

# Optimal Switching Control Via Direct Search Optimization

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Abstract—We demonstrate in this paper that for switched systems the optimal switching time instants and the optimal control policy can be solved readily by direct search optimization. General problem formulation and numerical procedures are presented together with two demonstrative examples previously considered in the literature using other numerical solution methods.

## I. Introduction

Hybrid control [1] has become a hot research topic since it combines the standard control, where dynamic systems are typically described by differential or difference equations, with discrete logic or discrete events [2], [3]. When digital computers, digital networks, and embedded systems involved in control systems become ubiquitous and increasingly complex, understanding the coupling between logic-based components and continuous physical systems becomes important [4], [5]. Moreover, purposely making use of hybrid control strategies to achieve the control objective unachievable via conventional control methods has become practically appealing and feasible [6], [7], [8].

As a special type of hybrid system, a switched system [9], [10], [11] consists of several subsystems and a switching law specifying the active subsystem at each time instant. Many of the real-world processes, e.g., chemical processes, automotive systems, etc., can be modeled as switched systems. Optimal control problems are one of the most challenging and important classes of problems for switched systems [12], [13], [14], [15], since for an optimal control problem of a switched system, both an optimal continuous input and an optimal switching sequence have to be determined and the system dynamics may vary significantly before and after every switch.

Although theoretical investigation of hybrid system has been in persistent focus, see, e.g., [11], [17] and the references therein, the numerical solution issue in optimal switched systems is just a recent topic [18], [19], [20], [21], [22]. The general switched linear quadratic optimal control problem (GSLQ) discussed in [21] is particularly interesting and a two stage numerical optimization method was proposed with several illustrative examples. However, it is obvious that the optimization problem for the optimal switching time instants may be usually nonconvex which

will trouble the proposed Newton iteration based local optimization technique.

In this paper, we consider to apply the Luus-Jaakola optimization procedure [26] which is a well tested, efficient direct search method for the similar optimal switched systems. Our numerical experiments indicate that, the Luus-Jaakola optimization procedure is well suitable in numerically solving various types of optimal switched control problems. In principle, nonlinear systems can be easily handled. But due to the lack of suitable nonlinear optimal switched control benchmark problems, we only presented some results for the examples used in [19], [20], [21]. In our numerical experiments, we can consider simultaneously the optimal switching time instant and optimal control signals for each of the subsystems. Moreover, we can accommodate the constraints on time duration between two consecutive switching instants. Additionally, we considered a case with indefinite number of switching times, i.e., the number of switching instants is to be optimized.

## II. Optimal Control of Switched Systems: Problem Formulation

In general, a switched system can be described by a tuple  $\mathcal{S} = \{\mathcal{D}, \mathcal{F}\}$  where  $\mathcal{D} = (I, E)$  is a directed graph indicating the discrete structure of the system with its node set  $I = \{1, 2, \dots, M\}$  for subsystem indices and  $E$  is a subset of  $I \times I \setminus \{(i, i) \mid i \in I\}$  containing all valid events. So, if an event  $e = (i_1, i_2)$  occurs, the subsystem  $i_1$  switches to the subsystem  $i_2$ ;  $\mathcal{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m, i \in I\}$  is a set of vector fields with  $f_i$  describing the vector field for the  $i$ -th subsystem  $\dot{x} = f_i(x, u)$ .

As mentioned in Sec. I, the Luus-Jaakola optimization procedure is not limited to linear quadratic optimization problems. However, in this paper, due to the lack of suitable nonlinear optimal switched control benchmark problems, we focus on the general switched linear quadratic optimal control problems (GSLQ) as considered in [21]. The GSLQ considers a switched system  $\mathcal{S}$  with all linear subsystems  $\dot{x} = A_i x + B_i u, i \in I$ . Given a fixed time interval  $[t_0, t_f]$ , find a continuous input signal  $u(t), t \in [t_0, t_f]$  and a switching sequence  $\sigma = ((t_0, i_0), (t_1, e_1), (t_2, e_2), \dots, (t_K, e_K))$  with  $K$  the total number of switching,  $t_0 \in I$ ,  $t_{i-1} < t_i$  and  $e_i =$

$(t_{i-1}, t_i) \in E$  for  $i = 0, 1, \dots, K$ , such that the following general quadratic cost function is minimized:

$$J = \frac{1}{2}x^T(t_f)Q_f x(t_f) + M_f x(t_f) + W_f + \int_{t_0}^{t_f} \left( \frac{1}{2}x^T Q x + x^T V u + \frac{1}{2}u^T R u + M x + N u + W \right) dt \quad (1)$$

where  $t_0$ ,  $t_f$  and  $x(t_0) = x_0$  are given;  $Q_f, M_f, W_f, Q, V, R, M, N, W$  are matrices of appropriate dimensions;  $Q_f$  and  $Q$  are positive semidefinite and  $R$  is positive definite.

### III. Luus-Jaakola Optimization Procedure and Iterative Dynamic Programming

Let us first consider the general problem of minimizing the real-valued performance index

$$I = f(x_1, x_2, \dots, x_n) \quad (2)$$

subject to the set of inequality constraints

$$g_j(x_1, x_2, \dots, x_n) \leq 0, \quad j = 1, 2, \dots, s. \quad (3)$$

through the appropriate choice of the variables  $x_1, x_2, \dots, x_n$ .

Let us now consider an optimization procedure that does not require any auxiliary variables to be introduced to solve steady-state optimization problems as given by Eqs. (2)-(3). We will deal with difficult equality constraints later. The direct search optimization procedure suggested by Luus and Jaakola[23] may then be used. Conceptually, the optimization procedure is very simple, involving only three steps:

1. Given some initial point  $x^*$ , choose  $R$  random points in the  $n$ -dimensional space through the equation

$$x = x^* + D r \quad (4)$$

where  $D$  is a diagonal matrix, where randomly chosen diagonal elements lie in the interval  $[-1, +1]$ , and  $r$  is the region size vector.

2. Check the feasibility of each such randomly chosen point with respect to the inequality constraint in Eq. (2). For each feasible point evaluate the performance index  $I$  in Eq. (1), and keep the best  $x$ -value.

3. An iteration is defined by Steps 1 and 2. At the end of each iteration,  $x^*$  is replaced by the best feasible  $x$ -value obtained in step 2, and the region size vector  $r$  is reduced by  $\gamma$  through

$$r^{j+1} = \gamma r^j \quad (5)$$

where  $\gamma$  is a region contraction factor such as 0.95, and  $j$  is the iteration index.

This procedure is continued for a number of iterations and the results are examined.

To increase the efficiency of this optimization procedure and to make it applicable to high-dimensional optimization problems, it was found by Luus et al. [24] in solving an

84-dimensional cancer chemotherapy optimization problem, that the use of a multi-pass procedure (in which a relatively small number of randomly chosen points is used in each iteration) improved the computational efficiency. In the multi-pass method the three-step procedure is repeated after a given number of iterations, usually with a smaller initial region size than that in the previous pass.

The procedure is easy to program and, with the availability of very fast personal computers, a reasonable amount of computational inefficiency can be tolerated. One of the great advantages of the method is that no auxiliary variables are required, so that the user is closer to the problem at hand.

Details of the LJ optimization procedure in solving a wide variety of optimization problems are given in references [25], [26] and in Chapter 2 of [27]. The method can be used for optimization of systems of very high dimension [28].

For optimal control problems iterative dynamic programming (IDP) [27] is considerably more efficient, because instead of solving a complex problem all at once, a sequence of simpler problems are solved. Like LJ optimization procedure IDP requires no auxiliary variables such as adjoint variables to be evaluated, and therefore keeps the user close to the problem.

Remark 3.1: In solving the optimal switching control problem we shall attempt to combine LJ optimization procedure with IDP, where LJ optimization procedure is used for the determination of switching times and IDP is used for the determination of the performance index. These computations are attempted simultaneously inside each iteration.

### IV. Two examples

#### A. Example 1

This example was considered in [20], [21].

Consider a switched systems consisting three subsystems

$$\dot{x} = A_i x + B_i u, \quad i = 1, 2, 3, \quad (6)$$

where

$$A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.5 & 5.3 \\ -5.3 & 0.5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this case,  $t_0=0$ ,  $t_f = 3$  and the system switches at  $t = t_1$  from subsystem 1 to 2 and at  $t = t_2$  from subsystem 2 to 3 ( $0 \leq t_1 \leq t_2 \leq 3$ ).

Find the optimal  $t_1$  and  $t_2$  and control signal  $u$  such that

$$J = [(x_1(3) + 4.1437)^2 + (x_2(3) - 9.3569)^2] + 0.5 \int_0^3 u^2(t) dt \quad (7)$$

is minimized. The initial states are given by  $x(0) = [4, 4]^T$ . No constraint is assumed on  $u(t)$ . This is a generalized LQR problem [19], [21].

To run this problem, we used LJ optimization procedure for the searching of the switching times  $t_1$  and  $t_2$  and simultaneously used IDP for the determination of the control policy  $u$ . The time interval was divided into 60 stages, each of length 0.05, and piecewise constant control was determined for each time stage. As starting points we chose  $t_1 = 0.1$ ,  $t_2 = 2.9$ , and  $u = 0.5$  for each stage, and the initial region size 2.0 for control, and 0.2 for the switching times. For integration of the differential equations, we used the DVERK subroutine with local error tolerance of  $10^{-7}$ . We took as the region reduction factor  $\gamma = 0.95$ , by which the search region was reduced after each iteration, and region restoration factor  $\eta = 0.90$  by which the region was restored after every pass. We used a single grid point at each time stage and 40 iterations in each pass. We allowed the number of random points chosen in each iteration  $R$  to be a parameter that was varied from run to run.

After 100 passes, from the initial value  $J = 1134.41537$ , we reached  $J = 2.427 \times 10^{-9}$  with the use of  $R = 15$  random points per iteration. The use of a larger number of random points did not improve the performance index. At the optimum  $t_1 = 0.59395$ ,  $t_2 = 2.78328$ , and  $u = 0$  for each stage. The total computation time for the 100 passes with  $R = 15$  was 989 s on an AMD Athlon XP-2000 personal computer.

Since the reported values for the switching times are  $t_1 = 1$ ,  $t_2 = 2$ , we made a set of runs starting in the vicinity of these switching times, namely  $t_1 = 0.9$ ,  $t_2 = 2.1$ ,  $u = 0.5$ , keeping the other parameters the same as before. The convergence with  $R = 50$  from the initial value of  $J = 85.72995$  to  $J = 2.15730 \times 10^{-8}$ , with  $t_1 = 0.99996$ ,  $t_2 = 1.99998$ ,  $u = 0$ , is not as good as before and there is a levelling off.

To examine this convergence difficulty, we reduced the local error tolerance for integrating the differential equations to  $10^{-8}$ , and obtained with  $R = 25$  the performance index  $J = 3.3039 \times 10^{-9}$  with  $t_1 = 0.99998$ ,  $t_2 = 1.99999$ . Then the use of local error tolerance of  $5 \times 10^{-9}$  after 150 passes with  $R = 25$  random points per iteration yielded the performance index  $J = 3.787 \times 10^{-11}$  with  $t_1 = 1.00000$ ,  $t_2 = 2.00000$ . In switching problems the error tolerance is quite important, because this determines the precision with which the switching times can be calculated. When we used this smaller value for error tolerance in the previous case the result was improved to  $J = 1.142 \times 10^{-11}$  with  $t_1 = 0.59396$ ,  $t_2 = 2.78329$ .

## B. Example 2

This example was considered in [21].

Consider a switched system consisting of

$$\text{subsystem 1 : } \dot{x} = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u; \quad (8)$$

and

$$\text{subsystem 2 : } \dot{x} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u. \quad (9)$$

It is known that  $t_0 = 0$  and  $t_f = 2$ . The system switches once at time instant  $t = t_1$  and  $t_1 \in [0, 2]$  from “subsystem 1” to “subsystem 2”. The control objective here is to find the optimal  $t_1$  and the control signal  $u(t)$  ( $t \in [0, 2]$ ) such that

$$J = \frac{1}{2}(x_1(2) - 1)^2 + \frac{1}{2}(x_2(2) - 2)^2 + \frac{1}{2} \int_0^2 [(x_2(t) - 2)^2 + u^2(t)] dt \quad (10)$$

is minimized. The initial states are given as  $x(0) = [0, 2]^T$ .

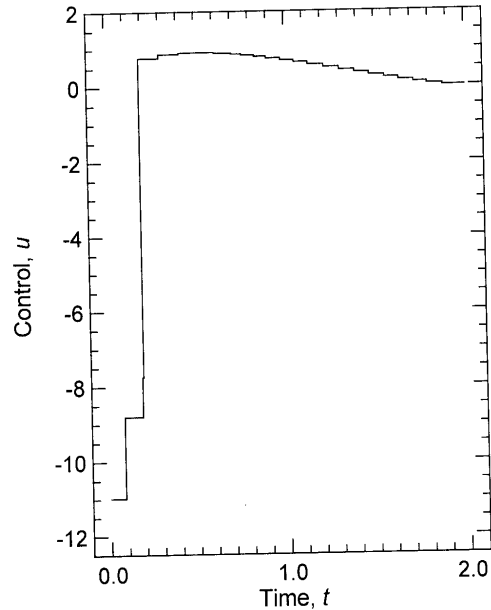


Fig. 1. Control policy for Example 2, with the use of 25 time stages, yielding  $J = 9.8027$

Although this problem appears to be simpler than Example 1, since there is only a single switch, the scheme used before does not work here because there is a very large discontinuity in the control policy in going from system 1 to system 2. Therefore, an easier way of obtaining the switching time and the control policy is to carry out the optimization separately. We used dichotomous search as outlined in [29] to establish the switching time  $t_1 = 0.190$  in 12 runs.

With this switching time, the optimal control policy was obtained with IDP, using 25 time stages of variable lengths as described in [30] and Chapter 9 of [27], yielding the control policy in Fig. 1, giving a performance index  $J = 9.8027$ . By using a larger number of stages then, a more refined optimal control policy is obtained, as is shown in Fig. 2. The use of  $t_1 = 0.190$  and 100 time stages of varying lengths gave a minimum performance index  $J = 9.7686$ .

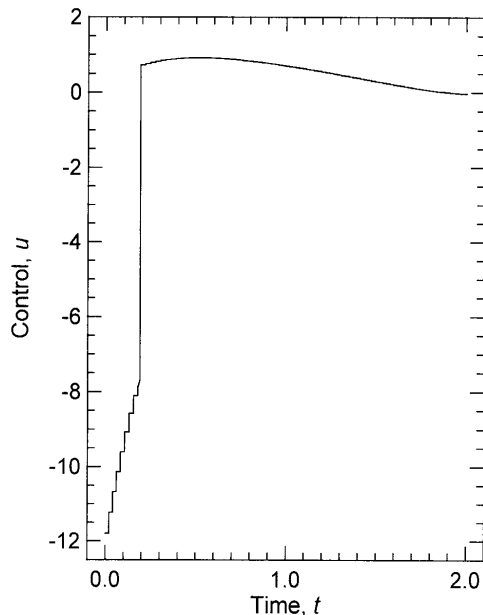


Fig. 2. Control policy for Example 2, with the use of 100 time stages, yielding  $J = 9.7686$

### V. Concluding Remarks

We demonstrated in this paper that for switched systems the optimal switching times and the optimal control policy can be established by incorporating the Luus-Jaakola optimization procedure into the iterative dynamic programming formulation. This works well if there is not a very large change in the control policy when the switching from one system to another takes place. However, if there is a very large change in the optimal control policy when switching from one system to another, the optimization can be done separately, as illustrated by the second example. The approaches presented here provide alternative procedures to existing methods and are applicable to nonlinear systems, and constraints can be readily handled.

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