Boundary Control of Wave Equations with Delayed Boundary Measurement

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***Abstract*— Smith predictor and its variant are applied to the boundary control of wave equation with delayed boundary measurement. The well-known instability problem due to a small time delay is solved for wave equations in this paper. Simulation results demonstrate the effectiveness of the proposed method. The effect of the time delay size and the boundary feedback gain has also been demonstrated.**

***Index Terms*— Boundary control; Smith predictor; wave equation; delay.**

I. INTRODUCTION

In recent years, the boundary control of flexible systems has become an active research area [1], [2], [3], [4], [5], [6], [7], [8], [9], due to the increasing demand on the high precision control of many mechanical systems, such as spacecraft with flexible attachment or robots with flexible links, which are governed by partial differential equations (PDE's) rather than ordinary differential equations (ODE's). For biomimetic actuator control, due to its inherent nature of distributed parameter systems (DPS), boundary stabilization control will be an important task. Two fundamentally important research topics are the boundary control of wave equation and beam equation, which are often encountered in the practical engineering designs. It is well-known that using a simple velocity feedback boundary controller at the boundaries is sufficient to stabilize the displacement of the string or the beam [7] [8]. However, in [10] and [11], it was shown that these systems become unstable when an arbitrary small time delay is introduced into the feedback loop, whereas such a time delay is unavoidable in practice due to measurement lags, analysis times, or computation lags. Although this delay induced instability phenomenon has been discovered for more than ten years, to our best knowledge, no efficient solution has been proposed so far except [13].

Based on our previous papers [12] and [13], this paper introduces the Smith predictor and its variant to the boundary control of wave equation with a delayed boundary measurement. The instability problem can be solved even for relatively large time delays. Simulation results are presented to demonstrate the effectiveness of the proposed method. The effect of the time delay size and the boundary feedback gain has also been demonstrated.

II. A BRIEF INTRODUCTION TO THE SMITH PREDICTOR

The Smith predictor is probably the most famous method for the control of systems with time delays [14] [15]. Consider a typical feedback control system with time delays in Fig. 1, where $C(s)$ is the controller; $P_0(s)e^{-\theta s}$ is the plant with a time delay θ and d is the disturbance.

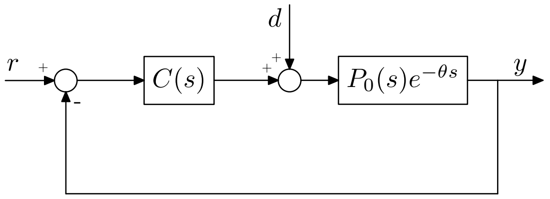

Fig. 1. A feedback control system with a time delay

With the presence of time delay, the transfer function of the closed-loop system relating the output $y(s)$ to the reference $r(s)$ is

$$\frac{y(s)}{r(s)} = \frac{C(s)P_0(s)e^{-\theta s}}{1 + C(s)P_0(s)e^{-\theta s}}. \tag{1}$$

Obviously, the time delay θ directly changes the closed-loop poles.Usually, the time delay reduces the stability margin of the control system, or more seriously, destabilizes the system.

The Smith predictor was proposed by Smith in [16]. The classical configuration of a system containing a Smith predictor is depicted in Fig. 2, where $P(s) = P_0(s)e^{-\theta s}$. $\hat{P}_0(s)$ and $\hat{P}(s)$ are nominal models of $P_0(s)$ and $P(s)$, respectively. The block $C(s)$ combined with the block $\hat{P}_0(s) - \hat{P}(s)$ is called the "Smith predictor". If we assume the perfect model matching, *i.e.*, $\hat{P}(s) = P(s)$, the closed-loop transfer function becomes

$$\frac{y(s)}{r(s)} = \frac{C(s)P(s)}{1 + C(s)P_0(s)}. \tag{2}$$

Now, it is clear what the underlying idea of the Smith predictor is. With the perfect model matching, the time delays can be removed from the denominator of the transfer function, making the closed-loop stability irrelevant to the time delays. Even if the model is not perfectly matched, the effect of the time delays can be mitigated to certain degree.

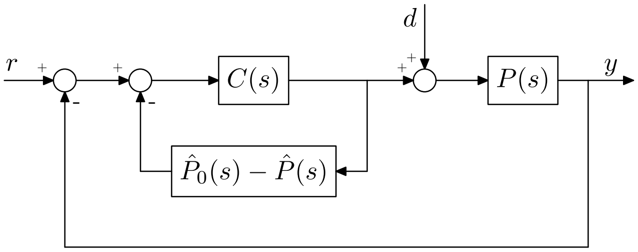

Fig. 2. Smith Predictor

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An alternative implementation of the Smith predictor is shown in Fig. 3. Since this configuration makes the design of the Smith predictor more convenient, we will use this implementation for the rest of this paper.

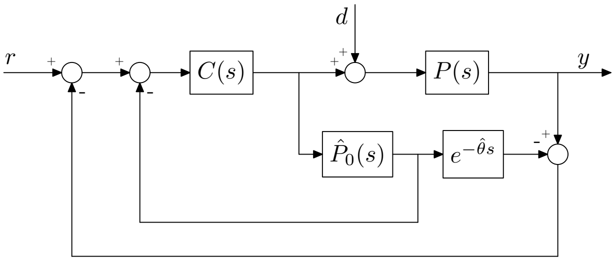

Fig. 3. An alternative Smith predictor implementation

III. BOUNDARY CONTROL OF WAVE EQUATION WITH DELAYED BOUNDARY MEASUREMENT USING THE SMITH PREDICTOR

Consider a string clamped at one end and is free at the other end. We denote the displacement of the string by $u(x,t)$, where $x \in (0,1)$ and $t \geq 0$. The string is controlled by a boundary control force at the free end. The governing equations are given as

$$u_{tt} - u_{xx} = 0, \tag{3}$$

$$u(0,t) = 0, \tag{4}$$

$$u_x(1,t) = f(t), \tag{5}$$

where $f(t)$ is the boundary control force applied at the free end of the string.

It is well-known that the following boundary controller is a stabilizing controller [8]:

$$f(t) = -ku_t(1,t) \tag{6}$$

where $k > 0$ is the constant gain.

However, in [11], it was shown that the system based on the boundary controller (6) becomes unstable when an arbitrary small time delay is introduced into the feedback loop as follows:

$$f(t) = -ku_t(1, t-\theta), \tag{7}$$

where θ is the time delay.

The same problem exists in other forms of boundary controller laws. Moreover, this delay induced instability exists as well in the boundary control of beam equations with a time delay in the feedback loop [10], [17]. In [17], a class of dynamic boundary controllers were designed to stabilize the damped wave equation with a time delay, but failed to stabilize the conservative wave equation with a time delay.

So far, the instability problem remains unsolved. In this paper, we introduce the Smith predictor to solve this problem.

Comparing the equation (7) with Fig. 2, it is clear that the output y is the delayed velocity of the tip end, $C(s)$ is the static controller k, and $P_0(s)$ is the transfer function from the control force $f(t)$ to the un-delayed velocity of the tip end.

If we assume $\hat{P}_0(s) = P_0(s)$ and the time delay θ is known, the remaining problem is how to get $P_0(s)$. In [12], a method combining the symbolic algebra and numerical method was designed to simulate some typical boundary control problems including the boundary control of wave equation studied in this paper. One of the contributions of [12] is that, in the intermediate steps, $U(x,s)$, the Laplace transform of $u(x,t)$ with respect to t, can be explicitly obtained, which can be used in this paper to get a $P_0(s)$. Since wave equation is relatively simple, $P_0(s)$ can be obtained manually, however, for higher order systems, manual derivation can be very difficult and the method presented in [12] is very easy to implement and powerful.

Here we summarize the steps to get $P_0(s)$.

- Assuming zero initial conditions for $u(x,0)$ and $u_t(x,0)$, take the Laplace transform of (3), (4), and (5) with respect to t. Thus, the original PDE of $u(x,t)$ with initial and boundary conditions is transformed into ODE of $U(x,s)$ with boundary conditions.
- Call the Matlab Symbolic Math Toolbox function `dsolve()` to symbolically solve the ODE(s) and the boundary or initial condition(s). Although `dsolve()` is able to determine the arbitrary constants in the solution using the boundary or initial condition(s), we find that its capability is very weak. Here, we feed only the ODE of $U(x,s)$ to `dsolve()` rather than provide both the ODE of $U(x,s)$ and the boundary conditions. The expression of $U(x,s)$ with two arbitrary constants $C1$ and $C2$, which are to be determined later, can be obtained.
- Using Matlab Symbolic Math Toolbox function `diff()`, differentiate $U(x,s)$ with respect to x to get the first order derivative of $U(x,s)$. Substituting $U(x,s)$ and its derivative into the Laplace transform of (4) and (5), we can get two equations with two unknowns $C1$ and $C2$.
- Passing the two equations obtained in the last step to the Matlab Symbolic Math Toolbox function `solve()` to determine the constants $C1$ and $C2$. Now, we have obtained the explicit expression of $U(x,s)$.
- Dividing $U(x,s)$ by $F(s)$, the Laplace transform of $f(t)$, we can get $G(x,s)$, the transfer function from $f(t)$ to the displacement $u(x,t)$ at any point $x \in [0,1]$.
- Multiply $G(x,s)$ by s, the Laplace transform variable, and substitute x for 1 to get $sU(1,s)$. Divide $sU(1,s)$ by $F(s)$, the Laplace transform of boundary force $f(t)$, we get the transfer function $P_0(s)$ from $f(t)$ to $u_t(1,t)$, the velocity of the free end of the string.

Using the above procedure, the obtained expression of $P_0(s)$ is shown below

$$P_0(s) = \frac{\sinh(s)}{\cosh(s)}. \tag{8}$$

To help further understand the properties of $P_0(s)$, its Bode plot is shown in Fig. 4.

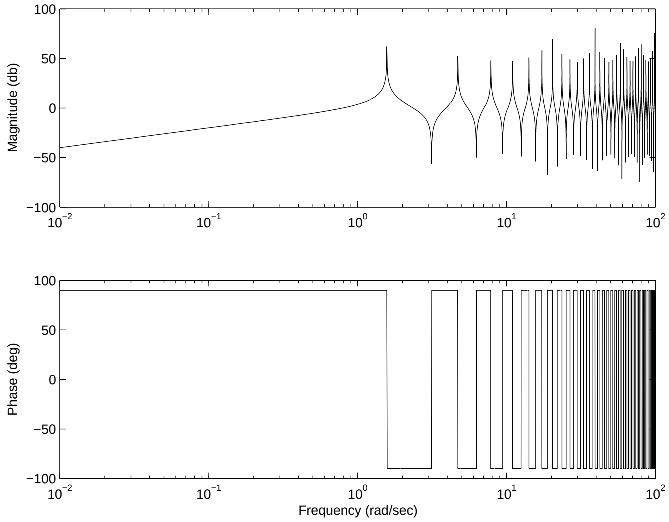

Fig. 4. Bode plot of $P_0(s)$

At this point, we have almost finished the controller design based on the configuration shown in Fig. 3. As commented in [15], although less effective in disturbance rejection, the Smith predictor is best suitable for tracking problems rather than regulation problems. In our case in this paper, we are more concerned with the regulation performance. Even worse is the situation that the initial conditions actually act as disturbances to the Smith predictor shown in Fig. 3. In section IV, we will demonstrate how the initial conditions, acting as disturbances, deteriorate severely the performance of the Smith predictor. To improve the regulation performance, some modified Smith predictors have been designed [14] [18], [19], [20]. In this paper, we use one of the modified Smith predictor schemes: the lead-lag compensation scheme shown in Fig. 5. We also applied a simple optimization search routine to determine the suitable parameters. We will explain implementation details of the two schemes and demonstrate the achieved performance in the next section.

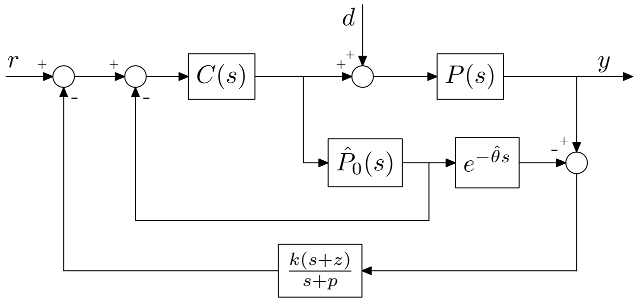

Fig. 5. Modified Smith predictor with a lead-lag filter

IV. SIMULATION RESULTS AND ANALYSIS

In our simulation study, we choose $k = 1$ and $\theta = 0.1$. The initial conditions are chosen as

$$u(x,0) = -\sin(0.5\pi x), \quad (9)$$

$$u_t(x,0) = 0. \quad (10)$$

If only the static feedback controller (7) is used, the simulation results are shown in Fig. 6 and Fig. 7. We can observe that the controller is working at the beginning, driving the tip end to the zero position. However, the frequency of the vibration is increasing over time. When the frequency is high enough, the time delay causes the control force to be in phase rather than out of phase with the tip velocity, thus making the system unstable.

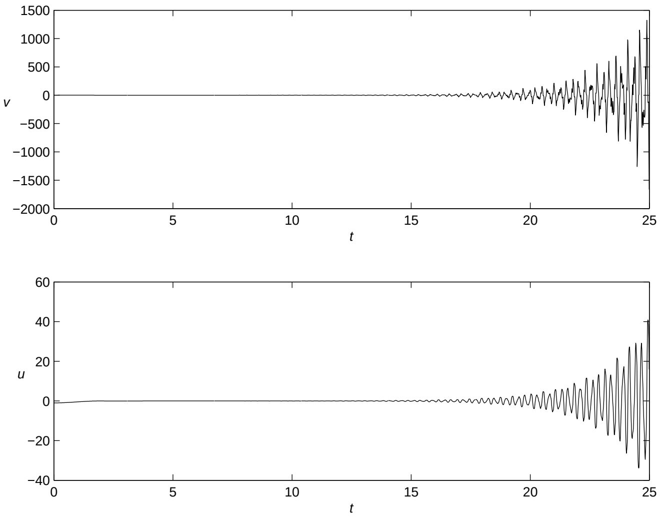

Fig. 6. Tip velocity and displacement without the Smith predictor

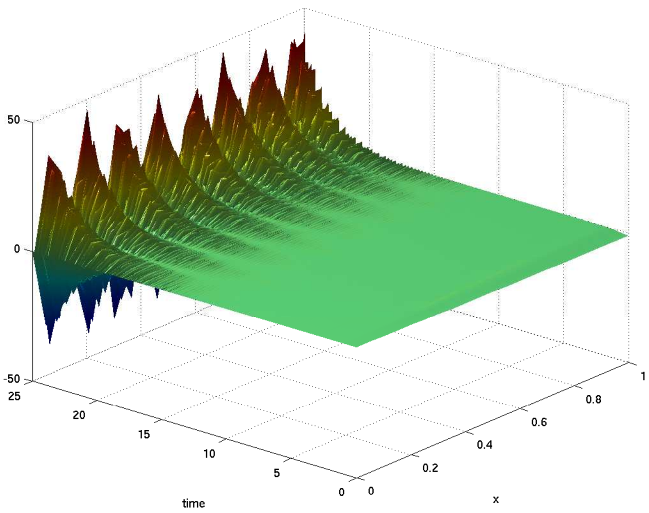

Fig. 7. Displacement of the whole string without the Smith predictor

After the Smith predictor shown in Fig. 2 is added, the simulation results are shown in Fig. 8 and Fig. 9. We can see that the velocity and displacement are not increasing to infinity over time, which is a great improvement over the results shown in Fig. 6 and Fig. 7. However, the string vibrates with a relatively large yet non-decreasing magnitude, rather than converging to zero position as we expected. As pointed out in [14], the open-loop poles are presented in G_d, the transfer function from the response y to the disturbance

d. These poles are excited by input disturbances but not by the reference. Depending on their locations relative to the closed-loop poles, these poles may dominate the response. In our case, the non-zero initial conditions act as disturbances, which deteriorate severely the regulation performance of the Smith predictor.

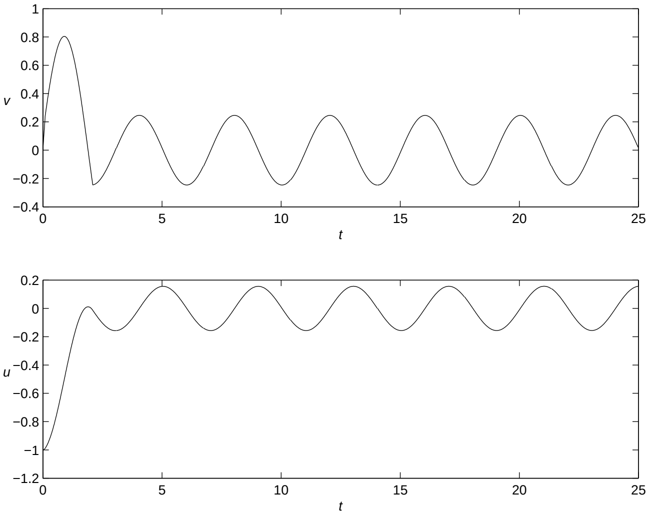

Fig. 8. Tip velocity and displace, the conventional Smith predictor

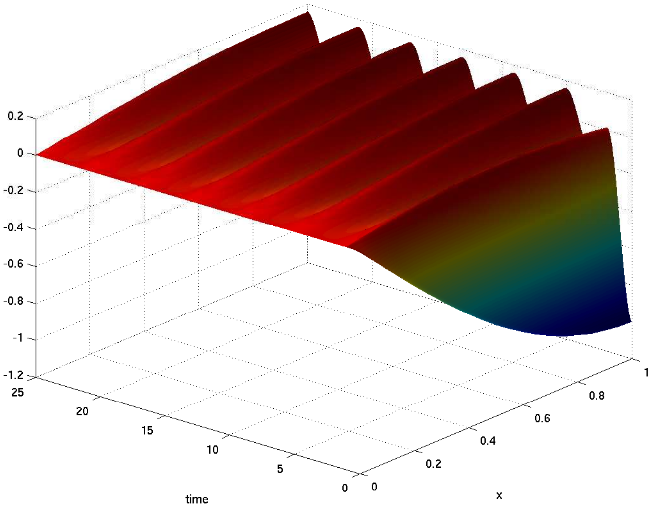

Fig. 9. Displacement of the whole string, the conventional Smith predictor

To get a better performance, we choose the modified Smith predictor in [18] and [19], shown in Fig. 5. The filter parameters, k, z, and p, are determined by an optimization search procedure. The optimal values are found to be that $k = 1.1118$, $z = 0.8864$, and $p = 1.2467$. The optimal filter turns out to be a lead filter. The simulation results are shown in Fig. 10 and Fig. 11.

Compared to the results in Fig. 8 and Fig. 9, we can see that the addition of an optimal lead-lag filter attenuates almost completely the output oscillation. To help further understand how the lead filter is working, its Bode plot is shown in Fig. 12. We can see that it is actually a high pass filter.

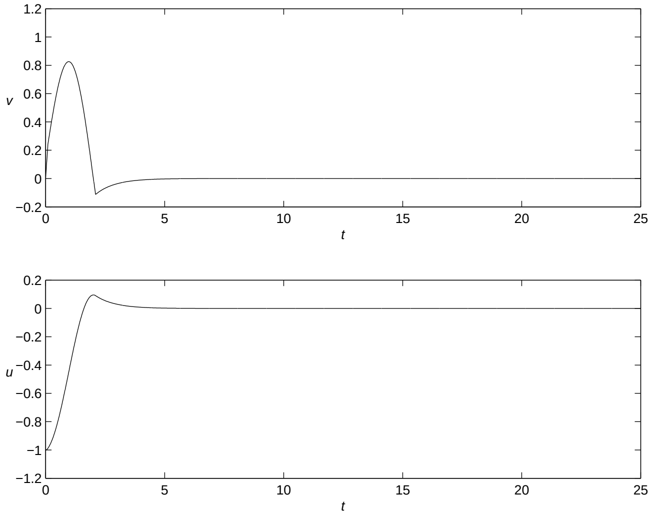

Fig. 10. Tip velocity and displace, the modified Smith predictor with a lead filter

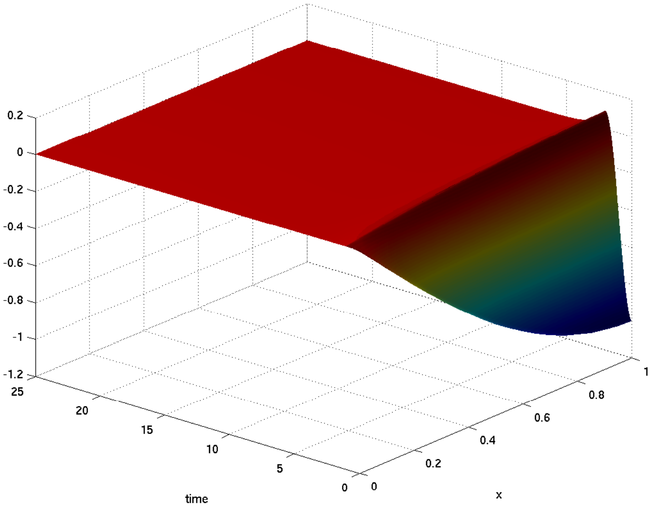

Fig. 11. Displacement of the whole string, the modified Smith predictor with a lead filter

Since the underlying idea of the Smith predictor is to remove the time delay terms from the denominator of the transfer function and the function of the optimal lead-lag filter is to attenuate the oscillation of the response, the final response is expected to be similar to the response of the system without delay using the simple static controller. The comparison of the responses to different time delays is shown in Fig. 13. For the same controller gain, larger time delay only increases the overshoot, rather than making the system unstable. As usual, since k is the derivative gain, small k makes the system under-damped and large k makes the system over-damped. These properties make controller design simpler. Also important to notice is that large time delay can change an over-damped system to an

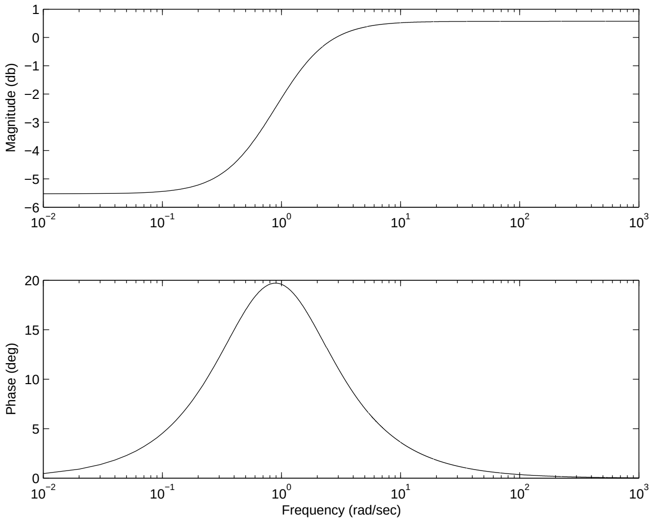

Fig. 12. Bode plot of the optimal lead filter

under-damped one.

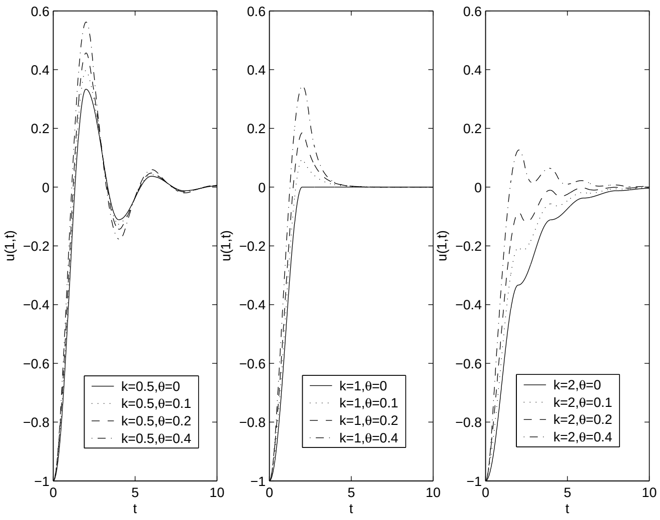

Fig. 13. Comparison of responses to different time delays

As a final remark, from Fig. 13, it is not hard to get a 3D plot of the decay rate versus delay θ and the boundary control gain k.

V. CONCLUDING REMARKS

With the introduction of the Smith predictor, the instability problem with the boundary control of wave equations with time delays has been solved for the first time in this paper. Although the boundary control of wave equation is similar to the boundary control of a beam equation in the presence of delayed boundary measurement, the wave equation case exhibits better performance using the same controller. The robustness issues and better boundary controller designs will be our future research efforts.

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