}ds \tag{28}$$

a function which vanishes identically if α is an integer, and is negative for all s if $1 < \alpha < 2$, and the function $g_\alpha(t)$ the corresponding to relevant poles (those situated in the main Riemann sheet defined by $-\pi < \arg s < \pi$), a vanishing oscillating function that can be expressed as:

$$g_\alpha(t) = \frac{2}{\alpha}e^{t\cos(\pi/\alpha)}\cos\left[t\sin\left(\frac{\pi}{\alpha}\right)\right] \tag{29}$$

So, the unit step response, $y_\alpha(t)$, for the normalized case $\omega_g = 1$ (it must be noted that $\omega_g \neq 1$ is, in the time domain just a scale factor), can be formulated as:

$$y_\alpha(t) = 1 + \frac{1}{\pi}\int_0^\infty \frac{s^{\alpha-1}\sin(\alpha\pi)}{s^{2\alpha} + 2s^\alpha\cos(\alpha\pi) + 1}e^{-st}ds \\ -\frac{2}{\alpha}e^{t\cos(\pi/\alpha)}\cos\left[t\sin\left(\frac{\pi}{\alpha}\right)\right] \tag{30}$$

It is clear that even expression (30) is not adequate for deriving the time domain usual characteristics and specifications. For the moment, it can be established that, since $g_\alpha(t)$ is the vanishing oscillating part, the atenuation of (30) is $\sigma_\alpha = -\omega_g\cos(\pi/\alpha)$, the damping ratio is $\delta_\alpha = -\cos(\pi/\alpha)$, the damped frequency is $\omega_{\alpha d} = \omega_g \sin\left(\frac{\pi}{\alpha}\right)$, and the natural frequency is $\omega_{\alpha n} = \omega_g$. As can be observed, these expressions are the same obtained in complex plane analysis.

The problem now is to derive some useful expressions for time domain specifications: overshoot (M_p), rise time (t_r), peak time (t_p) and settling time (t_s). In (Manabe 1961) an approximated expression is given for the overshoot, obtained by step response simulation and curve fitting. In (**?**) other approximated formulae are given for overshoot and time specifications, based on curve fitting by polynomial (M_p) or rational (t_p, t_r) functions of α, or, in the case of settling time, by functions analogous to the obtained for the servo with the corresponding expression of damping ratio (19) as a function of α. Though that approximated equations are useful enough for typical design problems, at least some of these parameters can be analytically obtained. It must be noted that in (30) the oscillating part is (29), so an analytical expression for t_p can be obtained by differentiating this oscillating part and next equating the result to zero, that is:

$$\frac{d}{dt}\left[\frac{2}{\alpha}e^{t\cos(\pi/\alpha)}\cos\left[t\sin\left(\frac{\pi}{\alpha}\right)\right]_{t=t_p} = 0\right]$$

By doing so, and taking into account the periodicity of the trigonometric functions, the expression for the peak time is (see figure 5)

$$t_p = \frac{\tan^{-1}(1/\tan(\pi/\alpha)) + \pi}{\omega_g\sin(\pi/\alpha)} \tag{31}$$

For obtaining a quasi-analytical expression for overshoot, the function (28) can be approximated (least squares minimization) by $\widehat{f}_a(t) = 0.9(2 - \alpha)e^{-t}$, and then the approximated expression for the unit step response becomes

$$\widehat{y}_\alpha(t) = 1 + 0.9(2-\alpha)e^{-t} \\ -\frac{2}{\alpha}e^{t\cos(\pi/\alpha)}\cos\left[t\sin\left(\frac{\pi}{\alpha}\right)\right] \tag{32}$$

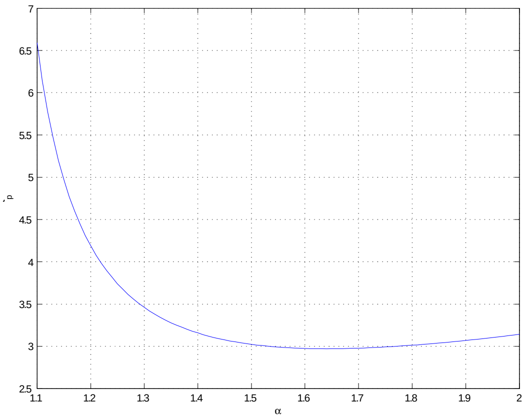

Fig. 5. Peak time, t_p, as a function of α.

In figure 6 this approximated step responses are given with the corresponding error, $y_\alpha(t) - \widehat{y}_\alpha(t)$. As can be observed, there are several differences with the step responses of the servo, mainly, we can remark that in this case there is a crossing point near $y_\alpha(t) = 0.75$, and the slopes near the origin are decreasing with increasing values of α (reverse for the servo). Furthermore, the peak time is not uniformely decreasing as α increase. These facts are in correspondence with the high frequencies behaviour in figure 4.

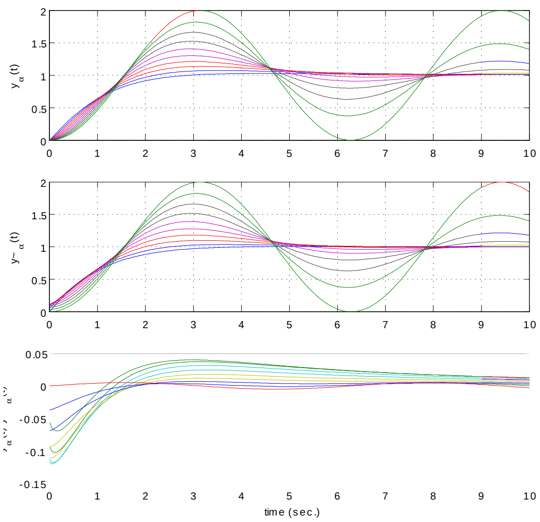

Fig. 6. Analytical and approximated step responses and corresponding errors.

By using this expression the overshoot can be obtained by substituting (31) into (32):

$$M_p \simeq \widehat{y}_\alpha(t_p) - 1 \tag{33}$$

In figure 7 the overshoot is represented as a function of α.

In order to obtain an expression for settling time, it must be noted that in (32) the first exponential term has few influence for long times, and the envelopes of the vanishing oscillation term are defined by $1 \pm (2/\alpha)\, e^{t\cos(\pi/\alpha)}$. So, a time constant

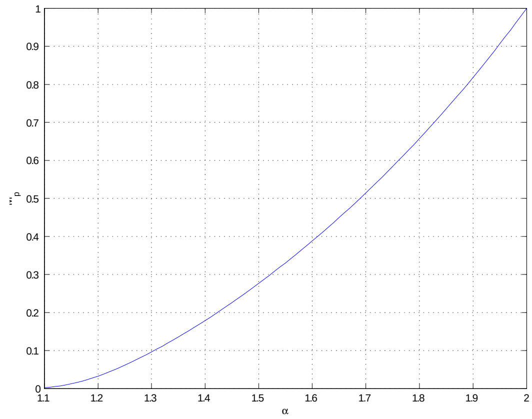

Fig. 7. Overshoot, M_p, as a function of α.

can be defined as $\varsigma_\alpha = 1/(-\cos(\pi/\alpha))$, and the general expression for the settling time becomes:

$$t_s = k\varsigma_\alpha = \frac{k}{\omega_g(-\cos(\pi/\alpha))} \tag{34}$$

where k defines the tolerance margin.

4. CONCLUSIONS

In this paper the fractional integrator as reference system for control has been analyzed by comparison to the position servo in order to find their similitudes and differences, has been made a review of the approaches previously made for design specifications in every domain when using the fractional integrator as reference system, and new approaches have been proposed. By taking into account the comments and results of the former sections, the following guidelines can be stablished for proper use of the fractional integrator as reference system:

(1) The fractional integrator (Bode's ideal function) can be used as an alternative reference system for control, but considering its own properties, not by extrapolating the properties of the position servo, because: (a) relations between parameters in different domains are not the same; (b) the shapes of the frequency and step responses are qualitatively different though equal for the limit oscillating case, $\alpha = 2, \delta = 0$ (For the other limit case, $\alpha = \delta = 1$, the critically damped step response of the servo ($y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$), is faster than the one of the fractional integrator ($y_\alpha(t) = 1 - e^{-\omega_g t}$), which is, in fact, a first order system.)
(2) Expressions for characteristic parameters and specifications must be taken from their own domain of definition: (a) damping, attenuation and natural frequency, from complex plane or, alternatively, from time domain; (b) phase margin, crossover frequency and resonance characteristics from frequency domain;

(c) overshoot and time specifications from time domain.

(3) The basic design parameters defining the reference model are not damping ratio and natural frequency, but integration order and crossover frequency.

(4) The basic design equations are: (9), (17) to (20), (22) to (24), and (31), (33) and (34).

Of course, analytical or graphycal relationships could be stablished between the different parameters and domains for design purposes.

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